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Control Ecological Model for the Survival of Biological Species which are Directly Affected by Pollutants Emitted from External Sources: A Qualitative Approach

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Abstract: In this paper, a non-linear control ecological model is proposed and analyzed to study the survival of biological species which is directly affected by a toxicant emitted into the environment by external sources. The model is formulated by using system of non-linear ordinary differential equations. In the analysis, all the feasible equilibrium points of the system have been obtained. The conditions for local and non-linear stability of the non-trivial equilibrium points have been carried out using a suitable Lyapunov function. Also, a region of attraction has been found for non-linear asymptotic stability of the equilibrium point. It has been shown that the density of the biological species decreases with the increase in the total emission rate of pollutant in the environment. The analysis of the non linear stability shows that the system settles at much lower density of the biological species when the concentration of the pollutants in the environment and in the uptake phase of the species is high.It has been found that the control measures (carbon emitter taxes) prove as disincentive to the emitters of the pollutants and its emission at source is checked and reduced and the biological species can be maintained at a desired level. The results are illustrated with the help of a numerical example and computer simulation.

Key Words: Biological species,Toxicity, Emission, Control measure.

2000 Mathematics Subject classification No.: 92D40.

1. Introduction

Due to various industrial and other activities of human beings, the pollutants and the toxicants are continuously emitted in the environment. These activities have increased at enormous level due to the rapid growth of the industrialization. In the second half of the 20^{th} century and especially from the last decade of this century due to globalization, this rate of industrialization had increased phenomenally. The urbanization and increase in the living standard of the societies has augmented the emissions of the pollutants and toxicants. Incessant use of all the natural resources without recharging and cleaning the same has further created a grave and dangerous situation. For example, the industrial pollutants have affected the water, air and the land which in turn have affected the large number of the biological species. The domestic and transportation uses of the fossils fuels have created the air pollution, the discharges of the industries and households of the cities have damagingly polluted the water of rivers, lakes and coastal seas. The pollutants have affected the biological species directly as well as indirectly by deteriorating the resource biomass on which some biological species are dependent. In this paper, we have studied the direct impact of the pollutants on the biological species. The examples of this kind of problem may be found in the eco-systems are the forests as are directly affected by the air pollutants like oxides of sulphur or oxides of carbons etc. The models relating to the biological species like forest animals and birds etc. affected by the air pollutants like oxides of sulphur or oxides of carbon may also fall in this category. In recent years some investigations have been conducted to study the effects of toxicants (pollutants) on biological species[1-3]. Hallam and Luna[4] proposed and analyzed a mathematical model to study the effects of toxicants on the biological population when the toxicants are emitted into the environment from external sources and they have also discussed at length the effect of a toxicant on population and showed that population will only persist if it exhibits a consistent potential for growth. Freedman and Shukla[5] in their model considered that the intrinsic growth rate of biological species decreases as the uptake concentration of the toxicant increases, while the carrying capacity of the species decreases as the concentration of the toxicant in the environment increases. Shukla and Dubey[6,7] proposed the model to study the simultaneous effects of two toxicants with different toxic concentrations on a biological population, in their model they assume that both toxicants are emitted from external sources. Shuklaet al.[8] also modeled the effects of primary and secondary toxicants on renewable resources. Srinivasu[9] and Thomas et.al[10] proposed model to study the control of environmental pollution to conserve a population. In view of this, in this paper we have proposed a dynamical model for conservation of biological species by controlling the emission rate of pollutant into the environment.

In this paper, the following assumptions have been made:

1. The densities of biological species as well as the concentration of the pollutants in the environment and in the uptake phase of the biological species are assumed to be governed by logistic equations.

2. The rate of emission of a pollutant or toxicant into the environment is a constant (i.e. the cumulative rate of production of a toxicant into the environment from the external sources such as Industrial, Chimneys, motor vehicles etc.).
3. The concentration of this toxicant in the environment decreases due to its assimilation, absorption, deposition, uptake, etc. by biological species, the amount being proportional to the biological species as well as environmental concentration of the toxicants. We assume that density a portion of the assimilated amount becomes part of the uptake phase in the biological species as this portion increases, the uptake concentration of the toxicant in the biological species increases. This up taken toxicant interacts with the biological species through bio-physical process leading to decrease in the growth rate of species density. It is also assumed that the remaining portion of this amount, which does not become part of the uptake, decreases the growth rate of the biological species directly.
4. It has been assumed that the carrying capacity of the biological species decreases as the concentration of the pollutants in the environment increases.
5. The toxicants in the environment as well as in uptake phase decrease due to natural factors by an amount which is proportional to its concentration in various cases.
6. Some of the pollutants in the uptake phase are decomposed and the same re-enter in the environment.
7. It has been assumed that environmental management systems (control measures), should be applicable/ imposed only when the concentration of the pollutants or the toxicants crosses the harmful limit. There may be some practical difficulties in implementation of the full proof environmental management system, due to some natural and administrative problem and faults of the system.

2. Mathematical Model

The following system of differential equations is considered to study the effects of toxicants emitted into the environment from various sources on the biological species present in the ecological system. In view of the above assumptions, we propose the following model, governing the dynamics of the biological species, concentrations of the toxicant/ pollutant and the control mechanism.

$$
(2.1)\qquad \begin{cases} \dfrac{dB}{dt} = s(U)B - \dfrac{s_0 B^2}{L(C)} - k\alpha BC \\ \dfrac{dC}{dt} = Q - \delta C - \alpha BC + \pi \nu UB - \mu F \\ \dfrac{dU}{dt} = (1-k)\alpha BC - \phi U - \nu UB \\ \dfrac{dF}{dt} = \theta_1 \left(C - C_p\right) - \theta_0 F \\ B(0) = B_0 \ge 0, \quad C(0) = C_0 \ge 0, \quad U(0) = U_0 \ge 0, \\ F(0) = F_0 \ge 0, \quad 0 \le k \le 1, \quad 0 \le \pi \le 1. \end{cases}
$$

Here $B(t)$ is the density of biological species , $C(t)$ and $U(t)$ are the concentrations of the pollutants in the environment and in the uptake phase of the biological species respectively at any time $t > 0$. Q is the Cumulative rate of production of a toxicant into the environment from the external sources.. The constants $\delta > 0$ and $\phi > 0$ are the natural wash out rate coefficients of toxicants and uptake phase respectively, $\alpha > 0$ is the rate of depletion of pollutant in the environment due to uptake of pollutant by the biological species. Also some amount of the biological species may die out at a rate ν due to excessive and unbearable presence of the toxicant and a fraction π of this may again re-enter into the environment. In (2.1) $k\alpha BC > 0$ is a fraction of αBC directly affecting biological species and remaining $(1-k)\alpha BC$ of it is up taken by the biological species which decreases the intrinsic growth rate of B. $F(t)$ denotes the control measure for reducing the concentration of toxicant by govt./ NGOs/ Education awareness, reforestation, taxation(carbon emitter tax), etc.

μF is a pollution control device which controls the growth of emission of the toxicant in the environment. Cp is the permissible level of the concentration of toxicant (C), which is harmless to the biological species.

The term $\theta_0 F$ is to account for some practical difficulties in implementing the fool proof environmental management system. $\alpha, \delta, \phi, \nu, \theta_1, \theta_0, Q, C_p$ are all positive constants.

In the model (2.1), the function $s(U)$ represents the growth rate coefficient of biological species which decreases with the increases of U, and hence

(2.1a) $s(0) = s_0 > 0, \quad \dfrac{ds(U)}{dU} < 0$, for $U \geq 0$.

Similarly, the function $L(C)$ denotes the carrying capacity (i.e. the maximum density of biological species which the environment can support).

We assume that $L(C)$ decreases as C increases, hence we have

(2.1b) $L(0) = L_0 > 0, \quad \dfrac{dL(C)}{dC} < 0, \text{ for } \quad C \geq 0.$

where L_0 is the toxicant independent carrying capacity.

3. Equilibrium Analysis

The given model (2.1) has two non-negative real equillibria (Feasible equilibrium points) in $B\text{-}C\text{-}U\text{-}F$ space denoted by $E_0(0, C, 0, F)$ and $E^*(B^*, C^*, U^*, F^*)$. For, $E_0(0, C, 0, F)$

$$C = \frac{\left[Q\theta_0 + \mu\theta_1 C_p\right]}{\delta\theta_0 + \mu\theta_1},$$

$$F = \frac{\left[Q\theta_1 - \delta\theta_1 C_p\right]}{\delta\theta_0 + \mu\theta_1}, \text{provided } Q > \delta C_p$$

where $\delta\theta_0 + \mu\theta_1 \neq 0$.

The existence of E_0 is obvious.

Existence Of Internal Equilibrium Point $\boldsymbol{E^*(B^*, C^*, U^*, F^*)}$:

The interior equilibrium $E^*(B^*, C^*, U^*, F^*)$ is the solution of the following system of equations:

(3.2a) $B = \dfrac{s(U)L(C) - k\alpha CL(C)}{s_0}$, provided $s(U) - k\alpha C > 0$

(3.2b) $$C=\frac{(\phi+\nu B)\left[Q\theta_0+\mu\theta_1 C_p\right]}{f_1(B)}=g(B),\quad \text{(assuming)}$$

(3.2c) $$U=\frac{(1-k)\alpha B\left[Q\theta_0+\mu\theta_1 C_p\right]}{f_1(B)}=h(B),\quad \text{(assuming)}$$

(3.2d) $$F=\frac{\theta_1}{\theta_0}(g(B)-C_p)=i(B),\ \text{provided}\quad g(B)>C_p\ \ \text{(assuming)}$$

where

$$f_1(B)=\phi(\delta\theta_0+\mu\theta_1)+\left[\phi\alpha\theta_0+\nu(\delta\theta_0+\mu\theta_1)\right]B+\alpha\nu\theta_0[1-\pi(1-k)]B^2$$

Rewrite equation(3.2a) as

(3.2e) $$s_0B=s(U)L(C)-k\alpha CL(C)\,.$$

Substituting the values of C and U from equations (3.2b) and (3.2c) in the above equation (3.2e) , we get

(3.2f) $$s_0B=s(h(B))L(g(B))-k\alpha g(B)L(g(B)).$$

To show the existence of the internal equilibrium point $E^*(B^*,\ C^*,\ U^*,\ F^*)$ its sufficient to show that equation (3.2f) has a unique positive solution in *B*.

Let us consider a function *F(B)* such that

(3.2g) $$F(B)=s_0B-s(h(B))L(g(B))+k\alpha g(B)L(g(B)).$$

Putting the value *B=0* and L_0 in equation (3.2g), we get

$$F(0)=-s(h(0))L(g(0))+k\alpha g(0)L(g(0)).$$

$$F(0)=-L(g(0))\left[\,s(h(0))-k\alpha g(0)\,\right].$$

L(g(0)) being carrying capacity is always positive and from equation(3.2a).

$$s(U)-k\alpha C>0\,.$$

This gives that at $B=0$,

$$s_0-k\alpha g(0)>0,$$

and hence

(3.2h) $F(0) < 0.$

Now, to find the nature of $F(L_0)$:
In this case L_0 has been taken at $C = 0$, therefore $g(L_0) = 0$.

$$F(L_0) = L_0\left[s_0 - s(h(L_0))\right] > 0,$$

here s_0 has been taken as maximum of s, therefore

(3.2i) $F(L_0) > 0.$

From equations (3.2h) and (3.2i), it is clear that there exists a root B^* in the interval $0 < B^* < L_0$ such that $F(B^*) = 0$.
For B^* to be unique we must have $F'(B) > 0$ in the interval $0 < B^* < L_0$.
From (3.2g), we get

$$F'(B) = s_0 + \frac{dL}{dg}\frac{dg}{dB}[-s(h(B) + k\alpha g(B)] + L(g(B)[-\frac{ds}{dh}\frac{dh}{dB} + k\alpha\frac{dg}{dB}] > 0.$$

Since $B = B^*$ is the root of the equation (3.2g), we get

$$F'(B) = s_0 - \frac{s_0 B}{L(g(B))}\frac{dL}{dg}\frac{dg}{dB} + L(g(B))\left(-\frac{ds}{dh}\frac{dh}{dB} + k\alpha\frac{dg}{dB}\right) > 0. \quad \text{(3.2j)}$$

Thus, the condition for unique and positive B^* is $F'(B) > 0$ .Once B^* is determined and then, C^*, U^* can be found from equations (3.2b)- (3.2d)
In view of the above, we have the following conditions: Let F(B) and g(B) be given by equations (3.2g) and (3.2b) respectively.
If

$$\begin{cases} (i) \quad s(U) - k\alpha C > 0 \\ (ii) \quad g(B) > C_p \\ (iii) \; F'(B) > 0 \quad for \;\; 0 < B < L_0, \end{cases} \quad \text{(3.2k)}$$

then there exists a unique interior equilibrium $E^*(B^*, C^*, U^*, F^*)$ for the model (2.1).

Now let us examine the effect of Q on B i.e. the cumulative rate of production of the pollutant on the density of the resource biomass.
From equation (3.2f), we have

$$s_0 B = s(h(B))L(g(B)) - k\alpha g(B)L(g(B)).$$

Differentiating with respect to Q, we get

(3.2l) $$s_0 \frac{dB}{dQ} = s(h)\frac{dL}{dg}\frac{dg}{dQ} + L(g)\frac{dS}{dh}\frac{dh}{dQ} - k\alpha L(g)\frac{\mathrm{d}g}{dQ} - k\alpha g\frac{dL}{\mathrm{d}g}\frac{dg}{dQ}$$

Using the formulae

(3.2m) $$\begin{cases} \dfrac{dg}{dQ} = \dfrac{\partial g}{\partial B}\left(\dfrac{dB}{dQ}\right) + \dfrac{\partial g}{\partial Q}, \\ \dfrac{dh}{dQ} = \dfrac{\partial h}{\partial B}\left(\dfrac{dB}{dQ}\right) + \dfrac{\partial h}{\partial Q} . \end{cases}$$

Substituting the values from (3.2m) to (3.2l) and rearranging the terms, equation (3.2l) becomes

(3.2n) $$\frac{dB}{dQ}\left(s_0 - \frac{s_0 B}{L(g)}\frac{dL}{dg}\frac{\partial g}{\partial B} - L(g)\frac{dS}{dh}\frac{\partial h}{\partial B} + \alpha_2 L(g)\frac{\alpha_1 K}{r_0} + k\alpha L(g)\frac{\partial g}{\partial B}\right)$$

$$= \frac{s_0 B}{L(g)}\frac{dL}{dg}\frac{\partial g}{\partial Q} - k\alpha L(g)\frac{\partial g}{\partial Q} + L(g)\frac{dS}{dh}\frac{\partial h}{\partial Q}$$

In the condition of uniqueness (3.2j) i.e. $F'(B) > 0$ in the interval $0 < B < L_0$ and further from equations (3.2c) and (3.2d), we get

$$\frac{\partial g}{\partial Q} = \frac{\theta_0(\phi + \nu B)}{f_1(B)} > 0 ,$$

$$\frac{\partial h}{\partial Q} = \frac{(1-k)\alpha B\theta_0}{f_1(B)} > 0 .$$

And from conditions, we have

$$\frac{ds(U)}{dU} < 0 , \quad \text{for} \quad U \geq 0 \quad ,$$

$$\frac{dL(C)}{dC} < 0, \quad \text{for} \quad C \geq 0,$$

using these conditions and results, we analyze equation (3.2n) as

$$\frac{dB}{dQ}\left(\text{+ve funtion}\right)=\left(\text{-ve function}\right).$$

Therefore,

$$\frac{dB}{dQ} < 0.$$

This suggest that the density of biological species decreases with the cumulative rate of toxicants increases in the environment.
Again from equation (3.2d),

$$F = \frac{\theta_1}{\theta_0}(g(B) - C_p) = i(B),$$

On differentiating with respect to F, we get

$$1 = \frac{\theta_1}{\theta_0}\left[\frac{dg}{dB}\frac{dB}{dF}\right].$$

From equation (3.2b), we get $\frac{dg}{dB} > 0$
using this result in the above equation, we have

$$\frac{dB}{dF} > 0.$$

From the above, we can say that with the increase in the control measure (efforts), the density of the biological species increases, thus the control measure (carbon emitter taxes) has a positive impact in the eco-system and the biological species may be saved from going to extinction.

4. Stability Analysis

Here we shall discuss the local as well as global stability of the equilibrium points. The local stability of the equilibria can be studied from variational matrices corresponding to each equilibrium point and for the global stability, suitable Lyapunov functions are found in the interior of some region Ω.

4a. Local Stability Via Eigen Value Method:

To study the local stability behavior of equilibria, we compute the variational matrices corresponding to each equilibrium points. Let M_0 and M^* be the variational matrices corresponding to equilibrium points E_0 and E^* respectively.

$$M_0 = \begin{bmatrix} s_0 - k\alpha\left(\dfrac{Q\theta_0 + \mu\theta_1 C_p}{\delta\theta_0 + \mu\theta_1}\right) & 0 & 0 & 0 \\ -\alpha\left(\dfrac{Q\theta_0 + \mu\theta_1 C_p}{\delta\theta_0 + \mu\theta_1}\right) & -\delta & 0 & -\mu \\ (1-k)\alpha\left(\dfrac{Q\theta_0 + \mu\theta_1 C_p}{\delta\theta_0 + \mu\theta_1}\right) & 0 & -\phi & 0 \\ 0 & \theta_1 & 0 & -\theta_0 \end{bmatrix},$$

and

$$M^* = \begin{bmatrix} -\dfrac{s_0 B^*}{L(C^*)} & \dfrac{s_0 B^{*^2} L'(C^*)}{\left[L(C^*)\right]^2} - k\alpha B^* & s'(U^*)B^* & 0 \\ -\alpha C^* + \pi\nu U^* & -\delta - \alpha B^* & \pi\nu B^* & -\mu \\ (1-k)\alpha C^* - \nu U^* & (1-k)\alpha B^* & -\phi - \nu B^* & 0 \\ 0 & \theta_1 & 0 & -\theta_0 \end{bmatrix}.$$

From the matrix M_0, it is clear that $E_0(0, C, 0, F)$ is a saddle point with stable manifold locally in the $U-$direction and unstable manifold locally in the $B-$direction if $s_0 - k\alpha C > 0$.

The stability behavior of E^* is not obvious from M^*. However, in the following theorem we find sufficient condition for E^* to be locally asymptotically stable.

The following theorem gives the criteria for the local stability of E^* which can be proved by constructing a suitable Lyapunov function.

4b. Local Stability ViaLyapunov Method:

Theorem 4.1: *If the following inequalities hold:*

(4.1a) $$\left[\left(\frac{s_0 B^* L'(C^*)}{\left[L(C^*)\right]^2} - k\alpha\right) - C_2\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \frac{2C_2 s_0 \delta}{L(C^*)} ,$$

where

(4.1b) $$C_2 = \frac{-s'(U^*)(1-k)\alpha}{\left[(1-k)\alpha C^* - \nu U^*\right]\pi\nu} \text{ , provided } (1-k)\alpha C^* > \nu U^* .$$

Then $E^*(B^*,\ C^*,\ U^*,\ F^*)$ *is locally asymptotically stable.*

Proof: See the Appendix.

5. Region Of Attraction For Global Stability

The following theorem gives the globally asymptotically behavior of the equilibrium point E^*.For this we first need a lemma which establishes the region of attraction for the system (2.1).

Lemma 5.1: *The set*

$$\Omega = \left\{(B, C, U, F):\ \ 0 \le B \le L_0\,; 0 \le C + U \le \frac{Q}{\phi_1}\,; 0 \le F \le \frac{\theta_1}{\theta_0}\left(\frac{Q}{\phi_1}\right)\right\}$$

is a region of attraction for all solutions initiating in the region

$$R = \{(B, C, U, F) : B > 0, C > 0, U > 0, F > 0\}$$

where

$$\phi_1 = \min(\delta, \phi).$$

5.1. Global Stability Analysis ViaLyapunov Method:

Theorem 5.2: *In addition to the assumptions (2.1a) and (2.1b), let* L_m,p and q *be positive constants such that, in the region* Ω*,*

$$L_m \le L(C) \le L_0\ ;\quad 0 \le -L'(C) \le p\ ;\quad 0 \le -s'(U) \le q.$$

If the following inequalities hold:

(5.2a) $$\left[\left(\frac{s_0 B^* p}{L_m{}^2} + k\alpha\right) + C_2\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \frac{C_2 s_0 \delta}{L_0},$$

(5.2b) $$\left[q + \left((1-k)\alpha C^* - \nu U^*\right)\right]^2 < \frac{s_0\phi}{L_0},$$

where

$$C_2 = \frac{(1-k)\alpha}{\pi\nu},$$

then E is globally asymptotically stable with respect to all solutions initiating in the interior of the region* Ω .

Proof: See the Appendix.

6. Numerical Example

To explain the applicability of the result we give here numerical simulation of the equillibria and the stability conditions for the model. We assume

(6.1a) $$s(U) = s_0 - \frac{a_1 U}{1 + s_1 U}, \qquad L(C) = L_0 - \frac{b_1 C}{1 + m_1 C}.$$

where

$$s_0 = 18,\ a_1 = 1,\ s_1 = 4,\ L_0 = 5.8,\ b_1 = 1,\ m_1 = 1.02.$$

We note from the above that

(6.1b) $$s'(U) = \frac{-a_1}{\left(1 + s_1 U\right)^2}, \qquad L'(C) = \frac{-b_1}{\left(1 + m_1 C\right)^2}.$$

Choosing p and q as 1.0 each and

$$\alpha = 0.01,\ \delta = 12,\ k = 0.5,\ Q = 12,\ \pi = 0.03,\ \nu = 0.03,$$
$$\phi = 14,\ \mu = 6,\ C_p = 0.6,\ \theta_1 = 60,\ \theta_0 = 0.16,\ \pi_1 = 0.08.$$

It can be checked that the interior equilibrium $E^*\left(B^*, C^*, U^*, F^*\right)$ of model (2.1) exists and to find these values using software Mathematica 5.2, we get the equilibrium values B^*, C^*, U^* and F^* are,

$$B^* = 5.425727784709277', C^* = 0.6021075760008402',$$

$$U^* = 0.0011533306030363205', \; F^* = 0.7903410003150668'.$$

It can be verified that all the conditions in Theorem (4.1) are satisfied for the above set of parameters and hence E^* is locally asymptotically stable.

We note from (6.1b) that if

$$\text{(6.1c)} \qquad \frac{-\partial s}{\partial U} = \frac{1}{\left(1+s_1 U\right)^2} \le 1 \quad , \quad \frac{-\partial L}{\partial C} = \frac{1}{\left(1+m_1 C\right)^2} \le 1 .$$

Along with the value of the parameters chosen above then it can be checked that all the conditions of Theorem (5.2) are satisfied and hence E^* is globally asymptotically stable.

Therefore if the biological species under consideration are initially growing with the intrinsic growth rate of s_0 units where $s_0 = 18$ units and if the intrinsic growth rate of these species is dependent on the concentration of the pollutant at the uptake phase as

$$s(U) = s_0 - \frac{a_1 U}{1 + s_1 U} \;,$$

where $a_1 = 1.0$, $s_1 = 4.0$ and if the carrying capacity of the biological species is dependent on the concentration of the pollutant in the environment as under:

$$L(C) = L_0 - \frac{b_1 C}{1 + m_1 C},$$

where $L_0 = 5.8$, $b_1 = 1.0$, $m_1 = 1.02$ where L_0 is the carrying capacity at the beginning when the environment is pollution free, which is 5.8 units, and if

$$\mathrm{L}_m(=1) \le L(C) \le L_0(=5.8) \;;\; 0 \le \text{-s}'(U) \le \mathrm{q}(=1);\; 0 \le -L'(C) \le p(=1),$$

i.e. rate of change of intrinsic growth rate of the biological species w.r.t. concentration at uptake phase and change of carrying capacity of the biological species w.r.t. concentration of the pollutants in the environment are both less than 1, and if the emission of the pollutants at the beginning is 12 units per unit time and per unit volume and when the concentration of the pollutants in the environment crosses the limit of 0.6, the control measure applicable/imposed on the pollution emitting industries and people then the stable equilibrium is achieved for which density of the biological species is 5.43 units, concentration of the pollutants in the environment is slightly more than 0.6 units, the concentration of the pollutants in the uptake phase of the species is 0.00115333 units and the control measure system applicable/imposed per unit volume are 0.79 units.

7. Computer Simulation

The dynamics of the model (2.1) is simulated by computer, using the different values of biological species and pollution in the environment as shown in Table- I and Table-II.

Table-1

F^*	Q	B^*	C^*	U^*
12	5.42573	0.602108	0.00115333	0.790341
24	5.42369	0.607412	0.00116306	2.77969
36	5.42166	0.612717	0.00117278	4.76903
48	5.41965	0.618022	0.0011825	6.75837
100	5.41108	0.64101	0.00122457	15.3789

Table -2

Q	μ	B^*	C^*	U^*	F^*
12	6	5.42573	0.602108	0.00115333	0.790341
24	12	5.42511	0.603716	0.00115628	1.39355
36	18	5.42490	0.604254	0.00115727	1.59534
48	24	5.42480	0.604524	0.00115776	1.69637
100	50	5.42464	0.604944	0.00115853	1.85416

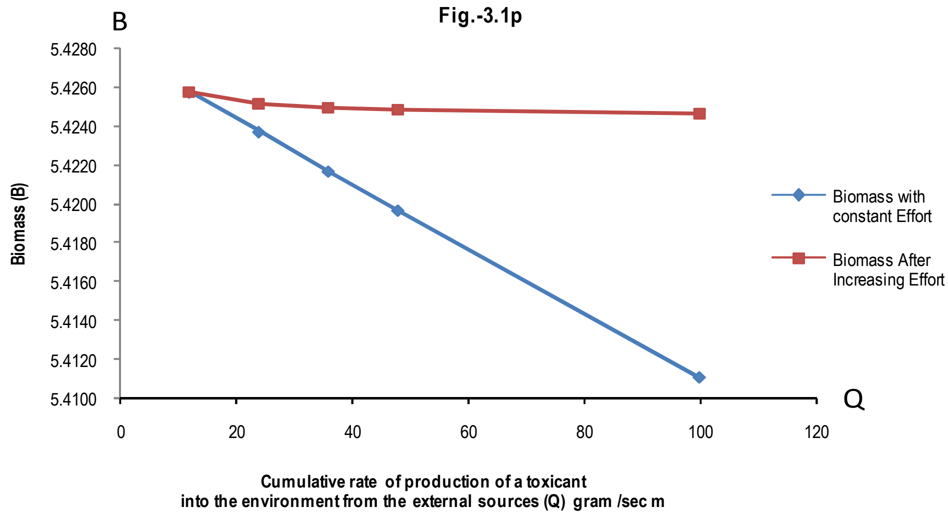

Figure 7.1

In figure (7.1) the cumulative rate of production of a toxicant into the environment from the external sources is plotted against the biological species, we find that at constant pollution control device($\mu = 6$) the biological species decrease rapidly and going to verge of extinction, but when pollution control device i.e. (μ) increases in the same ratio as Q, we see that biological species first decreases slowly at a certain level and then increases slowly to get desired level (i.e. nearly pollution free environment).

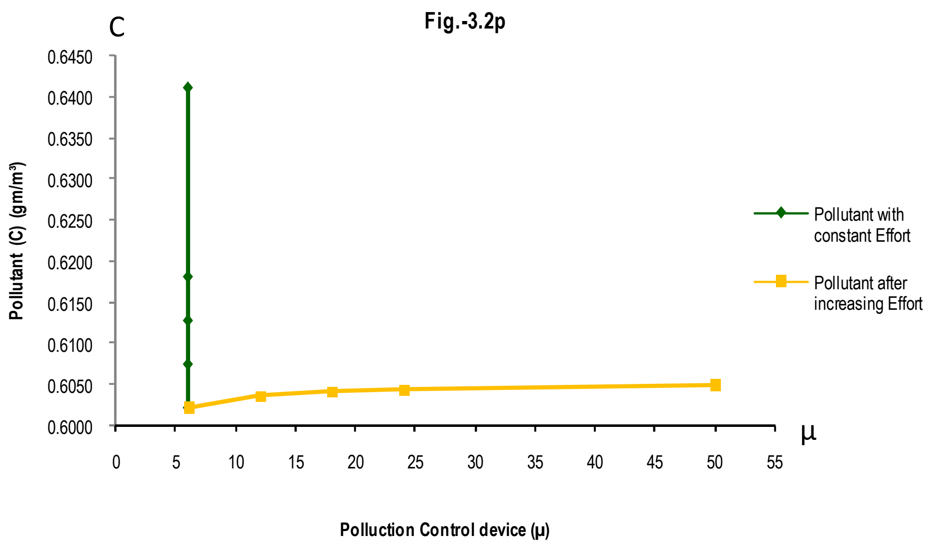

Figure7.2

In figure (7.2) the pollution control device is plotted against pollution in the environment, it is observed that at constant pollution device ($\mu = 6$) pollution is increasing due to cumulative rate of production is increasing in the environment (Q). But when pollution control device i.e. (μ) increases in the same ratio as the Q, it is observed that concentration of the pollution decreases as compare to pollution at constant efforts ($\mu = 6$), and after a certain level of efforts, pollution becomes steady. A comparative study is done by taking the value of removal coefficient $\mu = 6, 12, 18, 24, 50$. This gives that control measures (efforts) has a positive impact in the eco-system and the biological species may be saved from going to extinction.

Summary

In this paper, an ecological model has been proposed and analyzed to study the biological species which are directly affected by the pollutants emitted from the external sources. The existence and uniqueness of non-trivial equilibrium point has been discussed and its local and global stability behavior has been analyzed. Also, a region of attraction has been found for global asymptotic stability of the equilibrium point. It has been shown that the density of the biological species decreases with the increase in the total emission rate of pollutant in the environment. The analysis of the non linear stability shows that the system settles at much lower density of the biological species when the concentration of the pollutants in the environment and in the uptake phase of the species is high. It has been noted that the equilibrium level decreases as the toxicity and emission rates increases but with the increase of washout rates of the toxicants the equilibrium level is controlled to some extent from going down. It has been found that the control measures (carbon emitter taxes) imposed on the pollutant emitting industries and people, control concentration of the pollutants in the environment and due to this, the equilibrium point shifts in such a way that the density of the biological species is more near to the density when eco-system is pollution free. It has also been noted that toxicants emitted into the environment by population dependant human action are are not desirable and it must be controlled.

The conclusion drawn here suggests that emission of various kinds of toxicants in the environment must be controlled without further delay otherwise the survival of biological species will be threatened.

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Appendix

Proof of Theorem 4.1: First, we linearize the system (2.1) about E^* (B^*, C^*, U^*, F^*) by using the following transformations

$$B = B^* + b,\ C = C^* + c,\quad U = U^* + u,\quad F = F^* + f,$$

where b, c, u andf are small perturbations around $E^*$and on simplifying these equations, neglecting second order term of small perturbation, we get

$$(4.1c)\qquad \begin{cases} \dfrac{db}{dt} = -\dfrac{s_0 B^*}{L(C^*)} b + \left[\dfrac{s_0 B^{*2}\ \ L'(C^*)}{\left[L(C^*)\right]^2} - k\alpha B^* \right] c + \left(s'(U^*) B^* \right) u, \\ \dfrac{dc}{dt} = -\left(\alpha C^* - \pi\nu U^*\right) b - (\delta + \alpha B^*) c + \left(\pi\nu B^*\right) u - \mu f, \\ \dfrac{du}{dt} = \left((1-k)\alpha C^* - \nu U^*\right) b + \left((1-k)\alpha B^*\right) c - \left(\phi + \nu B^*\right) u, \\ \dfrac{df}{dt} = \theta_1 c - \theta_0 f. \end{cases}$$

Consider the positive definite function around E^*,

$$(4.1d)\qquad V = \frac{1}{2} C_1 \frac{b^2}{B^*} + \frac{1}{2} C_2 c^2 + \frac{1}{2} C_3 u^2 + \frac{1}{2} C_4 f^2$$

where C_1, C_2, C_3 and C_4 are positive constants, we can show that the derivative of V with respect to t along the linearized system(4.1c) is negative definite under the conditions of theorem(4.1).

Hence V is a Lyapunov function with respect to E^*, therefore E^* is locally asymptotically stable.

Proof of Theorem 5.2: Let us consider the positive definite function W around E^*

(5.2c) $$W(B,C,U,F)=C_1\left(B-B^*-B^*\log\frac{B}{B^*}\right)+\frac{1}{2}C_2(C-C^*)^2+\frac{1}{2}C_3(U-U^*)^2+\frac{1}{2}C_4(F-F^*)^2$$

where C_1, C_2 C_3 and C_4 are positive constants to be chosen such that it becomes a Lyapunov function, and its domain contains the region of attraction as defined by lemma(5.1).

On differentiating W with respect to t, we get

$$\dot{W}=C_1\left(\frac{B-B^*}{B}\right)\frac{dB}{dt}+C_2\left(C-C^*\right)\frac{dC}{dt}+C_3\left(U-U^*\right)\frac{dU}{dt}+C_4\left(F-F^*\right)\frac{dF}{dt}.$$

Substituting the values of $\dot{B},\dot{C},\dot{U},\dot{F}$ from (2.1), we have

$$\dot{W}=C_1\left(B-B^*\right)\left[s(U)-\frac{s_0B}{L(C)}-k\alpha C\right]+C_2\left(C-C^*\right)\left[Q-\delta C-\alpha BC+\pi\nu BU-\mu F\right]$$
$$+C_3\left(U-U^*\right)\left[(1-k)\alpha BC-\phi U-\nu UB\right]+C_4\left(F-F^*\right)\left[\theta_1\left(C-C_p\right)-\theta_0F\right].$$

After some algebraic manipulations, it can be written as

(5.2d) $$\begin{aligned}\dot{W}=&-C_1\frac{s_0}{L(C)}\left(B-B^*\right)^2-C_2\left(\delta+\alpha B\right)\left(C-C^*\right)^2\\&-C_3\left(\phi+\nu B\right)\left(U-U^*\right)^2-C_4\theta_0\left(F-F^*\right)^2\\&+\left(B-B^*\right)\left(C-C^*\right)\left[-C_1\left(s_0B^*\xi(C)+k\alpha\right)-C_2\left(\alpha C^*-\pi\nu U^*\right)\right]\\&+\left(B-B^*\right)\left(U-U^*\right)\left[C_1\eta(U)+C_3\left((1-k)\alpha C^*-\nu U^*\right)\right]\\&+\left(C-C^*\right)\left(U-U^*\right)\left[C_2\pi\nu B+C_3(1-k)\alpha B\right]\\&+\left(C-C^*\right)\left(F-F^*\right)\left[-\mu C_2+C_4\theta_1\right].\end{aligned}$$

where

(5.2e) $$\xi(\mathrm{C})=\begin{cases}\dfrac{\dfrac{1}{L(C)}-\dfrac{1}{L(C^*)}}{\mathrm{C}-\mathrm{C}^*} & ;\quad C\neq C^*\\[2ex] \dfrac{-L'\left(\mathrm{C}^*\right)}{\left[L\left(\mathrm{C}^*\right)\right]^2} & ;\quad C=C^*\end{cases},$$

$$\eta(\mathrm{U}) = \begin{cases} \dfrac{s(\mathrm{U}) - s(\mathrm{U}^*)}{(\mathrm{U} - \mathrm{U}^*)} & ; \quad \mathrm{U} \neq \mathrm{U}^* \\ s'(\mathrm{U}^*) & ; \quad \mathrm{U} = \mathrm{U}^* \end{cases},$$

For $\dot{W}$ to be negative definite the following inequalities must be satisfied :

(5.2f) $$\left[C_1\left(s_0 B^* \xi(C) + k\alpha\right) + C_2\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < 4\,\frac{1}{2}\,C_1 \frac{C_2 s_0}{L(C)}\left(\frac{\delta}{2}\right),$$

(5.2g) $$\left[C_1 \eta(U) + C_3\left((1-k)\alpha C^* - \nu U^*\right)\right]^2 < 4\,\frac{1}{2} C_1 \frac{C_3 s_0}{L(C)}\left(\frac{\phi}{2}\right), \quad ,$$

(5.2h) $$\left[C_2 \pi\nu B + C_3(1-k)\alpha B\right]^2 < 4C_2 C_3(\alpha B)\left(\frac{\phi}{2} + \nu B\right),$$

(5.2i) $$\left[-\mu C_2 + C_4\theta_1\right]^2 < 4C_2\left(\frac{\delta}{2}\right)C_4\theta_0\,,$$

since

$$\mathrm{L_m} \le \mathrm{L(C)} \le \mathrm{L_0}\,; \quad 0 \le -\mathrm{s'(U)} \le \mathrm{q} \quad ; \; 0 \le -\mathrm{L'(C)} \le \mathrm{p}.$$

From the mean value theorem,

(5.2j) $$\begin{cases} \left|\xi(\mathrm{C})\right| \le \dfrac{\mathrm{p}}{\mathrm{L_m}^2}, \\ \left|\eta(\mathrm{U})\right| \le \mathrm{q}. \end{cases}$$

for some positive constant L_m, p and q in the region Ω.

Keeping the maximum values of $\xi(C)$ and $\eta(U)$ on left hand sides from equation (5.2j) and rewriting the above inequalities as

(5.2k) $$\left[C_1\left(s_0 B^* \frac{p}{L_m^{\,2}} + k\alpha\right) + C_2\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \frac{C_1 C_2 s_0 \delta}{L(C)},$$

(5.2l) $$\left[C_1 q + C_3\left((1-k)\alpha C^* - \nu U^*\right)\right]^2 < \frac{C_1 C_3 s_0 \phi}{L(C)},$$

(5.2m) $$\left[C_2 \pi\nu B + C_3(1-k)\alpha B\right]^2 < 4C_2 C_3(\alpha B)\left(\frac{\phi}{2} + \nu B\right),$$

(5.2n) $$\left[-\mu C_2 + C_4\theta_1\right]^2 < 2C_2\delta C_4\theta_0 .$$

We rewrite the inequality (5.2m) as

$$\left[C_2\pi\nu B - C_3\left(1-k\right)\alpha B\right]^2 + 4C_2C_3\pi\nu\left(1-k\right)\alpha B^2 < 4C_2C_3\left(\alpha B\right)\left(\frac{\phi}{2}+\nu B\right).$$

In order to reduce the above inequality, we choose the value of constants

(5.2o) $$\begin{cases} C_2 = \dfrac{(1-k)\alpha}{\pi\nu} , & (\nu \neq 0) \\ C_3 = 1 . \end{cases}$$

Also we take $C_1 = 1$.

If we choose $C_4 = \dfrac{\mu C_2}{\theta_1}$ then (5.2n) will automatically be satisfied. This has been mentioned while discussing the local stability.

Further on keeping the minimum value of the variables on right hand sides, we get the remaining inequalities.

$$\left[\left(s_0 B^* \frac{p}{L_m{}^2} + k\alpha\right) + C_2\left(\alpha C^* - \pi\nu U^*\right)\right]^2 < \frac{C_2 s_0 \delta}{L_0},$$

$$\left[q + \left(\left(1-k\right)\alpha C^* - \nu U^*\right)\right]^2 < \frac{s_o\phi}{L_0},$$

where

$$C_2 = \frac{(1-k)\alpha}{\pi\nu} .$$

Which are same as mentioned in the Theorem 5.2. Hence W is Lyapunov function with respect to E^* whose domain contains Ω and therefore E^* is non-linearly stable and hence the theorem.

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