ISSN 0974 - 9373

Vol. 24 No. 2 (2020) **Journal of International Academy of Physical Sciences** pp. 129- 148

Generalized Symmetric Metric Connection on Semi-invariant Submanifolds of a Nearly Sasakian Manifold

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(Received April 17, 2020)

Abstract: We defined generalized symmetric metric connection of type (α, β) for a nearly Sasakian manifold and consider a semi-invariant submanifolds of a nearly Sasakian manifold endowed with a generalized symmetric metric connection of type (α, β). The object of this paper to study some properties of semi-invariant submanifolds of a nearly Sasakianm manifold endowed with a generalized symmetric metric connection of type (α, β).

Keywords: Nearly Sasakian Manifold, Generalized symmetric metric connection, totally geodesic, totally umbilical.
2000 Mathematics Subject Classification: 53D05, 53D25, 53D12.

1. Introduction

Let linear connection ∇ is said to be generalized symmetric connection if its torsion tensor T is of the form[1]

(1.1) $T(X, Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\phi X + u(Y)\phi X\},$

for any vector fields X, Y on a manifold, where α and β are smooth functions ϕ is a tensor of type $(1,1)$ and

α is a $1-$form associated with a non-vanishing smooth non-null unit vector field ξ. Moreover, the connection ∇ is said to be a generalized symmetric metric connection if there is Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric.

In the equation (1.1), if $\alpha = 0$ $(\beta = 0)$, then the generalised symmetric connection is called $\beta-$quarter-symmetric connection ($\alpha-$semi-symmetric connection), respectively. Moreover, if we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (1, 0)$, then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Hence a generalized symmetric connection can be viewed as a generalization of semi-symmetric and quarter-symmetric connection. They are most important for geometry study and application to physics.

It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. The idea of semi-symmetric linear connection and quarter-symmetric linear connections in differential manifold was introduced by S. Golab[2].

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form[3]

(1.2) $$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

The study of semi-invariant submanifolds in Sasakian manifolds were initiated by A. Bejancu and N. Papaghuic[4]. The notion of a nearly Sasakian manifolds was introduced by Blair et al[5]. CR-submanifolds of a nearly Sasakian manifold were studied by M. H. Shahid[6]. M. H. Shahid[7] investigated properties of semi-invariant submanifolds of a nearly Sasakian manifold. T. Khan[8], studied on semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifolds with semi-symmetric metric connection and Ahmed et al[9], studied on semi-invariant submanifolds of a nearly Kenmotsu manifold with semi-symmetric semi-metric connection. In this paper we study generalized symmetric metric connection on semi-invariant submanifolds of a nearly Sasakian manifold. The paper is organized as follows: In section 2, we give a brief introduction to nearly Sasakian manifolds. In section 3, we study semi-invariant submanifolds of a nearly Sasakian manifold. We find necessary conditions that induced connection on generalized symmetric metric connection on semi-invariant submanifolds of a nearly Sasakian manifold is also a generalized symmetric metric connection. In section 4, we discuss the Integrability condition of distributions of semi-invariant submanifolds.

2. Preliminaries

Let $\bar{M}$ be a $(2m+1)-$dimentional almost contact metric manifold with a metric tensor g, a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η which satisfies

(2.1) $$\phi^2 X = -X + \eta(X)\xi,$$

(2.1) $$\phi o \xi = 0,$$

(2.3) $$\eta o \phi = 0,$$

(2.4) $$\eta(\xi) = 0,$$

(2.5) $$\eta(X) = g(X, \xi),$$

(2.6) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector field X, Y in $\bar{M}$ If in addition to the condition for an almost contact metric structure we have $d\eta(X, Y) = g(X, \phi Y)$, the structure is said to be a contact metric structure[10].

The almost contact metric manifold $\bar{M}$ is called a nearly Sasakian manifold if it is satisfies the condition[5]

(2.7) $$\left(\bar{\bar{\nabla}}_X \phi\right)Y + \left(\bar{\bar{\nabla}}_Y \phi\right)X = \eta(Y)X + \eta(X)Y - 2g(X, Y)\xi,$$

where $\bar{\bar{\nabla}}$ denotes the Riemannian connection with respect to g. If, moreover, $\bar{M}$ satisfies

(2.8) $$\left(\bar{\bar{\nabla}}_X \phi\right)Y = \eta(Y)X - g(X, Y)\xi \quad ,$$

(2.9) $$\bar{\bar{\nabla}}_X \xi = \phi X \ ,$$

then it is called a Sasakian manifold[10]. Thus every Sasakian manifold is a nearly Sasakian manifold. The converse statement fails in general[5].

3. Semi-Invariant Submanifolds of a Nearly Sasakian Manifolds

Definition[11] 3.1: *An $n-$dimentional Riemannian submanifold M of a nearly Sasakian manifold $\bar{M}$ is called a semi-invariant submanifolds if ξ is tangent to M and there exists M a pair of orthogonal distributions $\left(D, D^{\perp}\right)$ such that*

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,

(ii) The distribution D is invariant under ϕ that is $\phi D_X = D_X$ for all $X \in M$,

(iii) the distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D_X^{\perp} \subset T^{\perp}M$ for all $X \in M$, where TM and $T^{\perp}M$ are the tangent space and normal space of M at X.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. A semi-invariant submanifold M is said to be an invariant (resp. anti-invariant) submanifold if $D_X^{\perp} = \{0\}$ (resp. $D_X = \{0\}$)for each $X \in M$. We also call M proper if neither D nor $D^{\perp}$ is null.

A vector field X tangent to M is given as

(3.1) $$X = PX + QX + \eta(X)\xi ,$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.
For any vector field N normal to M, we put

(3.2) $$\phi N = BN + CN ,$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

Now, we define a generalized symmetric metric connection $\bar{\nabla}$ of type (α, β)[1],

(3.3) $$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \alpha\{\eta(Y)X - g(X, Y)\xi\} - \beta\eta(X)\phi Y .$$

If we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (1, 0)$, generalized symmetric connection is reduced a semi-symmetric metric connection and quarter-symmetric metric connection as follows.

(3.4) $$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y) X - g(X, Y)\xi ,$$

(3.5) $$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)\phi Y ,$$

for all $X, Y \in TM$, where η is a $1-$form on M and $\bar{\bar{\nabla}}$ is the induced connection with respect to the metric g on M.
The covariant differential of the vector field ϕY is given by,

(3.6) $$(\bar{\nabla}_X \phi) Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y),$$

from (3.3), replace Y by ϕY, we have

(3.7) $$\bar{\nabla}_X \phi Y = \bar{\bar{\nabla}}_X \phi Y - \alpha g(X, \phi Y)\xi + \beta\eta(X) Y - \beta\eta(X)\eta(Y)\xi \ .$$

Operating ϕ both side in (3.3), we have

(3.8) $$\phi(\bar{\nabla}_X Y) = \phi(\bar{\nabla}_X Y) + \alpha\eta(Y)\phi X + \beta\eta(X) Y - \beta\eta(X)\eta(Y)\xi \ .$$

Using equations (3.7) and (3.8) in (3.6), we have

(3.9) $$(\bar{\nabla}_X \phi) Y = (\bar{\bar{\nabla}}_X \phi) Y - \alpha\{\eta(Y)\phi X + g(X, \phi Y)\xi\}.$$

Using equation (2.4) in (3.9), we have

(3.10) $$(\bar{\nabla}_X \phi) Y = \eta(Y) X - g(X, Y)\xi - \alpha\{\eta(Y)\phi X + g(X, \phi Y)\xi\}.$$

Interchanging X and Y in (3.10), we have

(3.11) $$(\bar{\nabla}_Y \phi) X = \eta(X) Y - g(X, Y)\xi - \alpha\{\eta(X)\phi Y + g(Y, \phi X)\xi\}.$$

Adding equations (3.10) and (3.11), we have

(3.12) $$(\bar{\nabla}_X \phi) Y + (\bar{\nabla}_Y \phi) X = \eta(X) Y + \eta(Y) X - 2g(X, Y)\xi$$
$$-\alpha\{\eta(X)\phi Y + \eta(Y)\phi X\}.$$

Now, taking $Y=\xi$ in (3.3), we have

$$\bar{\nabla}_X \xi = \bar{\bar{\nabla}}_X \xi + \alpha\{\eta(\xi)X - g(X, \xi)\xi\} - \beta\eta(X)\phi\xi$$

Using (2.1) and (2.4) in above equation, we have

$$\bar{\nabla}_X \xi = \phi X + \alpha X - \alpha\eta(X)\xi . \tag{3.13}$$

We denote by g the metric tensor of $\bar{M}$ and that induced on M. Let $\bar{\nabla}$ be the generalized symmetric metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to unit normal N.

Theorem 3.1: *(i) Let M be a semi-invariant submanifold. If $X, Y \in D$ and D is parallel with respect to ∇, then the connection induced on a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection is also a generalized symmetric metric connection.*

(ii) Let M be a semi-invariant submanifold. If $X, Y \in D^{\perp}$ and $D^{\perp}$ is parallel with respect to ∇, then the connection induced on a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection is also a generalized symmetric metric connection.

(iii) The Gauss formula with respect to a generalized symmetric metric connection is of the form

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

Proof: Let $\bar{\bar{\nabla}}$ be the induced connection with respect to the unit normal N on the semi-invariant submanifolds M of a nearly Sasakian manifold from a generalized symmetric metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y), \tag{3.14}$$

where m is a tensor field of type $(0, 2)$ in the semi-invariant submanifolds M. If ∇^* is the induced connection on semi-invariant submanifolds from the Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X, Y). \tag{3.15}$$

Using (3.14) and (3.15) in (3.16), we have

(3.16) $$\nabla_X Y + m(X, Y) = \nabla_X^* Y + m(X, Y) + \alpha\{\eta(Y)X - g(X, Y)\xi\} - \beta\eta(X)\phi Y .$$

Using (3.1) in (3.16), we have

$$P\nabla_X Y + Q\nabla_X Y + \eta(\nabla_X Y)\xi + m(X, Y)$$
$$= P\nabla_X^* Y + Q\nabla_X^* Y + \eta(\nabla_X^* Y)\xi + h(X, Y) + \alpha\eta(Y)PX$$
$$+\alpha\eta(Y)QX + \alpha\eta(X)\eta(Y)\xi - \beta\eta(X)\phi PY - \beta\eta(X)\phi QY$$
$$-\alpha g(X, Y)P\xi - \alpha g(X, Y)Q\xi - \alpha g(X, Y)\xi .$$

Equating tangential and normal components from both sides, we have

(3.17) $$P\nabla_X Y = P\nabla_X^* Y + \alpha\eta(Y)PX - \alpha g(X, Y)P\xi - \beta\eta(X)\phi PY ,$$

(3.18) $$Q\nabla_X Y = Q\nabla_X^* Y + \alpha\eta(Y)QX - \alpha g(X, Y)Q\xi - \beta\eta(X)\phi QY ,$$

(3.19) $$\eta(\nabla_X Y)\xi = \eta(\nabla_X^* Y)\xi + \alpha\eta(X)\eta(Y)\xi - \alpha g(X, Y)\xi ,$$

(3.20) $$m(X, Y) = h(X, Y).$$

In view of (3.17), if M is a semi-invariant submanifold, $X, Y \in D$ and D is parallel with respect to ∇, then the connection induced on semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection is also a generalized symmetric connection.

Similarly, In view of (3.18), if M is a semi-invariant submanifold, $X, Y \in D^\perp$ and $D^\perp$ is parallel with respect to ∇, then the connection induced on semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection is also a generalized symmetric connection.

Using (3.20), the Gauss formula for a semi-invariant submanifolds of a nearly Sasakian manifold with generalized symmetric metric connection is

(3.21) $$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) .$$

This prove (iii).

Now, for a semi-invariant submanifold M of a nearly Sasakian manifold with generalized symmetric metric connection, the Weingarten formula is given by

(3.22) $$\bar{\nabla}_X N = -A_N X + \alpha\eta(N)X - \beta\eta(X)\phi N + \nabla_X^{\perp} N,$$

for $X, Y \in TM$ and $N \in T^{\perp}M$, A_N is the tensor form of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection, on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten associated with N as

(3.23) $$g(h(X, Y), N) = g(A_N X - \alpha\eta(N)X + \beta\eta(X)\phi N, Y).$$

Definition[12] 3.2: *A semi-invariant submanifold is said to be mixed totally geodesic if $h(X, Y) = 0$, for all $X \in D$ and $Y \in D^{\perp}$. The Nijenhuis tensor $N(X, Y)$ for almost contract structure is expressed as*

(3.24) $$N(X, Y) = (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X,$$

for all $X, Y \in TM$.

From (3.12), replacing X and ϕX, we have

(3.25) $$(\bar{\nabla}_{\phi X}\phi)Y = -2g(\phi X, Y)\xi + \eta(Y)\phi X + \alpha\eta(Y)X$$
$$-\alpha\eta(X)\eta(Y)\xi - (\bar{\nabla}_Y\phi)\phi X.$$

From (2.1), again

$$\phi(\phi X) = -X + \eta(X)\xi.$$

Differentially covariantly along the vector and using (3.13), we have

$$(\bar{\nabla}_Y\phi)\phi X + \phi(\bar{\nabla}_Y\phi)X + \phi^2(\bar{\nabla}_Y X)$$
$$= -\bar{\nabla}_Y X + (\bar{\nabla}_Y\eta)(X)\xi + \eta(\bar{\nabla}_Y X)\xi$$
$$+\eta(X)\{\phi Y + \alpha Y - \alpha\eta(Y)\xi\}.$$

Using (2.1), we have

$$(3.26)\qquad \left(\bar{\nabla}_Y\phi\right)\phi X = \left(\bar{\nabla}_Y\eta\right)(X)\xi + \eta(X)\phi Y + \alpha\eta(X)Y$$
$$-\alpha\eta(X)\eta(Y)\xi - \phi\left(\bar{\nabla}_Y\phi\right)X\ .$$

Using (3.26) in (3.25), we have

$$(3.27)\qquad \left(\bar{\nabla}_{\phi X}\phi\right)Y = -2g(\phi X, Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y - \left(\bar{\nabla}_Y\eta\right)(X)\xi$$
$$+\alpha\{\eta(Y)X - \eta(X)Y\} + \phi\left(\bar{\nabla}_Y\phi\right)X\ .$$

Interchanging X and Y, we have

$$(3.28)\qquad \left(\bar{\nabla}_{\phi Y}\phi\right)X = -2g(\phi Y, X)\xi + \eta(X)\phi Y - \eta(Y)\phi X - \left(\bar{\nabla}_X\eta\right)(Y)\xi$$
$$-\alpha\{\eta(Y)X - \eta(X)Y\} + \phi\left(\bar{\nabla}_X\phi\right)Y\ .$$

Using (3.27) and (3.28) in (3.24), we have

$$N(X, Y) = 6g(\phi X, Y)\xi + 2\eta(Y)\phi X - 2\eta(X)\phi Y + 2\alpha\eta(Y)X$$
$$-2\alpha\eta(X) - 2\phi\left\{\left(\bar{\nabla}_X\phi\right)Y + \left(\bar{\nabla}_Y\phi\right)X\right\} + 4\phi\left(\bar{\nabla}_Y\phi\right)X\ .$$

Using (3.12) in above, we have

$$(3.29)\qquad N(X, Y) = 6g(\phi X, Y)\xi + 4\phi\left(\bar{\nabla}_Y\phi\right)X - 4\eta(X)\phi Y$$
$$-4\alpha\eta(X)Y + 4\alpha\eta(X)\eta(Y)\xi\ .$$

As we know that

$$\left(\bar{\nabla}_Y\phi\right)X = \bar{\nabla}_Y\phi X - \phi\left(\bar{\nabla}_Y X\right).$$

Using Gauss formula in above equation, we have

$$\left(\bar{\nabla}_Y\phi\right)X = \nabla_Y\phi X + h(Y, \phi X) - \phi\nabla_Y X - \phi h(Y, X).$$

Operating ϕ both side in above equation, we have

$$\phi\left(\bar{\nabla}_Y\phi\right)X=\phi\left(\nabla_Y\phi X\right)+\phi h\left(Y,\phi X\right)+\nabla_Y X-\eta\left(\nabla_Y X\right)\xi+h\left(Y,X\right).$$

Using (3.30) in above equation, we have

(3.30) $$N\left(X,Y\right)=4\phi\left(\nabla_Y\phi X\right)+4\phi h\left(Y,\phi X\right)+4h\left(Y,X\right)+4\nabla_Y X-4\eta\left(\nabla_Y X\right)\xi$$
$$-4\eta\left(X\right)\phi Y-4\alpha\eta\left(X\right)Y+4\alpha\eta\left(X\right)\eta\left(Y\right)\xi+6g\left(\phi X,Y\right)\xi .$$

Lemma 3.1: *Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then*

(3.31) $$2\left(\bar{\nabla}_X\phi\right)Y=\nabla_X\phi Y-\nabla_Y\phi X+h\left(X,\phi Y\right)-h\left(Y,\phi X\right)$$
$$-\phi\left[X,Y\right]-2g\left(X,Y\right)\xi\text{, for all } X,Y\in D$$

Proof: By Gauss Formula, replacing Y by ϕY, we have

$$\bar{\nabla}_X\phi Y=\nabla_X\phi Y+\phi\left(\bar{\nabla}_X Y\right),$$

Similarly, $$\bar{\nabla}_Y\phi X=\nabla_Y\phi X+\phi\left(\bar{\nabla}_Y X\right).$$

From both above equation, we have

(3.32) $$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=\nabla_X\phi Y-\nabla_Y\phi X+h\left(X,\phi Y\right)-h\left(Y,\phi X\right).$$

By, covariantly differentiation, we have

$$\bar{\nabla}_X\phi Y=\left(\bar{\nabla}_X\phi\right)Y+\phi\left(\bar{\nabla}_X Y\right),$$

Similarly, $$\bar{\nabla}_Y\phi X=\left(\bar{\nabla}_Y\phi\right)X+\phi\left(\bar{\nabla}_Y X\right).$$

From both above equation, we have

(3.33) $$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=\left(\bar{\nabla}_X\phi\right)Y-\left(\bar{\nabla}_Y\phi\right)X+\phi\left[X,Y\right].$$

From equation (3.32) and (3.33), we have

(3.34) $$\left(\bar{\nabla}_X\phi\right)Y-\left(\bar{\nabla}_Y\phi\right)X=\nabla_X\phi Y-\nabla_Y\phi X+h\left(X,\phi Y\right)$$
$$-h\left(Y,\phi X\right)-\phi\left[X,Y\right].$$

Adding equation (3.12) and (3.34), we have

$$2\left(\bar{\nabla}_X \phi\right)Y = \nabla_X \phi Y - \nabla_Y \phi X + h\left(X, \phi Y\right) - h\left(Y, \phi X\right) - \phi\left[X, Y\right] - 2g\left(X, Y\right)\xi ,$$

for all $X, Y \in D$

Lemma 3.2: *Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then*

(3.35) $$2\left(\bar{\nabla}_X \phi\right)Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X - h\left(Y, \phi X\right) - \phi\left[X, Y\right]$$

for all $X \in D$ and $Y \in D^{\perp}$.

Proof: By Weingarten Formula, replacing N by ϕY, we have

(3.36) $$\begin{aligned} \bar{\nabla}_X \phi Y &= -A_{\phi Y} X + \alpha\eta\left(\phi Y\right)X - \beta\eta\left(X\right)\phi^2 N + \nabla_X^{\perp} \phi Y , \\ \bar{\nabla}_X \phi Y &= -A_{\phi Y} X + \nabla_X^{\perp} \phi Y + \beta\eta\left(X\right)Y - \beta\eta\left(X\right)\eta\left(Y\right)\xi . \end{aligned}$$

As we know that

(3.37) $$\bar{\nabla}_Y \phi X = \nabla_Y \phi X + h\left(Y, \phi X\right).$$

From (3.36) and (3.37), we have

(3.38) $$\begin{aligned} \bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = &-A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X - h\left(Y, \phi X\right) \\ &+\beta\eta\left(X\right)Y - \beta\eta\left(X\right)\eta\left(Y\right)\xi . \end{aligned}$$

Comparing (3.33) and (3.38), we have

(3.39) $$\begin{aligned} \left(\bar{\nabla}_X \phi\right)Y - \left(\bar{\nabla}_Y \phi\right)X = &-A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X - h\left(Y, \phi X\right) \\ &-\phi\left[X, Y\right] + \beta\eta\left(X\right)Y - \beta\eta\left(X\right)\eta\left(Y\right)\xi . \end{aligned}$$

Adding (3.12) and (3.39), we have

$$2\left(\bar{\nabla}_X \phi\right)Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X - h\left(Y, \phi X\right) - \phi\left[X, Y\right],$$

for all $X \in D$ and $Y \in D^{\perp}$.

Lemma 3.3: Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then

(3.40) $$P\nabla_X(\phi PY)+P\nabla_Y(\phi PX)-PA_{\phi QY}X-PA_{\phi QX}Y=\eta(X)PY$$
$$+\eta(Y)PX+\phi P\nabla_X Y+\phi P\nabla_Y X-\alpha\{\eta(X)\phi PY+\eta(Y)\phi PX\},$$

(3.41) $$Q\nabla_X(\phi PY)+Q\nabla_Y(\phi PX)-QA_{\phi QY}X-QA_{\phi QX}Y=\eta(X)QY$$
$$+\eta(Y)QX-\alpha\{\eta(X)\phi QY+\eta(Y)\phi QX\}+2Bh(X,Y)$$
$$-\beta\{\eta(X)QY+\eta(Y)QX\},$$

(3.42) $$h(X,\phi PY)+h(Y,\phi PX)+\nabla_X^{\perp}\phi QY+\nabla_Y^{\perp}\phi QX$$
$$=2ch(X,Y)+\phi Q\nabla_X Y+\phi Q\nabla_Y X,$$

(3.43) $$\eta\left(\nabla_X\phi PY+\nabla_Y\phi PX-A_{\phi QY}X-A_{\phi QX}Y\right)\xi$$
$$=2\eta(X)\eta(Y)\xi-2g(X,Y)\xi,$$

for all $X,Y\in TM$.

Proof: By covariant differentiation, we have

$$(\bar{\nabla}_X\phi)Y+\phi(\bar{\nabla}_X Y)=\bar{\nabla}_X\phi Y.$$

Using (3.21) and (3.1), in above, we have

$$(\bar{\nabla}_X\phi)Y+\phi\nabla_X Y+\phi h(X,Y)=\bar{\nabla}_X\phi PY+\bar{\nabla}_X\phi QY$$

Using (3.1) and (3.22) in above, we have

(3.44) $$(\bar{\nabla}_X\phi)Y+\phi\nabla_X Y+\phi h(X,Y)=P\nabla_X(\phi PY)+Q\nabla_X(\phi PY)$$
$$+\eta(\nabla_X(\phi PY))\xi-PA_{\phi QY}X-QA_{\phi QY}X-\eta(A_{\phi QY}X)\xi$$
$$+h(X,\phi PY)+\nabla_X^{\perp}\phi QY+\beta\eta(X)QY,$$

Similarly,

(3.45) $$\left(\overline{\nabla}_Y\phi\right)X+\phi\nabla_Y X+\phi h\left(Y,X\right)=P\nabla_Y\left(\phi PX\right)+Q\nabla_Y\left(\phi PX\right)$$
$$+\eta\left(\nabla_Y\left(\phi PX\right)\right)\xi-PA_{\phi QX}Y-QA_{\phi QX}Y-\eta\left(A_{\phi QX}Y\right)\xi$$
$$+h\left(Y,\phi PX\right)+\nabla_Y^{\perp}\phi QX+\beta\eta\left(Y\right)QX\ .$$

Adding (3.34) and (3.35), we have

$$\left(\overline{\nabla}_X\phi\right)Y+\left(\overline{\nabla}_Y\phi\right)X+\phi\nabla_X Y+\phi\nabla_Y X+2\phi h\left(X,Y\right)$$
$$=P\nabla_X\left(\phi PY\right)+P\nabla_Y\left(\phi PX\right)+Q\nabla_X\left(\phi PY\right)$$
$$+Q\nabla_Y\left(\phi PX\right)+\eta\left(\nabla_X\left(\phi PY\right)\right)\xi+\eta\left(\nabla_Y\left(\phi PX\right)\right)\xi$$
$$-PA_{\phi QY}X-PA_{\phi QX}Y-QA_{\phi QY}X-\eta\left(A_{\phi QY}X\right)\xi-QA_{\phi QX}Y$$
$$-\eta\left(A_{\phi QX}Y\right)\xi-\eta\left(A_{\phi QY}X\right)\xi+h\left(X,\phi PY\right)+h\left(Y,\phi PX\right)$$
$$+\nabla_X^{\perp}\phi QY+\nabla_Y^{\perp}\phi QX+\beta\eta\left(X\right)QY+\beta\eta\left(Y\right)QX$$

Using (3.1), (3.2) and (3.12) in above equation, we have

$$\eta\left(X\right)PY+\eta\left(X\right)QY+\eta\left(Y\right)PX+\eta\left(Y\right)QX+2\eta\left(X\right)\eta\left(Y\right)\xi$$
$$-2g\left(X,Y\right)\xi-\alpha\eta\left(X\right)\phi PY-\alpha\eta\left(X\right)\phi QY-\alpha\eta\left(Y\right)\phi PX$$
$$-\alpha\eta\left(Y\right)\phi QX+\phi P\nabla_X Y+\phi Q\nabla_X Y+\phi P\nabla_Y X+\phi Q\nabla_Y X$$
$$+2Bh\left(X,Y\right)+2Ch\left(X,Y\right)=P\nabla_X\left(\phi PY\right)+P\nabla_Y\left(\phi PX\right)$$
$$+Q\nabla_X\left(\phi PY\right)+Q\nabla_Y\left(\phi PX\right)+\eta\left(\nabla_X\left(\phi PY\right)\right)\xi+\eta\left(\nabla_Y\left(\phi PX\right)\right)\xi$$
$$-PA_{\phi QY}X-PA_{\phi QX}Y-QA_{\phi QY}X-QA_{\phi QX}Y-\eta\left(A_{\phi QY}X\right)\xi$$
$$-\eta\left(A_{\phi QX}Y\right)\xi+h\left(X,\phi PY\right)+h\left(Y,\phi PX\right)+\nabla_X^{\perp}\phi QY+\nabla_Y^{\perp}\phi QX$$
$$+\beta\eta\left(X\right)QY+\beta\eta\left(Y\right)QX$$

Equation (3.40) to (3.43) following by comparing the tangential, normal and vertical parts.

Definition[12] 3.3: *The horizontal distribution* D *is said to be parallel with respect to the connection* ∇ *on* M *if* $\nabla_X Y$ *for all vector fields* $X, Y \in D$.

Proposition 3.1: *Let* M *be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection if the horizontal distribution* D *is parallel then*

$$h(X, \phi Y) = h(\phi X, Y) \quad \textit{for all } X, Y \in D.$$

Proof: As the horizontal distribution D is parallel, so for any $X, Y \in D$, we have

$$\nabla_X \phi Y \in D \quad \text{and} \quad \nabla_Y \phi X \in D$$

By virtue of the above fact, (3.41) gives

$$Bh(X, Y) = 0, \quad \text{for any } X, Y \in D$$

Next, since

$$\phi h(X, Y) = Bh(X, Y) + Ch(X, Y),$$

So, we get

$$\phi h(X, Y) = Bh(X, Y) + Ch(X, Y).$$

Further with the help of (3.42), we have

(3.48) $$h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y), \quad \text{for any } X, Y \in D.$$

Taking in (3.48), we get

(3.49) $$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y).$$

Again taking $Y = \phi Y$ in (3.48), we get

(3.50) $$h(\phi X, \phi Y) - h(X, Y) = 2\phi h(X, \phi Y).$$

Hence it follows from (3.49) and (3.50), we get

$$\phi h(X, \phi Y) = \phi h(\phi X, Y),$$

which is equivalent to

$$h(X, \phi Y) = h(\phi X, Y), \qquad \text{for any } X, Y \in D.$$

4. Integrability of Distribution

The purpose of this paragraph is to study the Integrability of distribution $D \oplus \{\xi\}$ and $D^{\perp}$ of semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection.

Theorem 4.1: *Let M be a semi-invariant submanifold for a nearly Sasakian manifold with generalized symmetric metric connection, then the distribution $D \oplus \{\xi\}$ is integrable if the following condition are satisfied*

(4.1) $$S(X, Y) \in D \oplus \{\xi\},$$

(4.2) $$h(X, \phi Y) = h(\phi X, Y),$$

for all $X, Y \in D \oplus \{\xi\}$

Proof: The torsion tensor $S(X, Y)$ of an almost contact structure (ϕ, ξ, η, g) is given by

$$S(X, Y) = N(X, Y) + 2d\eta(X, Y)\xi,$$

where $N(X, Y)$ is the Nijenhuis tensor of ϕ.
Thus we have,

(4.3) $$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi,$$

for any $X, Y \in TM$.
Suppose that the distribution $D \oplus \{\xi\}$ is Integrability, so far $X, Y \in D \oplus \{\xi\}$

(4.4) $$N(X, Y) = 0,$$

then, $$S(X, Y) = 2d\eta(X, Y) \in D \oplus \{\xi\},$$

therefore, $S(X, Y) \in D \oplus \{\xi\}$.

Using (3.30) and (4.4), we have

$$4\phi(\nabla_Y \phi X) + 4\phi h(Y, \phi X) + 4h(Y, X) + 4\nabla_Y X - 4\eta(\nabla_Y X)\xi$$
$$-4\eta(X)\phi Y - 4\alpha\eta(X)Y + 4\alpha\eta(X)\eta(Y)\xi + 6g(\phi X, Y)\xi .$$

Comparing normal part both sides, we have

$$4\phi Q(\nabla_Y \phi X) + 4Ch(Y, \phi X) + 4h(Y, X) = 0 ,$$

(4.5) $$\phi Q(\nabla_Y \phi X) + Ch(Y, \phi X) + h(Y, X) = 0 .$$

Replace Y by ϕZ in (4.5), we get

(4.6) $$\phi Q(\nabla_{\phi Z} \phi X) + Ch(\phi Z, \phi X) + h(\phi Z, X) = 0 .$$

Interchanging X and Z in (4.6), we have

(4.7) $$\phi Q(\nabla_{\phi X} \phi Z) + Ch(\phi X, \phi Z) + h(\phi X, Z) = 0 .$$

Subtracting (4.6) from (4.7), we have

$$\phi Q(\nabla_{\phi X} \phi Z) - \phi Q(\nabla_{\phi Z} \phi X) + h(\phi X, Z) - h(\phi Z, X) = 0$$

(4.8) $$\phi Q[\phi X, \phi Z] + h(\phi X, Z) - h(\phi Z, X) = 0 .$$

From which the assertion follows.

Lemma 4.1: *Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then*

$$2(\overline{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^{\perp} \phi Z - \nabla_Z^{\perp} \phi Y - 2g(X, Y)\xi - \phi[Y, Z],$$

for any $Y, Z \in D^{\perp}$.

Proof: Using Weingarten formula (3.22) and fact that ϕY and ϕZ are normal to M for

$Y, Z \in D^{\perp}$, we get

(4.9) $$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^{\perp} \phi Z - \nabla_Z^{\perp} \phi Y + \beta \{ \eta(Y) Z - \eta(Z) Y \}$$

On the other hand, we get

(4.10) $$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = (\bar{\nabla}_Y \phi) Z - (\bar{\nabla}_Z \phi) Y + \phi [Y, Z].$$

Now from (4.9), (4.10) and fact that $\eta(Y) = \eta(Z) = 0$, we have

(4.11) $$(\bar{\nabla}_Y \phi) Z - (\bar{\nabla}_Z \phi) Y = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^{\perp} \phi Z - \nabla_Z^{\perp} \phi Y - \phi [Y, Z].$$

Moreover from (3.22) and the fact that $\eta(Y) = \eta(Z) = 0$, for $Y, Z \in D^{\perp}$, we get

(4.12) $$(\bar{\nabla}_Y \phi) Z - (\bar{\nabla}_Z \phi) Y = -2 g(Y, Z) \xi .$$

Adding (4.11) and (4.12), we get our assertion.

Proposition 4.1: *Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then*

$$A_{\phi Y} Z - A_{\phi Z} Y = \frac{1}{3} \phi P [Y, Z]$$

for all $Y, Z \in D^{\perp}$.

Proof: As $Y, Z \in D^{\perp}$ and $X \in TM$, we have

$$2g(A_{\phi Z} Y, X) = g(h(X, Y), \phi Z) + g(h(Y, X), \phi Z),$$

$$2g(A_{\phi Z} Y, X) = g(\bar{\nabla}_X Y - \nabla_X Y, \phi Z) + g(\bar{\nabla}_Y X - \nabla_Y X, \phi Z),$$

$$2g(A_{\phi Z} Y, X) = g(\bar{\nabla}_X Y, \phi Z) + g(\bar{\nabla}_Y X, \phi Z)$$
$$- g(\nabla_X Y, \phi Z) - g(\nabla_Y X, \phi Z).$$

As $\nabla_X Y \in TM$, $\nabla_Y X \in TM$ and $\phi Z \in T^{\perp} M$, we have from above

$$2g\left(A_{\phi Z}Y,\, X\right)=g\left(\bar{\nabla}_X Y,\, \phi Z\right)+g\left(\bar{\nabla}_Y X,\, \phi Z\right),$$

$$2g\left(A_{\phi Z}Y,\, X\right)=-g\left(\phi\bar{\nabla}_X Y,\, Z\right)-g\left(\phi\bar{\nabla}_Y X,\, Z\right),$$

(4.13)
$$2g\left(A_{\phi Z}Y,\, X\right)=-g\left(\bar{\nabla}_X \phi Y,\, Z\right)-g\left(\bar{\nabla}_Y \phi X,\, Z\right)$$
$$-g\left(\left(\bar{\nabla}_X \phi\right)Y-\left(\bar{\nabla}_Y \phi\right)X,\, Z\right).$$

Now, As $Y\in T^{\perp}M \Rightarrow \phi Y\in T^{\perp}M$, so replace N by ϕY in (3.22), we have

$$\bar{\nabla}_X \phi Y=-A_{\phi Y}X+\alpha\eta\left(\phi Y\right)X-\beta\eta\left(X\right)\phi^2 Y+\nabla_X^{\perp}\phi Y\,.$$

Using equation (2.1) and (2.3) in above, we have

$$\bar{\nabla}_X \phi Y=-A_{\phi Y}X+\beta\eta\left(X\right)Y-\beta\eta\left(X\right)\eta\left(Y\right)\xi+\nabla_X^{\perp}\phi Y\,.$$

As $\eta\left(Y\right)=0$ for $Y\in D^{\perp}$, we have

(4.14)
$$\bar{\nabla}_X \phi Y=-A_{\phi Y}X+\beta\eta\left(X\right)Y+\nabla_X^{\perp}\phi Y\,.$$

Using equation (3.12) and (4.14) in (4.13), we have

$$2g\left(A_{\phi Z}Y,\, X\right)=-g\left(\phi\bar{\nabla}_Y Z,\, X\right)+g\left(A_{\phi Y}Z,\, X\right)-\beta\eta\left(X\right)g\left(Y,\, Z\right)$$
$$-g\left(\nabla_X^{\perp}\phi Y,\, Z\right)-\eta\left(X\right)g\left(Y,\, Z\right)-\eta\left(Y\right)g\left(X,\, Z\right)$$
$$+2g\left(X,\, Y\right)g\left(\xi,\, Z\right)+\alpha\eta\left(X\right)g\left(\phi Y,\, Z\right)$$
$$-\alpha\eta\left(Y\right)g\left(\phi X,\, Z\right).$$

As $\eta\left(Y\right)=\eta\left(Z\right)=0$ for $Y,Z\in D^{\perp}$, we have

$$2g\left(A_{\phi Z}Y,\, X\right)=-g\left(\phi\bar{\nabla}_Y Z,\, X\right)+g\left(A_{\phi Y}Z,\, X\right)-\beta g\left(Y,\, Z\right)\eta\left(X\right)$$
$$-g\left(Y,\, Z\right)\eta\left(X\right)+\alpha g\left(\phi Y,\, Z\right)\eta\left(X\right).$$

Transvecting X from both sides, we have

(4.15)
$$2A_{\phi Z}Y=-\phi\bar{\nabla}_Y Z+A_{\phi Y}Z-\beta g\left(Y,\, Z\right)\xi-g\left(Y,\, Z\right)\xi+\alpha g\left(\phi Y,\, Z\right)\xi\,,$$

(4.16) $$2A_{\phi Z}Y = -\phi\bar{\nabla}_Z Y + A_{\phi Z}Y - \beta g(Z, Y)\xi - g(Z, Y)\xi + \alpha g(\phi Z, Y)\xi .$$

Subtracting above two equation, we get

$$2\left(A_{\phi Y}Z - A_{\phi Z}Y\right) = \phi\left(\bar{\nabla}_Y Z - \bar{\nabla}_Z Y\right) - \left(A_{\phi Y}Z - A_{\phi Z}Y\right) + 2\alpha g(\phi Y, Z)\xi ,$$

$$2\left(A_{\phi Y}Z - A_{\phi Z}Y\right) = \phi[Y, Z] - \left(A_{\phi Y}Z - A_{\phi Z}Y\right) + 2\alpha g(\phi Y, Z)\xi .$$

Comparing the tangential component from both sides, we have

$$2\left(A_{\phi Y}Z - A_{\phi Z}Y\right) = \phi P[Y, Z] - \left(A_{\phi Y}Z - A_{\phi Z}Y\right)$$

(4.17) $$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z]$$

for all $Y, Z \in D^{\perp}$.

Theorem 4.2: *Let M be a semi-invariant submanifold of a nearly Sasakian manifold with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is intangible if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 ,$$

for all $Y, Z \in D^{\perp}$.

Proof: Suppose that distribution $D^{\perp}$ is intangible, then $[Y, Z] \in D^{\perp}$ for any
$Y, Z \in D^{\perp}$.
Therefore $P[Y, Z] = 0$ and from (4.17), we get

(4.18) $$A_{\phi Y}Z - A_{\phi Z}Y = 0 .$$

Conversely, let (4.18) hold. Then, by virtue of (4.17), we have $P[Y, Z] = 0$ for all $Y, Z \in D^{\perp}$. Since rank $\phi = 2n$, either $P[Y, Z] = 0$ or $P[Y, Z] = K\xi$. But $P[Y, Z] = K\xi$ is not possible as P being a projection operator on D. Hence $P[Y, Z] = 0$, which is equivalent to $[Y, Z] \in D^{\perp}$ for all $Y, Z \in D^{\perp}$ and $D^{\perp}$ is integrable.

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