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Design of Wavelet-Based Differentiator Filter

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Abstract: Differentiators are used in many analog and digital systems to take the derivative of a signal. The signal is affected by the presence of the machine epsilon of the computer which may be considered to be an extremely high frequency noise of very small amplitude. Perturbation is also caused by the presence of noise in the signal. It is the purpose of this work to construct a wavelet based band-pass filter that removes the noise and computes the derivative of the signal.

Keywords: High frequency noise, Band-pass filter, Extrapolation, Error Estimate.

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1. Introduction

It is interesting to note that sometimes process of differentiation reduces to integration as in the case of Cauchy integral formula which is applicable if the function is analytic. We know that in case of finite-difference method $f'(x)$ is approximated with an error of order h. In case of central difference, approximation to $f'(x)$ is better than the forward or backward difference approximation and the error is of order h^2 . One can see[1] and[2] for details. It is the purpose of this paper to construct a wavelet-based band pass filter that acts like a smooth difference quotient whose step size is of the same order as that of the usual difference quotient, but approximation of $f'(x)$ is more accurate with an error of order h^8. However in the presence of high-frequency noise the previous difference quotients, as well as their iterated counterparts, are essentially useless to compute $f'(x)$. We present in this work a technique based on the construction of an appropriate wavelet, with many vanishing moments which, when being convolved in a precise manner with the function, acts as a band pass filter and at the same time as a difference quotient. This technique has been already described explicitly by Maurice Hasson in his paper[3]. His construction is based on Mexican hat function. In this work the author uses derivative of the Gaussian Function in order to construct the wavelet filter. Of course, this technique to be effective requires that the convolution integrals be computed with high accuracy and at low cost. We may guess that there is a possibility of computing the convolution involving these wavelets by a quadrature rule with almost machine accuracy and at very low cost. By the appropriate use of Euler – McLaurin Summation formula in conjunction with the trapezoidal or simpson`s rule, one may compute the integrals involved in the process. One may go through the text[4] for detailed study of the formula and rules.

Through out this paper we use the following notations for Fourier transform $\widehat{f}(\omega)$ of a function f(x):

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

and the inversion formula takes the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega x} dw.$$

2. Construction of the wavelet filter and computation of derivatives

This section is devoted in building the wavelet $\psi_2(x)$ satisfying the following properties (2.1) and (2.2). Once $\psi_2(x)$ is built we will analyze the rate at which it approximates the derivative of a given function.

(2.1) $$\int_{-\infty}^{\infty} t^k \psi_2(t) dt = 0, \qquad k = 0,2,3,4$$

(2.2) $$\int_{-\infty}^{\infty} \psi_2(t) \ t dt = -1$$

2.1 Construction of the wavelet: We begin the required construction with function $\psi(x)$ which is derivative of the Gaussian function

$$\psi(x) = -xe^{-x^2/2}.$$

Its Fourier transform $\widehat{\psi}(\omega)$ is

(2.3) $$\widehat{\psi}(\omega) = \sqrt{2\pi}(i\omega) e^{-\omega^2/2}$$

One can see that

$$(e^{-x^2/2})' = -xe^{-x^2/2}.$$

Therefore, Fourier transform of $(-x)e^{-x^2/2}$ is equal to $(i\omega)\widehat{f}(\omega)$, Here $\widehat{f}(\omega)$ is the Fourier transform of $f(x) = e^{-x^2/2}$, We know that Fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}e^{-\omega^2/2}$. Thus Fourier transform of ψ(x) is given by (2.3).

Let us define,

$$\widehat{\psi}_1(\omega): \; = \frac{\widehat{\psi}(\omega)}{\sqrt{2\pi}}.$$

Now, we have

(2.4) $$\widehat{\psi}_1(\omega) = (i\omega)e^{-\omega^2/2} = (i\omega)(1 - \frac{\omega^2}{2} + \frac{\omega^4}{8} - \frac{\omega^6}{48} + \ldots\ldots\ldots\ldots).$$

By the above expansion, we have

$$\widehat{\psi}_1^{(k)}(0) = 0 \qquad for \quad k = 0,2$$

and

$$\widehat{\psi}_1^{(1)}(0) = i$$

Hence

$$\int_{-\infty}^{\infty} (\frac{x}{h})\frac{1}{h}\psi_1(\frac{x}{h})dx = \int_{-\infty}^{\infty} x\psi_1(x)dx = -1.$$

From (2.4), we have

(2.5) $$\widehat{\psi}(\omega/2) = (i\omega/2)(1 - \frac{\omega^2}{8} + \frac{\omega^4}{128} - \frac{\omega^6}{3072} + \ldots\ldots\ldots\ldots).$$

By following an adaptation of the classical Richardson extrapolation method[5]. We define

$$\widehat{\psi}_2(\omega): \; = \frac{\widehat{\psi}_1(\omega) - 8\widehat{\psi}_1(\omega/2)}{-3}.$$

Therefore, using (2.4) and (2.5), we have

(2.6) $$\widehat{\psi}_2(\omega) = (i\omega)(1 - \frac{\omega^4}{32} + \frac{5\omega^6}{768}\ldots\ldots\ldots\ldots)$$

This implies that

$$\widehat{\psi}_2^{(k)}(0) = 0 \qquad for \quad k = 0, 2, 3, 4$$

and

$$\widehat{\psi}_2^{(1)}(0) = i$$

Hence, we have again

$$\int_{-\infty}^{\infty} (\frac{x}{h}) \frac{1}{h} \psi_2 (\frac{x}{h}) dx = \int_{-\infty}^{\infty} x \psi_2 (x) dx = -1$$

and

$$\psi_2(x) = \frac{\psi_1(x) - 16\psi_1(2x)}{-3}.$$

Let us summarize the above construction in the form of

Theorem 2.1: *For the wavelet $\psi_2(x)$ defined by*

(2.7) $$\psi_2(x) = \frac{1}{3\sqrt{2\pi}} (xe^{-x^2/2} - 32xe^{-2x^2})$$

Equations (2.1) and (2.2) hold

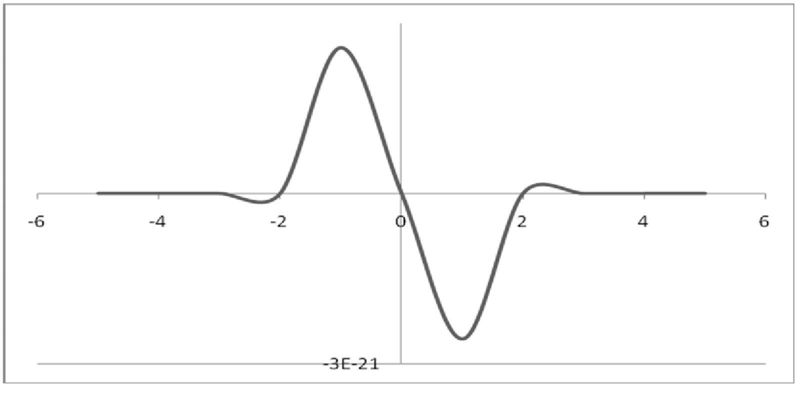

Figure 1: The Wavelet $\frac{1}{h}\psi_2(\frac{x}{h}), \qquad h = \frac{1}{10}$

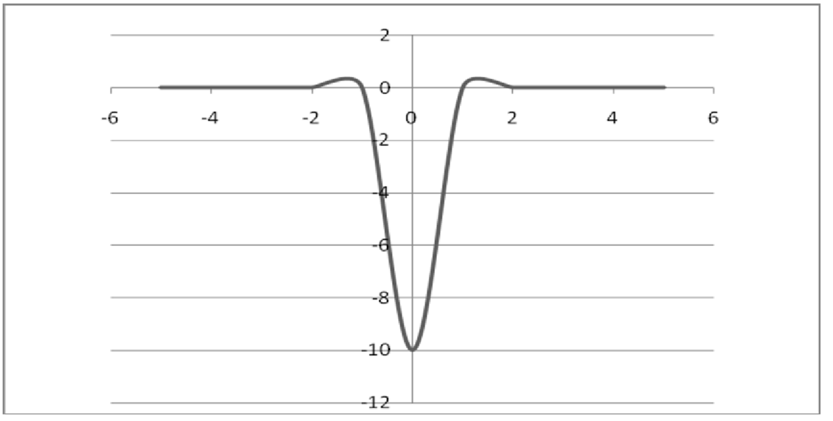

Figure 2: The Wavelet $\frac{1}{h}\psi_3(\frac{x}{h}), \qquad h = \frac{1}{10}$

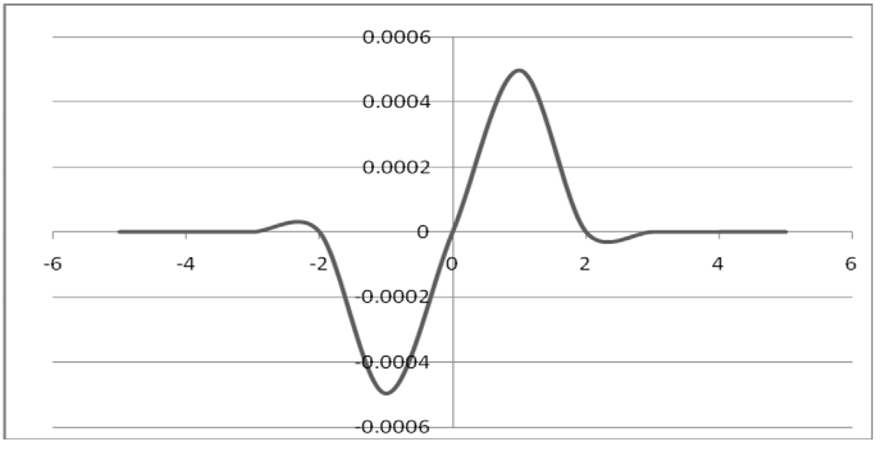

Figure 3: The Fourier transform $\widehat{\psi}_2(\omega)$ of $\psi_2(x)$, illustrating the band pass filter characteristics

Remark 2.1 By reiterating the Richardson extrapolation technique on $\psi_2(\omega)$, one can obtain

$$\psi_3(x) = \frac{1}{45\sqrt{2\pi}}(-xe^{-x^2/2} + 160xe^{-2x^2} - 4096xe^{-8x^2}).$$

Indeed the analysis of such a wavelet shows that the last term is identically zero in our experiment. This proves the inadequacy of the wavelet $\psi_3(x)$. This is why we do not reiterate $\psi_2(x)$.

2.2 Computation of derivatives

In the next theorem we will compute f `(x) and will present the analysis of the error estimate

Theorem 2.2: *Let f(x) be a smooth function, then*

(2.8)
$$\left| \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{h} f(x-t)\psi_2(\frac{t}{h})dt - f`(x) \right|$$

$$\leq C_1 f^{(5)}(x)h^4 + C_2 f^{(7)}(x)h^6 + O(h^8)$$

where $\psi_2(x)$ is given by (2.7) and

(2.9)
$$C_1 = \frac{1}{5!}\int_{-\infty}^{\infty} t^5\psi_2(t)dt = 0.03125$$

and

(2.10)
$$C_2 = \frac{1}{7!}\int_{-\infty}^{\infty} t^7\psi_2(t)dt = 0.0013021$$

Proof :
$$\frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{h} f(x-t)\psi_2(\frac{t}{h})dt$$

$$= \frac{1}{h}\int_{-\infty}^{\infty} f(x-ht)\ \psi_2(t)dt$$

(2.10)
$$= \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{h}(f(x) - thf`(x))\ \psi_2(t)dt + \frac{1}{h}O(h^2)$$

Now using (2.1) with k=0, 1 and (2.2) , we see from (2.10) that

(2.11) $$\lim_{h\to 0}\frac{1}{h}\int_{-\infty}^{\infty}\frac{1}{h}f(x-t)\ \psi_2(\frac{t}{h})dt = f`(x)$$

Now using (2.2) again

$$\frac{1}{h}\int_{-\infty}^{\infty}f(x-t)\frac{1}{h}\psi_2(\frac{t}{h})dt - f`(x)$$

$$=\frac{1}{h}\int_{-\infty}^{\infty}\left(f(x-t)\frac{1}{h}\psi_2(\frac{t}{h})dt+\frac{t}{h}\psi_2(\frac{t}{h})f`(x)\right)dt$$

$$=\frac{1}{h}\int_{-\infty}^{\infty}(f(x-th)+thf`(x))\ \psi_2(t)dt$$

We Know that

(2.12) $$f(x-th)=\sum_{k=0}^{4}(-1)^k\frac{f^{(k)}(x)}{k!}t^k h^k + O(t^5h^5)$$

Therefore, from (2.1) and (2.12) we have

$$\left|\frac{1}{h}\int_{-\infty}^{\infty}f(x-t)\frac{1}{h}\psi_2(\frac{t}{h})dt - f`(x)\right| \le \left|\frac{1}{h}\int_{-\infty}^{\infty}\left(\frac{t^5h^5}{5!}f^{(5)}(x)\right)\psi_2(t)dt\right|$$
$$+\left|\frac{1}{h}\int_{-\infty}^{\infty}\left(\frac{t^7h^7}{7!}f^{(7)}(x)\right)\psi_2(t)dt\right|+\frac{1}{h}O(h^9)$$

Here we have used the fact that $\psi_2^{(k)}(0)=0$ if k is even.
Now let us calculate the values of the quantities C_1 and C_2.
We have,

$$\int_{-\infty}^{\infty}x^5\psi_2(x)dx = i\widehat{\psi}_2^{(5)}(0)$$

Using (2.6), we find that

$$\int_{-\infty}^{\infty} x^5 \psi_2(x)dx = 3.75$$

In a similar manner we find that

$$\int_{-\infty}^{\infty} x^7 \psi_2(x)dx = 6.5625$$

This completes the proof of the theorem

Remark 2.2: It is easy to check that $\psi_2(x)$ is a linear invariant system (operator). Now, let us examine the stability of the system $\psi_2(x)$. An arbitrary system is said to be stable if and only if every bounded input produces a bounded output. If the input signal f(x) is bounded, there exists a constant M_f such that

$|f(x)| < M_f$ for all x.

Now consider the convolution formula

$$g(x) = \int_{-\infty}^{\infty} f(x-t)\psi_2(t)dt$$

where g(x) is the output signal.

Therefore,

$$|g(x)| = \left|\int_{-\infty}^{\infty} f(x-t)\psi_2(t)dt\right|$$

$$\leq \int_{-\infty}^{\infty} |f(x-t)||\psi_2(t)|dt$$

$$\leq M_f \int_{-\infty}^{\infty} |\psi_2(t)|dt \tag{2.13}$$

Now

$$\int_{-\infty}^{\infty} |\psi_2(t)|dt = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|te^{-t^2/2} - 32te^{-2t^2}\right|dt$$

$$\leq \frac{1}{3\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} \left|te^{-t^2/2}\right|dt + 32\int_{-\infty}^{\infty} \left|te^{-2t^2}\right|dt\right]$$

$$\leq \frac{1}{3\sqrt{2\pi}}\left[2\int_0^\infty \left|te^{-t^2/2}\right|dt + 64\int_0^\infty te^{-2t^2}\,dt\right]$$

$$\leq \frac{1}{3\sqrt{2\pi}}\left[2+16\right] = \frac{6}{\sqrt{2\pi}}.$$

This together with (2.13) proves that output is bounded .Hence the system $\psi_2(x)$ is stable.

Remark 2.3 In the case of a polynomial $P_n(x)$ of degree n, we have the following result, regardless the value of h,

$$\frac{1}{h}\int_{-\infty}^{\infty}\frac{1}{h}P_n(x-t)\psi_2(\frac{t}{h})dt = P_n^{`}(x) \qquad \text{for } n = 0,1,2,3,4$$

3. Frequency Domain Characterization of the Filter

Let us consider the Fourier transform of the convolution integral

$$\frac{1}{h}\int_{-\infty}^{\infty} f(x-t)\frac{1}{h}\psi_2(\frac{t}{h})dt$$

$$= \frac{1}{h^2}\int_{-\infty}^{\infty} e^{-iwx}\left[\int_{-\infty}^{\infty} f(x-t)\ \psi_2(\frac{t}{h})dt\right]dx$$

$$= \frac{1}{h^2}\int_{-\infty}^{\infty} \psi_2(\frac{t}{h})\left[\int_{-\infty}^{\infty} e^{-iwx} f(x-t)dx\right]dt$$

$$= \frac{\widehat{f}(\omega)}{h^2}\int_{-\infty}^{\infty}\psi_2(\frac{t}{h})e^{-iwt}\,dt$$

$$= \frac{\widehat{f}(\omega)}{h}\widehat{\psi}_2(\omega h)$$

$$= \widehat{f}(\omega).(i\omega) \qquad as \quad h \to 0$$

Here we have used (2.6).

We know that $\widehat{f}(\omega).(i\omega)$ is the Fourier transform of the function f `(x). Hence using the above technique one can express the derivative of the signal in frequency domain. Thus differentiation is obtained through multiplication in frequency domain.

Example 3.1 Let us excite the system with the complex exponential

$$f(x) = Ae^{i\omega x} \qquad -\infty < x < \infty$$

where A is the amplitude and ω is any arbitrary frequency .
We obtain the response

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} Ae^{i\omega(x-t)}\psi_2(t)dt \\ &= Ae^{i\omega x}\int_{-\infty}^{\infty} e^{-i\omega t}\psi_2(t)dt \\ &= Ae^{i\omega x}\hat{\psi}_2(\omega). \end{aligned}$$

As a result of this characteristic behavior the exponential signal is an eigen function of the system. Moreover, one can see that our wavelet function is also useful for the purpose of amplitude modulation.

4. Conclusion

Our wavelet $\psi_2(x)$ is an odd function and it is linear time-invariant stable system. It is, of course, remarkable that differentiation reduces to multiplication in frequency domain. Our wavelet is also useful for amplitude modulation as shown in the example 3.1.

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