

Hyperconvex Metric Spaces for Set-Valued Mappings with Non-Compactness

Abid Khan and Deepti Shakti

Department of Mathematics
Amity University, Gwalior, Madhya Pradesh, India

Ramakant Bhardwaj

Department of Mathematics
Amity University, Kolkata, W.B., India
Email: rkbhardwaj100@gmail.com

Qazi Aftab Kabir

Department of Mathematics
Govt. Degree College Surankote, Jammu & Kashmir, India

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Abstract: In this paper, we proved the Browder-Fan fixed point theorem for set-valued mappings in non-compact hyperconvex metric spaces. We use the applications of the Knaster-Kuratowski and Mazurkiewicz in short (KKM), principle in the hyperconvex space. The results of this paper are the substantial improvements of the corresponding ones obtained by Wen Kai-ting, (Journal of Mathematical Research & Exposition, 28:1 (2005) pp. 161-168) and many others in the literature.

Keywords: Hyperconvex spaces, fixed point theorem, Non-compactness, KKM mapping, Browder-Fan theorem, Set-valued mappings.

1. Introduction

A metric space (X, d) is said to be hyperconvex if every collection of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha \in A}$ $\leq r_\alpha + r_\beta$ has non-empty intersection $\bigcap_\alpha B(x_\alpha, r_\alpha)$

$\neq \emptyset$. In 1956 Aronszajn and panitchpakdi¹, presented the thought of hyperconvex metric spaces, who demonstrated that they are equivalent to injective measurement spaces and afterward, Sine² and Soardi³ demonstrated freely that the fixed point property for the nonexpansive mapping holds in limited hyperconvex metric spaces. From that point forward hyperconvex metric spaces have been generally contemplated and many fascinating fixed point hypotheses have been built up. Kirk⁴ got fixed point hypotheses for nonstop mappings in compact hyperconvex metric space.

Khamsi⁵ set up fixed point hypotheses, KKM and Ky Fan hypotheses in hyperconvex metric spaces. Kirk et al.⁶ built up the portrayal of the KKM standard in hyperconvex metric spaces, and as utilizations of their outcomes, the hyperconvex rendition of Fan's minimax guideline, Fan's best estimation hypothesis for mappings, Nash harmony, Browder-Fan fixed point hypothesis and some different outcomes were given. Several papers were posted containing constant point consequences for a set-valued mapping with special fixed point theorem in hyper convex spaces⁵⁻¹⁰

In this paper we establish a new form of the Browder-Fan fixed point theorem for set-valued mappings in non-compact hyperconvex spaces which at the generalisations of the results of Wen¹³

2. Preliminaries

Let X be a nonempty set; we denote 2^X the family of all nonempty finite subsets of X , by $\langle X \rangle$ the family of all subsets of X . If A is a subset of a linear space H , the notion $\text{cov}(A)$ always the convex hull of A in H . Let X and Y be two nonempty sets and $T: X \rightarrow 2^Y$ be a set-valued mapping. Then the set valued mapping $T^{-1}: Y \rightarrow 2^X$ is defined by $T^{-1}(y) = \{x \in X : y \in T(x)\}$ for every $y \in Y$.

Definition⁴ 2.1: A metric space (H, d) is said to be a hyperconvex space if for any collection of points $\{x_\alpha\}$ of H and any collection $\{r_\alpha\}$ of non-negative real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. We have $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$, where $B(x, r)$ denotes the closed ball centered at $x \in X$ with the radius r .

Definition 2.2: Let X be a nonempty subset of a hyperconvex space (H, d) . Suppose $G: X \rightarrow 2^H$ is a set-valued mapping with nonempty values. Then, G is said to be a KKM mapping if $\text{co}(F) \subseteq \bigcup_{x \in F} G(x)$ for every $F \in \langle X \rangle$.

Definition¹³ 2.3: A hyperconvex space (H, d) is said to have the convex hull finite property if the closed convex hull of every nonempty finite subset of H has the fixed point property.

Lemma¹³ 2.1: Let (H, d) be a complete metric space with the convex hull finite property and X be a nonempty subset of H . Suppose that $G: X \rightarrow 2^X$ is a KKM mapping with closed values and $G(z)$ is compact for some $z \in X$. Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

Lemma 2.2: Let X and Y be topology spaces and $G: X \rightarrow 2^X$ be a mapping with nonempty values. Then the following conditions are equivalent:

- (a) G has the local intersection property.
- (b) For each $y \in Y$, there exist an open subset O_y of X such that $O_y \subset G^{-1}(y)$ and $X = \bigcup_{y \in Y} O_y$.
- (c) There exist a mapping $F: X \rightarrow 2^Y$ such that $F(x) \subset G(x)$ for each $x \in X$ and $X = \bigcup_{y \in Y} \text{int } XG^{-1}(y)$

3. Main Results

Theorem 3.1: Let (H, d) be a hyperconvex metric space with the convex hull finite property, C be a nonempty compact subset of H and $I, G: H \rightarrow 2^H$ be two set-valued mappings such that.

- (1) For every $y \in H$, $I(y) \subseteq G(y)$ and $G(y)$ is convex;
- (2) For every $x \in H$, $I^{-1}(x)$ is open in H ;
- (3) For every $y \in C$, $I(y) \neq \emptyset$;
- (4) One of the following conditions holds:
 - (a) For every $N \in \langle H \rangle$, there exists a nonempty compact convex subset H_N of H containing N such that

$$H_N - C \subseteq \bigcup_{x, y \in H_N} \text{int}_{H_N} (G^{-1}(x) \cap H_N) (G^{-1}(y) \cap H_N)$$

(b) There exists a point $x_0 \in H$ such that $(H - G^{-1}(x_0)) \subseteq C$.

Then there exists $\check{y} \in H$ such that $\check{y} \in G \left(\check{y} \right)$

Proof: We distinguish the following last two cases (a) and (b) for the proof.

Case (a): Suppose the contrary. Then, for every $y \in H$, we have $y \notin G(y)$

Define $G', I' : H \rightarrow 2^H$ by

$$G'(x) = (H - G^{-1}(x)) \cap C, \quad x \in H,$$

$$I'(x) = (H - I^{-1}(x)) \cap C, \quad x \in H.$$

We will prove that the family $\{G'(x) : x \in H\}$ has the finite intersection property. Let $N \in \langle H \rangle$ be given. Then, by (a), there exists a nonempty compact convex subset H_N of H containing N . Furthermore, we define two set-valued mappings $G'', I'' : H_N \rightarrow 2^{H_N}$ by

$$G''(x) = H_N - G^{-1}(x),$$

$$I''(x) = H_N - I^{-1}(x),$$

$x \in H_N$. By (1) and (2), $G''(x) \subseteq I''(x)$ for every $x \in H_N$. Since H_N is compact and every $G''(x)$ is relatively closed in H_N , it follows that every $G''(x)$ is compact. Now we show that the mapping $G^* : H_N \rightarrow 2^{H_N}$ defined by

$$G^*(x) = H_N - G^{-1}(x), \quad x \in H_N.$$

Is a KKM mapping. Suppose the contrary. Then there exist $A \in \langle H_N \rangle$ and $y \in \text{co}(A) \subseteq H_N$ such that

$$y \notin \bigcup_{x \in A} G^*(x) = H_N - G^{-1}(x)$$

Hence, we have $y \in \bigcap_{x \in A} G^{-1}(x)$ and $A \subseteq G(y)$. Therefore, we have $y \in co(A) \subseteq G(y)$ by (1), which is contraction. Hence G^* is a KKM mapping and so is G'' . By lemma 2.1 and (b), we have.

$$\phi \neq \bigcap_{x, y \in H_N} G''(x) = \bigcap_{x, y \in H_N} H_N - G^{-1}(x) G^{-1}(y) \subseteq H_N \cap C.$$

Taking $\bigvee y \bigcap_{x \in H_N} G''(x)$ leads to

$$\begin{aligned} \bigvee y \in \bigcap_{x \in H_N} G''(x) &\subseteq \bigcap_{x \in N} (G''(x) \cap C) \subseteq \bigcap_{x \in N} (H - G^{-1}(x) \cap C) \\ &= \bigcap_{x \in N} G'(x) G^{-1}(y), \end{aligned}$$

which implies that the family $\{G'(x) : x \in H\}$ has a finite intersection property. By the compactness of C , we have $\bigcap_{x \in H} G'(x) \neq \phi$. Since $G'(x) \subseteq I'(x)$ for every $x \in H$ it follows that

$$\begin{aligned} \phi \neq \bigcap_{x \in H} I'(x) &= \bigcap_{x \in H} (H - I^{-1}(x)) \cap C \\ &= \left(H - \bigcup_{x \in H} I^{-1}(x) \right) \cap C \\ &= C - \bigcup_{x \in H} I^{-1}(x) \end{aligned}$$

By (3), for every $y \in C$, $I(y) \neq \phi$ and so, $C - \bigcup_{x \in H} I^{-1}(x)$, which is contradiction. Therefore, there exist $\bigvee y \in C$ such that $\bigvee y \in G\left(\bigvee y\right)$.

Now case (b). Suppose the contrary then for every $y \in H$, $x \in G(y)$. Hence G' is a KKM mapping. By the definition G' , $G'(x)$ is closed in H for every $x \in H$. By (b) there exists a point $x_0 \in H$ such that

$G'(x_0) = H - G^{-1}(x_0) \subseteq C$, which implies that $G'(x_0)$ is compact. Then by lemma 2.1, we get

$$\phi \neq \bigcap_{x \in H} G'(x) \subseteq G'(x_0) \subseteq C.$$

Therefore, we have

$$\phi \neq C \cap \left(\bigcap_{x \in H} G'(x) \right) \subseteq C \cap \left(\bigcap_{x \in H} I'(x) \right).$$

Taking $y_0 \in C \cap \left(\bigcap_{x \in H} I'(x) \right)$ we have $y_0 \in C$ and $x \notin I(y_0)$ for every $x \in H$.

This completes the proof.

Corollary 3.1: *Let (H, d) be a complete hyperconvex space and E a closed convex subset of H . Assume $G: E \rightarrow 2^H$ is a set-valued mapping with $G(x)$ and $G(y)$ a compact convex subset of H and $G(x), G(y) \cap E \neq \phi$ for each $x, y \in E$. Then the mapping G has a fixed point.*

4. Conclusion

In this paper we proved a fixed point theorem for set-valued mappings in non-compact hyperconvex metric spaces with intersection property and we use the applications of the KKM mappings in hyper convex spaces.

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