ISSN 0974 - 9373

Vol. 24 No. 2 (2020) **Journal of International Academy of Physical Sciences** pp. 115- 127

Common Fixed Point Theorems on Cone b-Metric Spaces

Yumnam Rohen

Department of Mathematics
National Institute of Technology Manipur, Langol, Imphal-795004, India
Email: ymnehor2008@yahoo.com

(Received March 30, 2020)

Abstract: In this note we prove some common fixed point theorems by using commuting and non-commuting mappings in cone b-metric spaces. We consider a more generalised inequality in order to establish our problems and to generalise earlier results available in the literature.
Keywords: Weakly compatible mappings, Compatible mappings, Common fixed point, Cone b-metric space.
2000 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The study on fixed point of weakly commuting mappings had started as early as 1982. Since the introduction of weakly commuting mappings by S. Sessa[1] as generalisation of commuting mappings. A quantum of research work follows the work of Sessa by generalisation and extension. Among them the most important work is the introduction of compatible mappings by G. Jungck[2] as generalisation of commuting mappings. Compatible mappings are more generalised than commuting and weakly commuting mappings. Huang and Zhang[3] introduced the concept of cone metric space as generalisation of metric space, replacing the set of real numbers in metric space by an ordered Banach space. They also obtained some fixed point theorems in this space for mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck[4], Abbas and Rhoades[5] studied common fixed point theorems in cone metric spaces. Vetro[6] proved some fixed point theorem for two self mappings satisfying a contractive condition through weak compatibility in a normal cone metric space. Amit Singh, R. C. Dimri and Sandeep Bhatt[7] proved a unique common fixed point theorem for four weakly compatible self mappings in complete cone metric spaces without using the notion of continuity. Rezapour and Hamlbarani[8] omit the assumption of normality in cone metric space making another dimension in developing fixed point theory in cone metric space. Non normal cone metric space is found to be used in the research papers[9,10]. Johnson Olaleru[11] proved common fixed point theorems of three self mappings in non normal cone metric space. Jain et al.[12,13] also proved some fixed point theorems without using the normality of a cone metric space. Hui-Sheng Ding and Lu Li[14] proved some results in partially ordered metric space. Hussain and Shah[15] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces. For more results on different types of b-metric space on can see research papers in[16-18].

In this paper, we prove some common fixed point theorems by using commuting and non-commuting mappings in cone b-metric spaces.

2. Preliminaries

Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if

(i) P is closed, non-empty and $P \neq 0$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \bigcap (-P) = 0$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{Int}.P$, where $\text{Int}.P$ denotes the interior of the set P.

Definition[3] 2.1: *Let X be a non-empty set. Suppose that the mapping $d: X \times X \to E$ satisfies:*

(a) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition[15] 2.2: *Let X be a non-empty set and $b \geq 1$ be a real number. Suppose that the mapping $d: X \times X \to E$ satisfies:*

(a) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$ *;*
(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *;*
(c) $d(x, y) \leq b\left[d(x, z) + d(z, y)\right]$ *for all* $x, y, z \in X$ *.*
Then d *is called a cone* $b-$*metric on* X *and* (X, d) *is called a cone b-metric space.*

Definition[15] 2.3: *Let* (X, d) *be a cone b-metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$ *.*
(i) $\{x_n\}$ *converges to x if for every* $c \in E$ *with* $0 << c$*, there is an integer* N_c *such that for all* $n > N_c$*,* $d(x_n, x) << c$*. We denote this by* $\lim_{n \to \infty} x_n = x$ *or* $x_n \to x$*, (*$n \to \infty$*).*
(ii) If for any $c \in E$ *with* $0 << c$*, there is an integer* N *such that for all* $n, m \geq N$*,* $d(x_n, x_m) << c$*, then* $\{x_n\}$ *is called a Cauchy sequence in* X *.*
(iii) If every Cauchy sequence in X *is convergent in* X*, then* X *is called a complete cone* $b-$*metric space.*

Definition[10] 2.4: *Let* X *be any set. A pair of self maps* (A, S) *in* X *is said to be weakly compatible if* $u \in X$ *,* $Au = Su$ *imply* $SAu = ASu$*.*

Definition[12] 2.5: *Let* (X, d) *be a cone b-metric space. A pair of self maps* (A, S) *in* X *is said to be compatible if for* $\{x_n\}$ *in* X*,* $Ax_n \to u$ *and* $Sx_n \to u$*, for some* $u \in X$*, then for every* $c \in P_0$*, there is a positive integer* N_c *such that* $d(ASx_n, SAx_n) << c$*, for all* $n > N_c$*.*

Proposition[12] 2.6: *In a cone* $b-$*metric space every commuting pair of self maps is compatible.*

Proposition[12] 2.7: *In a cone* $b-$*metric space every compatible pair of self maps is weakly compatible.*

3. Main Results

We prove the following theorems.

Theorem 3.1: *Let* (X, d) *be a complete cone* $b-$*metric space and* P *a normal cone with normal constant* k*. Suppose that the mapping* T*, from* X *into itself satisfy the condition,* *(3.1)* $$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

for all $x, y \in X$ *and* $a_1, a_2, a_3, a_4, a_5 \ge 0$ *such that* $0 \le a_1 + a_2 + a_3 + 2ba_4 + a_5 < 1$. *Then* T *has unique fixed point in* X *.*

Proof: Let x_0 be an arbitrary element in X. Let us choose $x_1, x_2 \in X$ such that $Tx_0 = x_1$ and $Tx_1 = x_2$. In general we can define a sequence of elements of X such that

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}.$$

Now,

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}).$$

From (3.1)

$$d(Tx_{2n}, Tx_{2n+1}) \le a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, Tx_{2n}) + a_3 d(x_{2n+1}, Tx_{2n+1}) + a_4 d(x_{2n}, Tx_{2n+1}) + a_5 d(x_{2n+1}, Tx_{2n}),$$

$$d(x_{2n+1}, x_{2n+2}) \le a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) + a_4 d(x_{2n}, x_{2n+2}) + a_5 d(x_{2n+1}, x_{2n+1}),$$

$$d(x_{2n+1}, x_{2n+2}) \le a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) + a_4 d(x_{2n}, x_{2n+2}).$$

By using triangle inequality, we get

$$d(x_{2n+1}, x_{2n+2}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)) d(x_{2n}, x_{2n+1}).$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)) d(x_{2n-1}, x_{2n}).$$

In general we can show that,

$$d(x_{2n+1}, x_{2n+2}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4))^{2n+1} d(x_0, x_1).$$

On taking $\left(\left(a_1+\ a_2+\ ba_4\right)/\left(1-a_3-ba_4\right)\right)=\ h$

$$d\left(x_{2n+1},\ x_{2n+2}\right)\leq\ h^{2n+1}d\left(x_0,\ x_1\right).$$

For $n\leq m$, we have

$$d\left(x_{2n},\ x_{2m}\right)\leq bd\left(x_{2n},\ x_{2n+1}\right)+b^2d\left(x_{2n+1},\ x_{2n+2}\right)+....+b^{2(m-n)}d\left(x_{2m-1},\ x_{2m}\right),$$

$$d\left(x_{2n},\ x_{2m}\right)\leq\left\{bh^n+b^2h^{n+1}+\ b^3h^{n+2}+....+b^{2(m-n)}h^m\right\}d\left(x_0,\ x_1\right),$$

$$d\left(x_{2n},\ x_{2m}\right)\leq\left(h^nb/\left(1-bh\right)\right)d\left(x_0,\ x_1\right),$$

$$\left\|d\left(x_{2n},\ x_{2m}\right)\right\|\leq\left(h^nb/\left(1-bh\right)\right)\left\|d\left(x_0,\ x_1\right)\right\| \text{ as } n\ \to\infty,$$

$$lim_{n\to\infty}\left\|d\left(x_{2n},\ x_{2m}\right)\right\|\to 0.$$

In this way

$$lim\ n\to\infty\ d\left(x_{2n},\ x_{2m}\right)\to 0 \text{ as } n\ \to\infty.$$

Hence $\{x_n\}$ is Cauchy sequence which converges to $u\in X$. Since (X,d) is complete cone $b-$metric space. Thus $x_n\to u$ as $n\to\infty$, $Tx_{2n}\to u$ and $Tx_{2n+1}\to u$ as $n\to\infty$, u is fixed point of T in X. In order to prove the uniqueness of fixed point let v be another fixed point of T in X different from u. Then,

$$Tu=u \text{ and } Tv=v,$$

$$d\left(u,v\right)=d\left(Tu,Tv\right).$$

From (3.1)

$$d\left(Tu,\ Tv\right)\leq\ a_1d\left(u,\ v\right)+a_2d\left(u,\ Tu\right)+a_3d\left(v,\ Tv\right)+a_4d\left(u,\ Tv\right)+a_5d\left(v,\ Tu\right),$$

$$d\left(Tu,\ Tv\right)\leq\left(a_1+\ a_4+\ a_5\right)d\left(u,\ v\right).$$

which is a contradiction and hence u is unique fixed point of T in X.

Theorem 3.2: *Let (X, d) be a complete cone b-metric space and P a normal cone with normal constant k. Suppose that S and T be the mappings from X into itself satisfies the condition,*

(3.2)
$$d(Sx,\ Ty)\le a_1 d(x,\ y)+a_2 d(x,\ Sx)+a_3 d(y,\ Ty)$$
$$+a_4 d(x,\ Ty)+a_5 d(y,\ Sx),$$

for all $x,\ y\in X$ *and* $a_1,\ a_2,\ a_3,\ a_4,\ a_5\ge 0$ *such that* $0\le a_1+a_2+a_3+2ba_4+a_5$ <1. *Then* S *and* T *have unique fixed point in* X *. Further more if* $ST=TS$ *, then* S *and* T *have unique common fixed point in* X *.*

Proof: Let x_0 be an arbitrary element in X. Let us choose $x_1,\ x_2\in X$ such that $Sx_0=x_1$ and $Tx_1=x_2$. In general we can define a sequence of elements of X such that

$$x_{2n+1}=Sx_{2n} \quad \text{and} \quad x_{2n+2}=Tx_{2n+1}.$$

Now,
$$d(x_{2n+1},\ x_{2n+2})=d(Sx_{2n},\ Tx_{2n+1}).$$

From (3.2)

$$d(Sx_{2n},\ Tx_{2n+1})\le a_1 d(x_{2n},\ x_{2n+1})+a_2 d(x_{2n},\ Sx_{2n})+a_3 d(x_{2n+1},\ Tx_{2n+1})$$
$$+a_4 d(x_{2n},\ Tx_{2n+1})+\ a_5 d(x_{2n+1},\ Sx_{2n}),$$

$$d(x_{2n+1},\ x_{2n+2})\le a_1 d(x_{2n},\ x_{2n+1})+a_2 d(x_{2n},\ x_{2n+1})+a_3 d(x_{2n+1},\ x_{2n+2})$$
$$+a_4 d(x_{2n},\ x_{2n+2})+\ a_5 d(x_{2n+1},\ x_{2n+1}),$$

$$d(x_{2n+1},\ x_{2n+2})\le a_1 d(x_{2n},\ x_{2n+1})+a_2 d(x_{2n},\ x_{2n+1})+a_3 d(x_{2n+1},\ x_{2n+2})$$
$$+a_4 d(x_{2n},\ x_{2n+2}).$$

By using triangle inequality, we get

$$d(x_{2n+1},\ x_{2n+2})\le\left((a_1+\ a_2+\ ba_4)/(1-a_3-ba_4)\right)d(x_{2n},\ x_{2n+1}).$$

Similarly we can show that,

$$d\left(x_{2n}, x_{2n+1}\right) \le \left(\left(a_1 + a_2 + ba_4\right) / \left(1 - a_3 - ba_4\right)\right)\left(x_{2n-1}, x_{2n}\right).$$

In general we can show that,

$$d\left(x_{2n+1}, x_{2n+2}\right) \le \left(\left(a_1 + a_2 + ba_4\right) / \left(1 - a_3 - ba_4\right)\right)^{2n+1} d\left(x_0, x_1\right).$$

On taking $\left(\left(a_1 + a_2 + ba_4\right) / \left(1 - a_3 - ba_4\right)\right) = h$

$$d\left(x_{2n+1}, x_{2n+2}\right) \le h^{2n+1} d\left(x_0, x_1\right).$$

For $n \le m$, we have

$$d\left(x_{2n}, x_{2m}\right) \le bd\left(x_{2n}, x_{2n+1}\right) + b^2 d\left(x_{2n+1}, x_{2n+2}\right) + \dots + b^{2(m-n)} d\left(x_{2m-1}, x_{2m}\right),$$

$$d\left(x_{2n}, x_{2m}\right) \le \left\{bh^n + b^2 h^{n+1} + b^3 h^{n+2} + \dots + b^{2(m-n)} h^m\right\} d\left(x_0, x_1\right),$$

$$d\left(x_{2n}, x_{2m}\right) \le \left(h^n b / \left(1 - bh\right)\right) d\left(x_0, x_1\right),$$

$$\left\| d\left(x_{2n}, x_{2m}\right) \right\| \le \left(h^n b / \left(1 - bh\right)\right) \left\| d\left(x_0, x_1\right) \right\| \text{ as } n \to \infty,$$

$$lim_{n \to \infty} \left\| d\left(x_{2n}, x_{2m}\right) \right\| \to 0.$$

In this way

$$lim\ n \to \infty\ d\left(x_{2n}, x_{2m}\right) \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ is Cauchy sequence which converges to $u \in X$. Since (X, d) is complete cone b-metric space. Thus $x_n \to u$ as $n \to \infty$, $Sx_n \to u$ and $Tx_{2n+1} \to u$ as $n \to \infty$, u is fixed point of S and T in X. Since $ST = TS$, this give

$$u = Tu = TSu = STu = Su = u.$$

u is common fixed point of S and T. In order to prove the uniqueness of fixed point let v be another fixed point of S and T in X different from u. Then,

$$Tu = u \quad \text{and} \quad Tv = v.$$

Also,

$$Su = u \text{ and } Sv = v,$$

d(u, v) = d(Tu, Tv) $d(u, v) = d(Tu, Tv)$.

From (3.2)

$$d(Su,\ Tv) \leq\ a_1 d(u,\ v) + a_2 d(u,\ Su) + a_3 d(v,\ Tv) + a_4 d(u,\ Tv) + a_5 d(v,\ Su),$$

$$d(Su,\ Tv)\ \leq (a_1 +\ a_4 +\ a_5) d(u,\ v).$$

Which is a contradiction, hence u is unique fixed point of S and T in X.

Theorem 3.3: *Let (X, d) be a complete cone $b-$metric space and P a normal cone with normal constant k. Suppose that S, R and T be the mappings from X into itself satisfies the condition,*

(3.3) $$d(SRx,\ TRy) \leq a_1 d(x,\ y) + a_2 d(x,\ SRx) + a_3 d(y,\ TRy)$$
$$+ a_4 d(x,\ TRy) + a_5 d(y,\ SRx),$$

for all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ such that $0 \leq a_1 + a_2 + a_3 + 2ba_4 + a_5$ <1. Then S, R and T have unique fixed point in X. Furthermore either $SR = RS$ or $TR = RT$, then S, R and T have unique common fixed point in X.

Proof: Let x_0 be an arbitrary element in X. Let us choose $x_1, x_2 \in X$ such that $SRx_0 = x_1$ and $TRx_1 = x_2$. In general we can define a sequence of elements of X such that

$$x_{2n+1} = SRx_{2n} \quad \text{and} \quad x_{2n+2} =\ TRx_{2n+1}.$$

Now,

$$d(x_{2n+1},\ x_{2n+2}) = d(SRx_{2n},\ TRx_{2n+1}).$$

From (3.3)

$$(SRx_{2n},\ TRx_{2n+1}) \leq a_1 d(x_{2n},\ x_{2n+1}) + a_2 d(x_{2n},\ SRx_{2n}) + a_3 d(x_{2n+1},\ TRx_{2n+1})$$

$$+a_4 d(x_{2n}, TRx_{2n+1}) + a_5 d(x_{2n+1}, SRx_{2n}),$$

$$d(x_{2n+1}, x_{2n+2}) \le a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2})$$
$$+a_4 d(x_{2n}, x_{2n+2}) + a_5 d(x_{2n+1}, x_{2n+1}),$$

$$d(x_{2n+1}, x_{2n+2}) \le a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1})$$
$$+a_3 d(x_{2n+1}, x_{2n+2}) + a_4 d(x_{2n}, x_{2n+2}).$$

By using triangle inequality, we get

$$d(x_{2n+1}, x_{2n+2}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)) d(x_{2n}, x_{2n+1}).$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4))(x_{2n-1}, x_{2n}).$$

In general we can show that,

$$d(x_{2n+1}, x_{2n+2}) \le ((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4))^{2n+1} d(x_0, x_1).$$

On taking $((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)) = h$

$$d(x_{2n+1}, x_{2n+2}) \le h^{2n+1} d(x_0, x_1).$$

For $n \le m$, we have

$$d(x_{2n}, x_{2m}) \le bd(x_{2n}, x_{2n+1}) + b^2 d(x_{2n+1}, x_{2n+2}) + \ldots + b^{2(m-n)} d(x_{2m-1}, x_{2m}),$$

$$d(x_{2n}, x_{2m}) \le \{bh^n + b^2 h^{n+1} + b^3 h^{n+2} + \ldots + b^{2(m-n)} h^m\} d(x_0, x_1),$$

$$d(x_{2n}, x_{2m}) \le (h^n b/(1 - bh)) d(x_0, x_1),$$

$$\|d(x_{2n}, x_{2m})\| \le (h^n b/(1 - bh)) \|d(x_0, x_1)\| \text{ as } n \to \infty,$$

$$lim_{n\to\infty} \|d(x_{2n}, x_{2m})\| \to 0.$$

In this way

$$\lim n \to \infty \, d\left(x_{2n},\, x_{2m}\right) \to 0 \text{ as } n \to \infty .$$

Hence $\{x_n\}$ is Cauchy sequence which converges to $u \in X$. Since (X, d) is complete cone $b-$metric space. Thus $x_n \to u$ as $n \to \infty$, $SRx_n \to u$ and $TRx_{2n+1} \to u$ as $n \to \infty$, u is fixed point of S and T in X . Since $ST = TS$, this give $u = Tu = TSu = STu = Su = u$ is common fixed point of S and T .
In order to prove the uniqueness of fixed point let v be another fixed point of S and T in X different from u . Then,

$$Tu = u \text{ and } Tv = v .$$

Also,

$$Su = u \text{ and } Sv = v ,$$

$$d(u,\, v) = d(Su,\, Tv) .$$

From (4.5.3)

$$d(Su,\, Tv) \le a_1 d(u,\, v) + a_2 d(u,\, Su) + a_3 d(v,\, Tv) + a_4 d(u,\, Tv) + a_5 d(v,\, Su),$$

$$d(Su,\, Tv) \le (a_1 + a_4 + a_5) d(u, v) .$$

Which is a contradiction, hence u is unique fixed point of S and T in X .

Theorem 3.4: *Let (X, d) be a complete cone $b-$metric space and P a normal cone with normal constant k . Suppose that A, B, S and T be the mappings from X into itself satisfies the condition,*

(i) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$
(ii) $\{A, S\}$ *and* $\{B, T\}$ *are weakly compatible*
(iii) S *or* T *is continuous*
(iv) $d(Ax,\, By) \le a_1 d(Sx,\, Ty) + a_2 d(Sx,\, Ax) + a_3 d(Ty,\, By)$

$$+ a_4 d(Sx,\, By) + a_5 d(Ty,\, Ax)$$

for all $x, y \in X$ *and* $a_1, a_2, a_3, a_4, a_5 \ge 0$ *such that* $0 \le a_1 + a_2 + a_3 + 2ba_4 + a_5 < 1$. *Then* A, B, S *and* T *have a unique fixed point in* X .

Proof: For any arbitrary x_0 in X, we define sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for all $n = 0, 1, 2, ...$ $SRx_0 = x_1$ and $TRx_1 = x_2$. In general we can define a sequence of elements of X such that

$$x_{2n+1} = SRx_{2n} \quad \text{and} \quad x_{2n+2} = TRx_{2n+1}.$$

Now,

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}).$$

From (iv)

$$d(Ax_{2n}, Bx_{2n+1}) \leq a_1 d(Sx_{2n}, Tx_{2n+1}) + a_2 d(Sx_{2n}, Ax_{2n}) + a_3 d(Tx_{2n+1}, Bx_{2n+1})$$
$$+ a_4 d(Sx_{2n}, Bx_{2n+1}) + a_5 d(Tx_{2n+1}, Ax_{2n}),$$

$$d(y_{2n}, y_{2n+1}) \leq a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n}, y_{2n+1})$$
$$+ a_4 d(y_{2n-1}, y_{2n+1}),$$

$$d(y_{2n}, y_{2n+1}) \leq \left((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)\right)^{2n+1} d(y_0, y_1).$$

On taking $\left((a_1 + a_2 + ba_4)/(1 - a_3 - ba_4)\right) = h$ and for $n \leq m$, we have

$$d(y_{2n}, y_{2m}) \leq \left\{bh^n + b^2 h^{n+1} + b^3 h^{n+2} + ... + b^{2(m-n)} h^m\right\} d(y_0, y_1),$$

$$d(y_{2n}, y_{2m}) \leq \left(h^n b/(1 - bh)\right)(y_0, y_1),$$

$$\left\|d(y_{2n}, y_{2m})\right\| \leq \left(h^n b/(1 - bh)\right)\left\|d(y_0, y_1)\right\| \text{ as } n \to \infty,$$

$$lim_{n\to\infty}\left\|d(y_{2n}, y_{2m})\right\| \to 0.$$

Hence $\{y_n\}$ is Cauchy sequence which converges to $u \in X$. By the continuity of S and T, $\{x_n\}$ is also convergent sequence which converges to $u \in X$. Hence (X, d) is complete cone $b-$metric space. Hence u is fixed point of A, B, S and T. Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible, implies that u is common fixed point of A, B, S and T. In order to prove the uniqueness of fixed point let v be another fixed point of A, B, S and T in X different from u. Then,

$$Au = u \text{ and } Av = v.$$

Also, $Bu = u$ and $Bv = v$.

$$d(u, v) = d(Au, Bv).$$

From (iv)

$$d(Au, Bv) \leq a_1 d(Su, Tv) + a_2 d(Su, Au) + a_3 d(Tv, Bv) + a_4 d(Su, Bv) + a_5 d(Tv, Au),$$

$$d(Au, Bv) \leq (a_1 + a_4 + a_5) d(u, v).$$

Which is a contradiction and hence u is unique fixed point of A, B, S and T in X.

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