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Newtonian Limit for the Curvature of Space Time

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Abstract: The curvature of space time expresses the tidal force that a body feels when moving along a geodesic. Harmann Weyl named Weyl[1] curvature tensor which is measure of the curvature of space time. In general relativity, A. Danehkar[2] studied that the curvature of space time is a solution of vacuum Einstein equation and it governs the propagation of gravitational waves through area of space devoid of matter. Hermann Klaus Hugo Weyl (1955), one of the German Mathematician of 19th century, published technical and some general work on space, time, matter, philosophy, logic symmetry and visualized general relativity with the laws of electromagnetism.

In the present manuscript, we have tried to draw our focus on properties of conformal Weyl curvature tensor i.e. curvature of space time and its applications in the modern literature of relativity and cosmology. However, in this note we wish to compliment some recent enhancements in the cosmological literature by implementing notions of Weyl's conformal curvature tensor and its recurrence properties. In particular, we shall outline some generalized recurrence properties of Weyl's curvature tensor in the Weyl's space and then delineate its Newtonian limit. Besides this, we shall discuss some relativistic equations under Newtonian limit. It is shown that for the space-time having dimensions less than 4, needed a tensor (called Cotton tensor), other than Weyl's tensor to check out the conformal flatness of the space-time and its recurrent nature. Moreover, a relativistic form of Weyl's tensor and relativistic equation evolved due to its parts (namely, electric and magnetic) has been studied.

Keywords: Relativistic, Curvature, Weyl space, Weyl tensor, Newtonian limit, Cotton tensor, Eulerian, Newtonian tidal tensor.

1. Introduction

H. Weyl's at al[1] sums up his efficiency, perfection and curiosity, not only in Mathematics, but in Physics of space-time, matter and Philosophy also. In search of "what is the role of electricity in the geometry of space-time"? Weyl concentrated to this topic in paper after paper and book after book. In a 1918 article, Hermann Weyl[3] tired to combine electromagnetism and gravity by requiring the theory to be invariant under a local scale change of the metric, i.e., $g_{uv} \to g_{uv} e^{\alpha(\chi)}$, where χ is a 4-vector. This attempt was successful and was characterized by Einstein for being inconsistent with observed physical results. It predicted that a vector parallel transported from point p to q would have a length that was path dependent. Now, in order to pursue our proposed study "Newtonian limit for the curvature of space time", we briefly introduce some notions on Weyl's space, Generalized Weyl's space and Newtonian limit of general relativity.

Weyl's space: An $n-$dimensionl differentiable manifold M_n is said to be a Weyl space if it has a symmetric connection ∇ and a symmetric conformal metric tensor g_{ij} preserved by ∇ satisfying the compatibility condition given by the equation[3-5]

(1.1) $$\nabla_k g_{ij} - 2F_k g_{ij} = 0,$$

Or in extended form

(1.2) $$\frac{\partial}{\partial x^k} g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2F_k g_{ij} = 0,$$

where F_k represents a covariant vector filed and Γ^i_{kl} are the connection coefficients of the symmetric connection ∇ and are defined as;

(1.3) $$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ k\,l \end{matrix} \right\} - g^{im}\left(g_{mk} F_l + g_{ml} F_k - g_{kl} F_m\right).$$

Moreover, under the renormalization condition;

(1.4) $$\tilde{g}_{ij} = \lambda^2 g_{ij},$$

of the metric tensor g_{ij}, the covariant vector field F_k is transformed by the law;

(1.5) $$\tilde{F}_k = F_k + \frac{\partial}{\partial x^k} \ln \lambda ,$$

where λ is a scalar function defined on M_n.
Thus the space M_n satisfying all the foregoing condition will be symbolized by $M_n\left(\Gamma^i_{jk}, g_{ij}, F_k\right)$ or $M_n(g, F)$. Also a geometric object Ω defined on $M_n\left(\Gamma^i_{jk}, g_{ij}, F_k\right)$ is called a satellite of weight $\{w\}$ of the tensor g_{ij}, if it admits a transformation of the form:

(1.6) $$\tilde{\Omega} = \lambda^w \Omega ,$$

under the renormalization condition of the metric tensor g_{ij} [3,6].
Further the prolonged covariant derivative of a satellite Ω is defined by;

(1.7) $$\dot{\nabla}_k \Omega = \nabla_k \Omega - w F_k \Omega .$$

It is remarkable that the prolonged covariant derivative preserves the weight.

Generalized Weyl's space: An $n-$dimensional differentiable manifold GM_n having an anti-symmetric connection ∇^* and anti-symmetric conformal metric tensor g^*_{ij} preserved by ∇^* is called a "generalized Weyl space"[7]. For such a space, in local co-ordinate system, we have a compatibility condition as below:

(1.8) $$\nabla^*_k g_{ij} - 2F^*_k g^*_{ij} = 0 ,$$

where F^*_k are the components of a covariant vector filed called the complementary vector filed of the GM_n space.
Using the concept of covariant differentiation[8,9], the compatibility condition (1.8) can be written in extended form as;

(1.9) $$\frac{\partial^*}{\partial x^k} g^*_{ij} - g^*_{hj} L^h_{ik} - g^*_{ih} L^h_{jk} - 2F_k^* g^*_{ij} = 0 ,$$

where L^i_{kl} are the connection coefficients of the anti-symmetric connection ∇^*_k and are obtained from the compatibility condition as[10];

(1.10) $$L^i_{kl} = \Gamma^i_{kl} + \frac{1}{2}\left[\chi^h_{km} g^*_{lh} + \chi^h_{ml} g^*_{kh} + \chi^h_{kl} g^*_{hm}\right] g^{*mi} .$$

Now, putting

(1.11) $$\chi^i_{kl} = \frac{1}{2}\left[\chi^h_{km} g^*_{lh} + \chi^h_{ml} g^*_{kh} + \chi^h_{kl} g^*_{hm}\right] g^{*mi} ,$$

we obtain

(1.12) $$L^i_{kl} = \Gamma^i_{kl} + \chi^i_{kl} ,$$

where Γ^i_{kl} and χ^i_{kl} are respectively the coefficients of a Weyl connection and the torsion tensor of GM_n space and are expressed as;

(1.13) $$\Gamma^i_{kl} = \frac{1}{2}\left(L^i_{kl} + L^i_{lk}\right) = L^i_{(kl)} ,$$

(the round bracket stands for symmetry)

and

(1.14) $$\chi^i_{kl} = \frac{1}{2}\left(L^i_{kl} - L^i_{lk}\right) = L^i_{[kl]} .$$

(The square bracket stands for anti-symmetry).
A generalized Weyl space satisfying all the aforementioned condition is symbolized as $GM_n\left(L^i_{jk}, g^*_{ij}, F^*_k\right)$.

Newtonian Limit in General Relativity: The concept of Newtonian limit in theory of relativity has been introduced to focus on two major aspects; one of them concerns to the presentation of a precise derivation of the Newtonian limit of fluid evolution equation in a 4-dimensional "frame theory" developed by [11,12]. This theory covers both Einstein and Newtonian theory of gravitationally interacting matter. On the other hand, second aspect pertains to the discussion general relativistic equations to describe a closed Newtonian system.

2. Recurrence Properties of Space-Time/Weyl Curvature Tensor in M_n & Gm_n Spaces

According to ref.[3], under a renormalization condition of the fundamental metric tensor g_{ij} of the form (1.4), an object Ω defined on GM_n space, admits a transformation of the form (1.6) is called a satellite with weight $\{w\}$ of the metric tensor and the prolonged covariant derivative of the satellite Ω relative to the symmetric connection ∇ is defined by[13]:

(2.1) $$\dot{\nabla}_k \Omega = \nabla_k \Omega - wF_k \Omega .$$

Whereas the same relative to anti-symmetric connection ∇^* is defined as;

(2.2) $$\dot{\nabla}_k^* \Omega = \nabla_k^* \Omega - wF_k^* \Omega ,$$

which evinces that the prolonged derivative preserves the weight of satellite. Now the conformal Weyl curvature tensor C^i_{jkl} of the generalized Weyl space GM_n is given by[14] as

(2.3) $$C^i_{jkl} = R^i_{jkl} + \frac{2}{n(n-2)}\Big(\delta^i_l R_{[jk]} - \delta^i_k R_{[jl]} - g_{jl} g^{im} R_{[mk]} + g_{jk} g^{im} R_{ml} - (n-2)\delta^i_j R_{[lk]}\Big) - \frac{1}{n-2}\Big(\delta^i_l R_{jk} - \delta^i_k R_{jl} - g_{jl} g^{im} R_{mk} + g_{jk} g^{im} R_{ml}\Big) + \frac{R}{(n-1)(n-2)}\Big(g_{jk}\delta^i_l - g_{jl}\delta^i_k\Big),$$

where the square bracket stands for the anti-symmetrization.
This $n-$dimensional GM_n space is said to be conformally recurrent Weyl space if its conformal curvature tensor (2.3) of weight 0, on taking the prolonged derivative, satisfies the condition;

(2.4a) $$\dot{\nabla}_m^* C^i_{jkl} = \phi_m C^i_{jkl} ,$$

where $\phi_m \neq F_m$ is a non-zero covariant recurrence vector field of weight zero. The conformal curvature tensor given by (2.3) can be re-defined in the purely contravariant pattern as below:

(2.4b) $$C^{ijkl} = R^{ijkl} - \frac{1}{n-2}\left(g^{il}S^{kj} - g^{il}S^{jk} - g^{jk}S^{li} + g^{jl}S^{ki}\right),$$

where $S^{lj} = R^{lj} - \frac{1}{2(n-1)}g^{lj}R$. R^{lj} and R are the Ricci quantities, while C^{ijkl} bears the following main properties:

(a) It is purely covariant against conformal redefinition of the metric, i.e., $g_{kl}(x) \to \lambda(x) g_{kl}$

(b) It vanishes if and only if the GW_n space is conformally flat, i.e., g_{kl} is diffeomorphic to $\lambda\eta_{kl}$, where η_{kl} is flat.

(c) It possesses the symmetries of Riemannian tensor and also is traceless in each index pair.

(d) Evidently, from the properties (a) and (b), the Weyl tensor acts as a template for conformal flatness. Thus by evaluating it on a specific metric tensor, one can distinguish, whether the space-time under consideration is conformally flat or not.

In three dimensions the Weyl tensor vanishes identically and the Riemannian tensor is given by the last term in (2.4) (at $n=3$). But not all three dimensional space-times are conformally flat.
Now, as it is well known that in dimensions greater than three, the conformal tensor (2.3) or (2.4) is the Weyl tensor, then what about the three dimensional space-time? To overcome from this difficulty, one need a substitute for Weyl tensor, which would act as a template for the conformal flatness?

Indeed there is a crucial substitution for Weyl tensor, known as "*Cotton Tensor*" which is delineated as;

(2.5) $$C^{ij} = \frac{1}{2\sqrt{g}}\left(\varepsilon^{ikl}D_k R_l^j + \varepsilon^{jkl}D_k R_l^i\right).$$

This Cotton tensor serves that role, as it possesses conformal template properties (a) and (b). C^{ij} is symmetric in its indices and like the Weyl tensor, it is traceless. Therefore, in case of the space-time having dimension less than 4, one can have the conformal recurrence properties for such conformally non-flat continuum by taking the prolonged derivative of Cottons tensor (2.5) of weight zero. The conformally non-flat space-time having dimension $n=3$ will be called conformally recurrent if the Cotton tensor satisfy the following:

(2.6) $\dot{\nabla}^{*}_{m} C^{ij} = \phi_m C^{ij}$,

where ϕ_m is a non-zero recurrence vector filed of weight zero.
It is remarkable that for the space-time having dimension less than three, there is no need of a conformal tensor and indeed none exists as all such spaces are locally conformally flat.We, now, study conformal Weyl tensor, its Newtonian limit and some relativistic equations in general relativity due to Newtonian limit.

3. Conformal Weyl's Curvature Tensor, Its Newtonian Limit and Relativistic Equations

In the modern cosmological literature, role of conformal Weyl's curvature tensor has been adequately discussed. Especially, concentration upon the two crucial parts, namely "electric" and "magnetic" parts of conformal Weyl's curvature tensor has been drawn by[11,12] individually. Further[11,12] have discussed some ideas on the magnetic part of Weyl tensor under Newtonian limit of general relativity.

In order to pursue significance of Weyl tensor in cosmological structures, the "frame-theory" for a general 4-dimensional space-time continuum is employed which encapsulate both the "Newton's theory" as well as "Einstein's theory". In this frame theory, a single parameter $\varepsilon = c^2$ is introduces in such a way that it distinguishes between Newton's and Einstein's theory. The limit for this parameter is take as $\varepsilon \to 0$. Moreover, in the formalism of Einstein's theory, a temporal metric t_{ij} and an inverse spatial metric s^{ij} are used which are related by an expression of the form:

(3.1) $t_{ij} s^{jk} = -\varepsilon \delta^{k}_{i}$.

In case of general relativity, the parameter ε is taken to be greater than zero, i.e., $\varepsilon > 0$ and the Riemannian as well as inverse Riemannian metric g_{ij} & g^{ij} are used such that;

(3.2) $g_{ij} = -\varepsilon^{-1} t_{ij}$ & $g^{ij} = s^{ij}$.

On the other hand, in Newton's theory, the parameter ε is taken to be zero and the temporal metric is defined as;

(3.3) $$t_{ij} = t_{,i} t_{,j} ,$$

where t is the absolute time and the comma $(,)$ denotes the derivative with respect to the Eulerian co-ordinates which will be discussed further.
In the conception of frame theory, the conformal Weyl's curvature tensor (2.3), for $\varepsilon > 0$ takes the following form:

(3.4) $$C^{i}_{jkl} = R^{i}_{jkl} - \delta^{i}_{k} R_{lj} - \varepsilon^{-1} \left\{ t_{jk} R_{lm} s^{mi} + \frac{1}{3} \delta^{i}_{k} t_{lj} R_{pr} z^{pr} \right\}.$$

This expression becomes worthless for $\varepsilon = 0$. However, if one uses the Einstein's field equation;

(3.5) $$R_{ij} = 8\pi G \left(t_{ik} t_{jl} - \frac{1}{2} t_{ij} t_{kl} \right) F^{kl} - \Lambda t_{ij} \quad ,$$

of the frame theory (valid for $\varepsilon \geq 0$), to eliminate R_{ij} from the equation (3.4), we obtain;

(3.6) $$C^{i}_{jkl} = R^{i}_{jkl} - 8\pi G \left\{ \delta^{i}_{k} t_{lp} t_{jr} T^{pr} - t_{jk} t_{lm} T^{mi} - \frac{2}{3} \delta^{i}_{k} t_{lj} t_{pr} T^{pr} \right\}.$$

Now this formula is noteworthy, even for $\varepsilon = 0$. Thereby, one can define the conformal Weyl's curvature tensor in the frame theory by (3.6). This expression for the Weyl's curvature tensor is quite suitable, for instance, if a sequence of general relativistic solutions has a Newtonian solution as a limit, then the limit of conformal Weyl curvature tensor is surely produced by (3.6). Furthermore, in case of Newton's theory, $(\varepsilon = 0)$, (3.6) due to (3.3) reduces to the form:

(3.7) $$C^{i}_{jkl} = R^{i}_{jkl} - \frac{8\pi G}{3} \rho t_{,j} \delta^{i}_{k} t_{,l} .$$

Also, the "electric" and "magnetic" part of the conformal Weyl curvature tensor with respect to any 4-velocity vector v^{i} can be obtained from (3.7) and these are respectively defined like below:

(3.8) $$E^{i}_{k} = R^{i}_{kjl} v^{j} v^{l} - \frac{4\pi G}{3} \rho \left(\delta^{i}_{k} - v^{i} t_{,k} \right),$$

(3.9) $$H_{ik} = \frac{1}{2}\eta_{ijpr} s^{ru} C^{p}_{ukl} v^{j} v^{l} = 0 .$$

Here it is very observable that the magnetic part H_{ik} of the conformal Weyl tensor vanishes in the Newtonian limit. This fact can be more precisely justified as follows:

In the general relativity, H_{ik} measures the relative rotation of nearby freely falling gyroscopes due to gravito-magnetism. This effect has nonexistence in case of Newtonian theory in which the parallelism of spatial vectors is path independent. It means, parallel gyroscopes will always remain parallel if subjected to nothing except inertia and gravity. Here, we now discuss some relativistic equations in Newtonian limit.
In three dimensional Weyl space, (3.8) yields a new kind of tensorial quantity called "Newtonian tidal tensor" which is trace free part of gravitational field tensor $\left(g_{i,j}\right)$ (here comma denotes derivative with respect to Eulerian co-ordinate system) and is given by

(3.10) $$E_{ij} = g_{i,j} - \frac{1}{3}\delta_{ij} g_{l,l} ,$$

Provided that

(3.11) $$E_{[ji]} = 0 , \quad E_{ii} = 0 .$$

Likewise any Eulerian field, the Newtonian tidal tensor of the gravitational field strength $\vec{g}$ can be written in terms of Lagrangian co-ordinates as below:

(3.12) $$E_{ij} = g_i \,|_k \, J^{-1}_{kj} - \frac{1}{3}\delta_{ij} g_l \,|_k \, J^{-1}_{kl} ,$$

where a vertical slash stands for the derivative with regard to Lagrangian co-ordinate system.
In order to discuss evolution of tidal tensor as relativistic equation, we introduce a diffeomorphism $\vec{f}_t : \vec{x} = \vec{f}\left(\vec{X}, t\right)$, which sends fluid elements from their initial Lagrangian position $\vec{X}$ to a point $\vec{x}$ in the Eulerian space at time t. Also, we use an expression for the jacobian of the inverse transformation;

(3.13) $$\vec{h} = \vec{f}^{\,-1},\ \vec{g}\left[\vec{x}, t\right] = \ddot{\vec{f}}\left(\vec{h}\left[\vec{x}, t\right], t\right),\quad J = \det\left(f_i\,|_k\right)$$

and

(3.14) $$h_j\,|_l = \frac{1}{2J}\varepsilon_{jpq}\tau\left(\ddot{f}_l,\ f_p,\ f_q\right) - \frac{1}{3}\varepsilon_{opq}\tau\left(\ddot{f}_o,\ f_p,\ f_q\right)\delta_{ij},$$

so that the Newtonian tidal tensor could explicitly be expressed in terms of $\vec{f}$ as:

(3.15) $$E_{ij} = \frac{1}{2J}\left(\varepsilon_{jpq}\tau\left(\ddot{f}_l,\ f_p,\ f_q\right) - \frac{1}{3}\varepsilon_{opq}\tau\left(\ddot{f}_o,\ f_p,\ f_q\right)\delta_{ij}\right).$$

Therefore any trajectory field $\vec{f}$, which obeys the Lagrange-Newtonian system is given as;

(3.16) $$\tau\left(\ddot{f}_J,\ f_j,\ f_k\right) = 0$$

and

(3.17) $$\tau\left(\ddot{f}_1,\ f_2,\ f_3\right) + \tau\left(\ddot{f}_2,\ f_3,\ f_1\right) + \tau\left(\ddot{f}_3,\ f_1,\ f_2\right) - \Lambda J = -4\pi G\rho^0,$$

where Λ is a cosmological constant.
The last two equations determine the evolution of tidal tensor in the form of relativistic equation of general relativity via (3.12). In (3.16) and (3.17), the symbol $\tau(A, B, C)$ denote the functional determinant of any functions $A\left(\vec{X}, t\right)$, $B\left(\vec{X}, t\right)$ and $C\left(\vec{X}, t\right)$ with respect to Lagrangian co-ordinate system $\vec{X}$ and ρ^0 denotes the initial density field[15].

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