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Certain Properties and Integral Transforms of the *k*-Generalized Mittag-Leffler Type Function $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$*

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Abstract: In this paper, we have introduced and study of the k-generalized Mittag-Leffler type function $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$. Moreover, we proved some of its properties including usual Differential, Integration, Euler (Beta) transforms, Laplace transforms and Whittaker transforms. Some special cases have also been discussed.

Keywords: *k*-Generalized Mittag-Leffler function, *k*-Mittag-Leffler function, *k*-Pochhammer symbol, *k*- Gamma function, *k*- Beta function, Euler (Beta) transform, Laplace transform, and Whittaker transform.

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1. Preliminaries

The *k*-Pochhammer symbol $(x)_{n,k}$ was introduced by Diaz and Pariguan[1] in the form as

(1.1) $$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \text{ where } x \in C,\ k \in R,\ n \in N .$$

k – Gamma function $\Gamma_k(x)$ was defined by Diaz and Pariguan[1] as

(1.2) $$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1} dt, \quad x \in C, \quad k \in R, \ \mathrm{Re}(x) > 0,$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x).$$

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k – Beta function $B_k(x,y)$ was also defined by Diaz and Pariguan[1] as

(1.3) $$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,\ x\in C,\ k>0,\ \text{Re}(x)>0,\ \text{Re}(y)>0,$$

and

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}.$$

The well known Gamma function $\Gamma(n)$ is defined as

(1.4) $$\Gamma(n)=\int_0^\infty e^{-t}t^{n-1}dt,\quad \text{Re}(n)>0.$$

The Beta function $B(m,n)$ is defined as

(1.5) $$B(m,n)=\int_0^1 t^{n-1}(1-t)^{m-1}dt=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \text{Re}(m)>0,\ \text{Re}(n)>0.$$

The Fox- Wright function ${}_p\psi_q(z)$ was defined by Srivastava and Karlsson[2]as

(1.6) $${}_p\psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p); \\ \\ (b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\ ;z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_in)_n}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)_n}\frac{z^n}{n!},$$

where

$$z,\ a_i,\ b_j,\ A_i,\ B_j\in C,\ \text{Re}(a_i)>0,\ \text{Re}(A_i)>0,\ i=1,..,p,\ \text{Re}(b_j)>0,\ \text{Re}(B_j)>0,$$

$$j=1,.,q\ \text{and}\ 1+\text{Re}\left(\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\right)\geq 0.$$

The following known identities are required in our seque[3].

Result I: Let $\gamma\in C,$ and $k,\ s\in R,$ then the following identity holds

(1.6) $$\Gamma_s(\gamma)=\left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right),$$

and in particular case

(1.7) $$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\,\Gamma\left(\frac{\gamma}{k}\right).$$

Result II: Let $\gamma \in C,\ k,\ s \in R,$ and $n \in N,$ then the following identity holds

(1.8) $$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq,k},$$

and in particular case

(1.9) $$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq}.$$

2. A New Generalized Mittag-Leffler Type Function

In this section, we introduce and define a new Generalized Mittag-Leffler type function, called as $k-$Generalized Mittag-Leffler function $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$ and mention some of its special cases.

Definition 2.1: Let $k \in R,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \upsilon,\ \rho,\ \sigma \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,$ $\mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(\delta) > 0,\ \mathrm{Re}(\mu) > 0,\ \mathrm{Re}(\upsilon) > 0,\ \mathrm{Re}(\rho) > 0,\ \mathrm{Re}(\sigma) > 0$ and $p,\ q > 0,$ $q \le \mathrm{Re}(\alpha) + p,$ the $k-$Generalized Mittag-Leffler function denoted by $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$ and is defined as

(2.1) $$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}\,(\gamma)_{qn,k}\,z^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}\,(\delta)_{pn,k}},$$

where $(x)_{\tau n,k}$ is the $k-$Pochhammer symbol given by (1.1) and $\Gamma_k(x)$ is the $k-$Gamma function given by (1.2).

Special cases of $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$:

On giving some particular values to the parameters $k, \alpha, \beta, \gamma, \delta, \mu, \upsilon, \rho, \sigma,$ we can obtain certain Mittag-Leffler functions defined earlier

(a) For $k=1$, Eqn. (2.1) reduces to Mittag-Leffler function defined by Khan and Ahmed[4],

(2.2)
$$E_{1,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,1}(\gamma)_{qn,1}\,z^n}{\Gamma_1(\alpha n+\beta)(\upsilon)_{\sigma n,1}(\delta)_{pn,1}}$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}\,z^n}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$=E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)\,.$$

(b) Putting $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1, \delta=1$, Eqn. (2.1) reduces to Mittag-Leffler function defined by Shukla and Prajapati[5],

(2.3)
$$E_{1,\alpha,\beta,\mu,\rho,1,1}^{\mu,\rho,\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,1}(\gamma)_{qn,1}\,z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^n}{\Gamma(\alpha n+\beta)\,n!}=E_{\alpha,\beta}^{\gamma,q}(z)\,.$$

(c) On taking $k=1,\mu=\upsilon,\rho=\sigma, p=1, q=1,\delta=1$ Eqn. (2.1) reduces to Mittag-Leffler function defined by Prabhakar[6],

(2.4)
$$E_{1,\alpha,\beta,\mu,\rho,1,1}^{\mu,\rho,\gamma,1}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,1}(\gamma)_{1n,1}\,z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}\,z^n}{\Gamma(\alpha n+\beta)\,n!}=E_{\alpha,\beta}^{\gamma}(z)\,.$$

(d) Giving $k=1,\mu=\upsilon,\rho=\sigma, p=q=\delta=\gamma=1$, Eqn. (2.1) reduces to Mittag-Leffler function defined by Wiman[7],

(2.5)
$$E_{1,\alpha,\beta,\mu,\rho,1,1}^{\mu,\rho,1,1}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,1}(1)_{1n,1}\,z^n}{\Gamma_1(\alpha n+\beta)(\mu)_{\rho n,1}(1)_{1n,1}}$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}=E_{\alpha,\beta}(z)\,.$$

(e) Assigning $k=1, \mu=\upsilon, \rho=\sigma, p=q=\delta=\gamma=\beta=1$, Eqn. (2.1) reduces to Mittag-Leffler function $E_{\alpha}(z)$ defined byGosta Mittag-Leffler[8],

(2. 6) $$E^{\mu,\rho,1,1}_{1,\alpha,1,\mu,\rho,1,1}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,1}(1)_{1n,1}\,z^n}{\Gamma_1(\alpha n+1)(\mu)_{\rho n,1}(1)_{1n,1}}$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}=E_{\alpha}(z).$$

3. Basic Properties

In this section, we introduce some new basic properties of the of $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$.

Theorem 3.1: *We introduced here an elegant relation between* $k-$ *Generalized Mittag-Leffler type function* $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$ *and Generalized Mittag-Leffler type function as follows*

(3.1) $$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=k^{1-\frac{\beta}{k}}E^{\frac{\mu}{k},\rho,\frac{\gamma}{k},q,}_{\frac{\alpha}{k},\frac{\beta}{k},\frac{\upsilon}{k},\sigma,\frac{\delta}{k},p,}\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k},}z\right).$$

Proof: From the definition of $k-$Generalized Mittag-Leffler function given by the Eqn. (2.1), we have

$$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\,z^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}},$$

On using Eqns. (1.7) and (1.9), we have

$$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=k^{1-\frac{\beta}{k}}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{k}\right)_{\rho n}\left(\frac{\gamma}{k}\right)_{qn}\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}z\right)^n}{\Gamma_k\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)\left(\frac{\upsilon}{k}\right)_{\sigma n}\left(\frac{\delta}{k}\right)_{pn}}$$

$$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=k^{1-\frac{\beta}{k}}E^{\frac{\mu}{k},\rho,\frac{\gamma}{k},q,}_{\frac{\alpha}{k},\frac{\beta}{k},\frac{\upsilon}{k},\sigma,\frac{\delta}{k},p,}\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k},}z\right).$$

This completes the proof of the theorem (3.1).

Theorem 3.2: *If* $k \in R,\ \alpha, \beta, \gamma, \delta, \mu, \upsilon, \rho, \sigma \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,$ $\mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(\delta) > 0,\ \mathrm{Re}(\mu) > 0,\ \mathrm{Re}(\upsilon) > 0, \mathrm{Re}(\rho) > 0\ , \mathrm{Re}(\sigma) > 0$ *and* $p, q > 0,$ $q \le \mathrm{Re}(\alpha) + p,$ *then*

$$E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z) = \beta E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\upsilon,\sigma,\delta,p}(z) + \alpha z \frac{d}{dz} E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\upsilon,\sigma,\delta,p}(z) \tag{3.2}$$

Proof: In the right-hand side of Eqn. (3.2), and making use of the series form of $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$ given by the Eqn. (2.1), we have

$$\begin{bmatrix} \beta E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\upsilon,\sigma,\delta,p}(z) \\ +\alpha z \dfrac{d}{dz} E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta+k,\upsilon,\sigma,\delta,p}(z) \end{bmatrix} = \begin{bmatrix} \beta \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta + k) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}} \\ +\alpha z \dfrac{d}{dz} \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta + k) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}} \end{bmatrix}$$

$$= \begin{bmatrix} \beta \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta + k) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}} \\ +(\alpha n) \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta + k) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}} \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha n + \beta)(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta + k) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}}$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k} (\gamma)_{q n,k}\, z^n}{\Gamma_k(\alpha n + \beta) (\upsilon)_{\sigma n,k} (\delta)_{p n,k}}$$

$$= E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z).$$

This completes the proof of the theorem (3.2).

Special Cases:

1. On Setting $k = 1,$ in Eqn. (3.2), takes the following result, we have

$$E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\beta E_{\alpha,\beta+1,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z),$$

which is the same result as obtained by Khan and Ahmed[4].

2. Putting $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1,$ and $\delta=1,$ Eqn.(3.2), reduces to the following result, we get

$$E_{\alpha,\beta}^{\gamma,q}(z)=\beta E_{\alpha,\beta+1}^{\gamma,q}(z)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,q}(z).$$

which is the same result as deduced by Shukla and Prajapati[5].

3. The Eqn. (3.2) reduces to the following form on taking $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1,\ q=1$ and $\delta=1,$ we get

$$E_{\alpha,\beta}^{\gamma}(z)=\beta E_{\alpha,\beta+1}^{\gamma}(z)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma}(z),$$

which is the same result as given by Prabhakar[6].

Theorem 3.3: *If* $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \upsilon,\ \rho,\ \sigma\in\mathbb{C}, \operatorname{Re}(\alpha)>0, \operatorname{Re}(\beta)>0,$ $\operatorname{Re}(\gamma)>0,\ \operatorname{Re}(\delta)>0, \operatorname{Re}(\mu)>0, \operatorname{Re}(\upsilon)>0, \operatorname{Re}(\rho)>0, \operatorname{Re}(\sigma)>0$ *and* $p,\ q>0,$ $q\le \operatorname{Re}(\alpha)+p,\ m\in N$, *then*

$$\left(\frac{d}{dz}\right)^m E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\frac{(1)_m(\mu)_{\rho m,k}(\gamma)_{qm,k}}{(\upsilon)_{\sigma m,k}(\delta)_{pm,k}} \tag{3.3}$$

$$\times\sum_{n=o}^{\infty}\frac{(\mu+\rho mk)_{\rho n,k}(\gamma+qmk)_{qn,k}(1+m)_n}{\Gamma_k(\alpha n+\alpha m+\beta)(\upsilon+\sigma mk)_{\sigma n,k}(\delta+pmk)_{pn,k}}\frac{z^n}{n!}.$$

Proof: In the left-hand side of the Eqn. (3.3), and making use of the series form of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$, given by the Eqn. (2.1), we have

$$\left(\frac{d}{dz}\right)^m E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}$$

$$=\sum_{n=m}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}z^{n-m}\quad n!}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}(n-m)!}$$

$$=\sum_{n=0}^{\infty}\frac{(1)_m(1+m)_n(\mu)_{\rho(n+m),k}(\gamma)_{q(n+m),k}}{\Gamma_k(\alpha n+\alpha m+\beta)(\upsilon)_{\sigma(n+m),k}(\delta)_{p(n+m),k}}\frac{z^n}{n!},$$

on using the relation $(\chi)_{n+m,k=}(\chi)_{m,k}(\chi+mk)_{n,k}$, we have

$$\left(\frac{d}{dz}\right)^m E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=\frac{(1)_m(\mu)_{\rho m,k}(\gamma)_{qm,k}}{(\upsilon)_{\sigma m,k}(\delta)_{pm,k}}$$

$$\times\sum_{n=o}^{\infty}\frac{(\mu+\rho mk)_{\rho n,k}(\gamma+qmk)_{qn,k}(1+m)_n}{\Gamma_k(\alpha n+\alpha m+\beta)(\upsilon+\sigma mk)_{\sigma n,k}(\delta+pmk)_{pn,k}}\frac{z^n}{n!}.$$

This completes the proof of the theorem (3.3).

Special Cases:

1. Taking $k=1$, Eqn. (3.3), reduces to the following result, we have

$$\left(\frac{d}{dz}\right)^m E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,p}(z)=\frac{(1)_m(\mu)_{\rho m}(\gamma)_{qm}}{(\upsilon)_{\sigma m}(\delta)_{pm}}$$

$$\times\sum_{n=o}^{\infty}\frac{(\mu+\rho mk)_{\rho n}(\gamma+qmk)_{qn}(1+m)_n}{\Gamma(\alpha n+\alpha m+\beta)(\upsilon+\sigma mk)_{\sigma n}(\delta+pmk)_{pn}}\frac{z^n}{n!},$$

which is the same result as deduced by Khan and Ahmed[4].

2. On setting $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1$and $\delta=1,$ Eqn. (3.3), given us

$$\left(\frac{d}{dz}\right)^m E^{\gamma,q}_{\alpha,\beta}(z)=(\gamma)_{qm}\sum_{n=o}^{\infty}\frac{(\gamma+qmk)_{qn}}{\Gamma(\alpha n+\alpha m+\beta)}\frac{z^n}{n!}$$

$$=(\gamma)_{qm}E^{\gamma+qm,\ q}_{\alpha,\ \beta+m\alpha}(z),$$

which is the same result as obtained by Shukla and Prajapati[5].

3. Putting $k=1,\mu=\upsilon,\rho=\sigma, p=1, q=1$ and $\delta=1,$ Eqn. (3.3) , takes the following form, we have

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}^{\gamma}(z) = (\gamma)_m \sum_{n=o}^{\infty} \frac{(\gamma+mk)_n}{\Gamma(\alpha n+\alpha m+\beta)} \frac{z^n}{n!}$$
$$= (\gamma)_m E_{\alpha,\,\beta+m\alpha}^{\gamma+m}(z),$$

which is the same result as given by Prabhakar[6].

Theorem 3.4: *If* $k \in R,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \upsilon,\ \rho,\ \sigma,\ \eta \in \mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$ $\mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,\ \mathrm{Re}(\upsilon)>0,\ \mathrm{Re}(\rho)>0,\ \mathrm{Re}(\sigma)>0$ *and* $p,\ q>0,$ $q \le \mathrm{Re}(\alpha)+p,$ *then*

(3.4)
$$\frac{1}{\Gamma_k(\eta)}\int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\frac{\eta}{k}-1} E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(zu^{\frac{\alpha}{k}})du = kE_{k,\alpha,\beta+\eta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$$

Proof: In the left-hand side of the Eqn. (3.4), and making use of the series form of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$, given by the Eqn. (2.1), we have

$$\frac{1}{\Gamma_k(\eta)}\int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\frac{\eta}{k}-1} E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(zu^{\frac{\alpha}{k}})du$$
$$= \frac{1}{\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\, z^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}\int_0^1 u^{\frac{\alpha n+\beta}{k}-1}(1-u)^{\frac{\eta}{k}-1}du$$
$$= \frac{k}{\Gamma_k(\eta)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\, z^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}\frac{\Gamma_k(\alpha n+\beta)\Gamma_k(\eta)}{\Gamma_k(\alpha n+\beta+\eta)}$$
$$= k\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\, z^n}{\Gamma_k(\alpha n+\beta+\eta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}} = kE_{k,\alpha,\beta+\eta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z).$$

This completes the proof of the theorem (3.4).

Special Cases:

1. The Eqn. (3.4) reduces to the following form on taking $k=1,$ we get

$$\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1} E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(zu^{\alpha})du = kE_{\alpha,\beta+\eta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z),$$

which is the same result as obtained by Khan and Ahmed[4].

2. Taking $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1$ and $\delta=1$, Eqn. (3.4), reduces to the following result, we have

$$\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E_{\alpha,\beta}^{\gamma,q}(z^{\alpha})du = kE_{\alpha,\beta+\eta}^{\gamma,q}(z),$$

which is the same result as given by Shukla and Prajapati[5].

3. On Setting $k=1, \mu=\upsilon, \rho=\sigma, p=1, q=1$ and $\delta=1$, Eqn. (3.4), takes the following form, we have

$$\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E_{\alpha,\beta}^{\gamma}(zu^{\alpha})du = kE_{\alpha,\beta+\eta}^{\gamma}(z),$$

which is the same result as deduced by Prabhakar[6].

4. Integral Transforms of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$

In this section, we discuss some useful integral transforms like Euler transforms, Laplace transforms, Whittaker transforms of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$.

Euler (Beta) Transform of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$:

Theorem 4.1: *If* $k\in R,\ a,b,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \upsilon,\ \rho,\ \sigma,\ \eta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,$ $\mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0, \mathrm{Re}(\upsilon)>0,\ \mathrm{Re}(\rho)>0,\ \mathrm{Re}(\sigma)>0$ *and* $p,q>0,\ q\le \mathrm{Re}(\alpha)+p,$ *then*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(xz^{\eta})dz = k^{1-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)\Gamma(b)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)} \tag{4.1}$$

$$\times\ {}_4\psi_4\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1), \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right),(a+b,\eta),\end{matrix};\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}\right)x\right].$$

Proof: In the Left-hand side of the Eqn. (4.1), and making use of the series form of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$, given by the Eqn. (2.1), we have

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(xz^\eta)dz$$

$$=\int_0^1 z^{a-1}(1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}(xz^\eta)^n dz}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\,x^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}\int_0^1 z^{a+\eta n-1}(1-z)^{b-1}dz$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\,x^n\quad B(a+\eta n,b)}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}$$

$$=\sum_{n=0}^{\infty}\frac{k^{(\rho+q-\sigma-p)n}\left(\frac{\mu}{k}\right)_{\rho n}\left(\frac{\gamma}{k}\right)_{qn}\;x^n}{k^{\left(\frac{\alpha n+\beta}{k}-1\right)}\Gamma_k\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\left(\frac{\upsilon}{k}\right)_{\sigma n,k}\left(\frac{\delta}{k}\right)_{pn}}\frac{\Gamma(a+\eta n)\,\Gamma(b)}{\Gamma(a+\eta n+b)}$$

$$=\frac{k^{1-\frac{\beta}{k}}\,\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)\Gamma(b)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+\rho n\right)\Gamma\left(\frac{\gamma}{k}+qn\right)\;\Gamma(n+1)\;\;\Gamma(a+\eta n)\;\;\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}x\right)^n}{\Gamma_k\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma\left(\frac{\upsilon}{k}+\sigma n\right)\Gamma\left(\frac{\delta}{k}+pn\right)\Gamma(a+\eta n+b)(n!)}$$

$$=k^{1-\frac{\beta}{k}}\frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)\Gamma(b)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times\,{}_4\psi_4\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1) \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right),(a+b,\eta)\end{matrix};\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}\right)x\right].$$

This completes the proof of the theorem (4.1).

Special Cases:

1. Taking $k=1,$ Eqn. (4.1), reduces to the following result, we have

$$\int_0^1 z^{a-1}(1-z)^{b-1} E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,p}(xz^{\eta})dz$$

$$=\frac{\Gamma(\upsilon)\Gamma(\delta)\Gamma(b)}{\Gamma(\mu)\Gamma(\gamma)}\,{}_4\psi_4\left[\begin{matrix}(\mu,\rho),(\gamma,q),(a,\eta),(1,1)\\ \\ (\beta,\alpha),(\upsilon,\sigma),(\delta,p),(a+b,\eta)\end{matrix};\ x\right],$$

which is the same result as given by Khan and Ahmed[4].

2. Putting $k=1,\ \mu=\upsilon,\ \rho=\sigma, p=1$ and $\delta=1,$ the Eqn. (4.1) gives us

$$\int_0^1 z^{a-1}(1-z)^{b-1} E^{\gamma,q}_{\alpha,\beta}(xz^{\eta})dz=\frac{\Gamma(b)}{\Gamma(\gamma)}\,{}_2\psi_2\left[\begin{matrix}(\gamma,q),(a,\eta)\\ \\ (\beta,\alpha),(a+b,\eta)\end{matrix};x\right],$$

which is the same result as obtained by Shukla and Prajapati[5].

3. On setting $k=1,\mu=\upsilon,\rho=\sigma, p=1, q=1$ and $\delta=1,$ Eqn. (4.1), takes the following form, we get

$$\int_0^1 z^{a-1}(1-z)^{b-1} E^{\gamma}_{\alpha,\beta}(xz^{\eta})dz=\frac{\Gamma(b)}{\Gamma(\gamma)}\,{}_2\psi_2\left[\begin{matrix}(\gamma,1),(a,\eta),\\ \\ (\beta,\alpha),(a+b,\eta)\end{matrix};x\right],$$

which is the same result as deduced by Saxena[9].

Laplace Transform of $E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(z)$:

Theorem 4.2: *If* $k\in R,\ a,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \upsilon,\ \rho,\ \sigma,\ \eta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$ $\mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,\ \mathrm{Re}(\upsilon)>0,\ \mathrm{Re}(\rho)>0,\ \mathrm{Re}(\sigma)>0,\ \mathrm{Re}(s)>0,$ $\left|\frac{x}{s^{\eta}}\right|<1,$ *and* $p,q>0,\ q\le\mathrm{Re}(\alpha)+p,$ *then*

(4.2)
$$\int_0^{\infty} z^{a-1}e^{-sz}E^{\mu,\rho,\gamma,q}_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}(xz^{\eta})dz=s^{-a}\,k^{1-\frac{\beta}{k}}\,\frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times {}_4\psi_3\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1), \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right),\end{matrix};\frac{x}{s^{\eta}}\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}\right)\right].$$

Proof : In the Left-hand side of the Eqn. (4.2), and making use of the series form of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$, given by the Eqn. (2.1), we have

$$\int_0^{\infty} z^{a-1}e^{-sz}E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(xz^{\eta})dz=\int_0^{\infty} z^{a-1}e^{-sz}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}(xz^{\eta})^n dz}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\,x^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}\int_0^{\infty} z^{a+\eta n-1}e^{-sz}dz$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}\quad x^n}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}}\frac{\Gamma(a+n\eta)}{s^{a+n\eta}}$$

$$=s^{-a}\sum_{n=0}^{\infty}\frac{k^{(\rho+q-\sigma-p)n}\left(\frac{\mu}{k}\right)_{\rho n}\left(\frac{\gamma}{k}\right)_{qn}\;x^n\;\;\Gamma(a+\eta n)\Gamma(n+1)}{k^{\left(\frac{\alpha n+\beta}{k}-1\right)}\Gamma_k\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\left(\frac{\upsilon}{k}\right)_{\sigma n,k}\left(\frac{\delta}{k}\right)_{pn}\;n!}\left(\frac{x}{s^{\eta}}\right)^n$$

$$=\frac{s^{-a}\,k^{1-\frac{\beta}{k}}\;\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+\rho n\right)\Gamma\left(\frac{\gamma}{k}+qn\right)\;\Gamma(n+1)\;\;\Gamma(a+\eta n)\;\left(\frac{xk^{\rho+q-\sigma-p-\frac{\alpha}{k}}}{s^{\eta}}\right)^n}{\Gamma_k\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma\left(\frac{\upsilon}{k}+\sigma n\right)\Gamma\left(\frac{\delta}{k}+pn\right)(n!)}$$

$$= s^{-a} k^{1-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times {}_4\psi_3 \left[\begin{matrix} \left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),(a,\eta),(1,1), \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right), \end{matrix} ; \frac{x}{s^{\eta}}\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}\right) \right].$$

This completes the proof of the theorem (4.2).

Special Cases:

1. The Eqn. (4.2) reduces to the following form on taking $k=1$, we have

$$\int_0^{\infty} z^{a-1} e^{-sz} E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(xz^{\eta})dz$$

$$= \frac{\Gamma(\upsilon)\Gamma(\delta)s^{-a}}{\Gamma(\mu)\Gamma(\gamma)} {}_4\psi_3 \left[\begin{matrix} (\mu,\rho),(\gamma,q),(a,\eta),(1,1), \\ \\ (\beta,\alpha),(\upsilon,\sigma),(\delta,p), \end{matrix} ; \frac{x}{s^{\eta}} \right],$$

which is the same result as obtained by Khan and Ahmed[4].

2. Putting $k=1, \mu=\upsilon, \rho=\sigma, p=1$ and $\delta=1$, Eqn. (4.2), takes the following form, we have

$$\int_0^{\infty} z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma,q}(xz^{\eta})dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\psi_1 \left[\begin{matrix} (\gamma,q),(a,\eta), \\ \\ (\beta,\alpha), \end{matrix} ; \frac{x}{s^{\eta}} \right],$$

which is the same result as deduced by Shukla and Prajapati[5].

3. Onsetting $k=1, \mu=\upsilon, \rho=\sigma, p=1, q=1$ and $\delta=1$, Eqn. (4.2), reduces to the following result, we get

$$\int_0^\infty z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma}(xz^\eta)dz = \frac{s^{-a}}{\Gamma(\gamma)}\, {}_2\psi_1\left[\begin{matrix}(\gamma,1),(a,\eta), \\ \\ (\beta,\alpha),\end{matrix}\; ;\frac{x}{s^\eta}\right],$$

which is the same result as given by Saxena[9].

Whittaker Transform of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$:

Theorem 4.3: *If* $k\in R$, $a, \alpha, \beta, \gamma, \delta, \mu, \upsilon, \rho, \sigma, \eta \in \mathbb{C}$, $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$, $\mathrm{Re}(\upsilon)>0$, $\mathrm{Re}(\rho)>0, \mathrm{Re}(\sigma)>0$, $\mathrm{Re}(s)>0$, $\mathrm{Re}(\lambda+\psi)>-1/2$ *and then* $p,q>0$, $q\le \mathrm{Re}(\alpha)+p$,

(4.3) $$\int_0^\infty t^{\xi-1} e^{\frac{-\phi t}{2}} W_{\lambda,\psi}(\phi t) E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\omega t^\eta)dt = \phi^{-\xi} k^{1-\frac{\beta}{k}} \frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\, {}_5\psi_4$$

$$\times\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1), \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right),(1-\lambda+\xi,\eta)\end{matrix}\; ;\left(\frac{\omega k^{\rho+q-\sigma-p-\frac{\alpha}{k}}}{\phi^\eta}\right)\right]$$

Proof: In the left-hand side of the Eqn. (4.3), and substituting $\phi t = v$ and making use of the series form of $E_{k,\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$, given by the Eqn. (2.1), we have

$$\int_0^\infty \left(\frac{v}{\phi}\right)^{\xi-1} e^{\frac{-v}{2}} W_{\lambda,\psi}(v) \sum_{n=0}^\infty \frac{(\mu)_{\rho n,k}(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\upsilon)_{\sigma n,k}(\delta)_{pn,k}} \left(\frac{v}{\phi}\right)^{n\eta} \frac{1}{\phi} dv$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}\,\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+\rho n\right)\Gamma\left(\frac{\gamma}{k}+qn\right)\left(k^{\rho+q-\sigma-p-\frac{\alpha}{k}}\right)^{n}}{\Gamma_{k}\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma\left(\frac{\upsilon}{k}+\sigma n\right)\Gamma\left(\frac{\delta}{k}+pn\right)}$$

$$\times\left(\frac{\omega}{\phi^{\eta}}\right)^{n}\int_{0}^{\infty}v^{\xi+n\eta-1}e^{\frac{-v}{2}}W_{\lambda,\psi}(v)dv$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}\,\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu}{k}+\rho n\right)\Gamma\left(\frac{\gamma}{k}+qn\right)\left(\frac{1}{2}+\psi+\xi+n\eta\right)\left(\frac{1}{2}-\psi+\xi+n\eta\right)\Gamma(n+1)}{\Gamma_{k}\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma\left(\frac{\upsilon}{k}+\sigma n\right)\Gamma\left(\frac{\delta}{k}+pn\right)\left(1-\lambda+\xi+n\eta\right)n!}$$

$$\times\left(\frac{\omega k^{\rho+q-\sigma-p-\frac{\alpha}{k}}}{\phi^{\eta}}\right)^{n}$$

$$=\phi^{-\xi}\,k^{1-\frac{\beta}{k}}\,\frac{\Gamma\left(\frac{\upsilon}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}{\Gamma\left(\frac{\mu}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)}\,{}_5\psi_4$$

$$\times\left[\begin{matrix}\left(\frac{\mu}{k},\rho\right),\left(\frac{\gamma}{k},q\right),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1), \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\left(\frac{\upsilon}{k},\sigma\right),\left(\frac{\delta}{k},p\right),\left(1-\lambda+\xi,\eta\right)\end{matrix};\left(\frac{\omega k^{\rho+q-\sigma-p-\frac{\alpha}{k}}}{\phi^{\eta}}\right)\right]$$

This completes the proof of the theorem (4.3).

Special Cases:

1. The Eqn. (4.3) reduces to the following form on taking $k=1$, we have

$$\int_0^\infty t^{\xi-1} e^{\frac{-\phi t}{2}} W_{\lambda,\psi}(\phi t) E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,p}(\omega t^\eta)dt = \phi^{-\xi} \frac{\Gamma(\upsilon)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}$$

$$\times {}_5\psi_4 \left[\begin{matrix} (\mu,\rho),(\gamma,q),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1), \\ \\ (\beta,\alpha),(\upsilon,\sigma),(\delta,p),(1-\lambda+\xi,\eta) \end{matrix} ;\left(\frac{\omega}{\phi^\eta}\right) \right].$$

which is the same result as obtained by Khan and Ahmed[4].

2. Taking $k=1,\mu=\upsilon,\rho=\sigma$ and $p=1$, Eqn. (4.3), reduces to the following result, we get

$$\int_0^\infty t^{\xi-1} e^{\frac{-\phi t}{2}} W_{\lambda,\psi}(\phi t) E^{\gamma,q}_{\alpha,\beta,\delta}(\omega t^\eta)dt$$

$$= \frac{\phi^{-\xi}\Gamma(\delta)}{\Gamma(\gamma)} {}_4\psi_3 \left[\begin{matrix} (\gamma,q),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1), \\ \\ (\beta,\alpha),(1-\lambda+\xi,\eta),(\delta,1), \end{matrix} ;\left(\frac{\omega}{\phi^\eta}\right) \right],$$

which is the same result as deduced by Shukla and Prajapati[5].

3. On Setting $k=1,\mu=\upsilon,\ \rho=\sigma,\ p=1,\ q=1$ and $\delta=1$. The Eqn. (4.3) takes the following form, we have

$$\int_0^\infty t^{\xi-1} e^{\frac{-\phi t}{2}} W_{\lambda,\psi}(\phi t) E^{\gamma}_{\alpha,\beta}(\omega t^\eta)dt$$

$$= \frac{\phi^{-\xi}}{\Gamma(\gamma)} {}_3\psi_2 \left[\begin{matrix} (\gamma,1),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right), \\ \\ (\beta,\alpha),(1-\lambda+\xi,\eta) \end{matrix} ;\left(\frac{\omega}{\phi^\eta}\right) \right],$$

which is the same result as given by Saxena[9].

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