ISSN 0974 - 9373

Vol. 14 No.2 (2010) **Journal of International Academy of Physical Sciences** pp. 251-264

An Unified Analytic Study of the Nucleus of Polytropic Stars*

J. P. Sharma, Sanish Thomas and W. David
Department of physics
College of Basic Sciences, Allahabad Agricultural Institute
Deemed University (AAI-DU), Allahabad, (U. P.) 211007
Email: sanishthomas1@gmail.com

(Received February 20, 2010)

Abstract : An aspect of nucleus of polytropes has been analytical studied under the concept of a sphere of uniform density defined by the polytrophic indices n tending to zero $(n \to 0)$ and to minus one $(n \to -1)$.The generalized form of the structure equation (2.9), (2.10) and (2.11) in (u_p, v_p), (u_ρ, v_ρ) and (u_θ, v_θ), respectively, represent the plane symmetric (*N=0*), cylindrical (*N=1*) and spherical (*N=2*) configurations. It describes the nucleus of polytropic stars incorporating the solutions for $n \to 0$ and $n \to -1$, and extents of the immediate neighborhood of the origin, (eqs. (2.13), (3.9) and (3.10)). And, more interestingly, the geometrical size of n=2 configuration is found to be $\xi_{p_1} = 3.86789962 \left(\gamma = 4/3\right)$, $3.39601184 \left(\gamma = 5/3\right)$ and 3.202300313 $(\gamma = 2)$ (eqn. (3.14)).

Keywords: Stellar structure/ polytropic stars / unified study / uniform density model / analytic study

1. Introduction

The study of self-gravitating polytropic and isothermal spheres, cylinders and sheets, in static equilibrium, under the concept of Newtonian gravitation theory (classical) dates back to the Emden's[1], Eddington's [2], Milne's[3], and Chandresekhar's[4] well known researches. The self-gravitating polytropic and isothermal sheets, such as, the Saturn ring system and the Laplacian-disc cosmogonies, have been considered by Spitzer[5], Ledoux[6], and Gold Reich and Lynden-Bell[7]. Some new ideas to the study of polytropic, and isothermal cylinders have been lately added by Ostriker [8, 9] and Srivastava[10]. Harrison and Lake[11] and the author [12] tackled the problem of polytropic and isothermal plane-symmetric configurations from viewpoint of the ease with which the equations of hydrostatic equilibrium could be solved.

*Presented at CONIAPS XI, University of Allahabad, Feb. 20-22, 2010.

The behavior of solutions of the Lane-Emden equations of index n, which controls the distribution of physical variables, has been studied in the immediate neighborhood of the origin by Hopf[13], Fowler[14] and Chandrasekhar for n<3, n=3, and n>3, respectively. It is well understood so far from some of these studies that the polytropic indices n=0 and 1 represent , respectively, the liquid and gaseous states of a polytrope of uniform density and governing the origin. And the behavior of solutions is the same whatever be the index of a polytrope. Since the nucleus includes the immediate neighborhood of the origin, it will be of our interest to study it in somewhat more detail. In the light of this new concept of uniform density $n \to 0$ and $n \to -1$, the author12 has been able to determine the maximum limiting density[15] and the maximum value of mass of a star[16]. He [17] has further extended the Lyttleton's work[18] pertaining to the structure of a planet.

Srivastava[10] studied the structure of a cylindrical polytrope and has shown that the natural variables "r" and "P" are, however, most suitable for studying the structure. Some valuable contributions have further been made by the author and others towards the discussion of classical (Newtonian) as well as relativistic aspects of problems on polytropic structures., for example, under Newtonian (classical) treatment- (i) Lane-Emden equation of index 5 and Pade`approximants[19] , (ii) transformation of the generalized isothermal Lane- Emden equation and Pade` approximation[20] , (iii) planetary structures: an analytical approach[21] , (iv) self- consistent fields in atoms[22], (v)structure of uniformly rotating isothermal gas cylinder[23] , (vi) the equilibrium of uniform slowly rotating polytropes[24], (vii) thermodynamic equilibrium of star clusters[25] , (viii) transformation of the equations of equilibrium for isothermal plane-symmetric configuration [26] ; and under relativistic treatment- (ix) structure of white dwarfs[27] , (x) neutron star models[28] , (xi)relativistic features of stellar structures[29], and (xii) relativistic non –rotating isothermal gas cylinders[30].These references contain a fair proportion of new materials. And, the above mentioned recent works bring out clearly the application and importance of Pade/(2,2) approximation technique in solving the differential equations (ordinary or partial) analytically which are of astrophysical importance.

2. Generalized Structure Equations in Different Planes

Our generalized form of the fundamental equation of hydrostatic equilibrium is given by

(2.1) $$\frac{1}{r^N}\frac{d}{dr}\left(\frac{r^N}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho,$$

where N is constant and it takes the values 0, 1, and 2 respectively, for plane-symmetric, cylindrical and spherical configurations and other symbols have their usual meanings. Define the variables ρ (density) and P (pressure) by

(2.2) $$P = \lambda\theta^n,\ P \equiv K\rho^{1+\frac{1}{n}} = K\lambda^{1+\frac{1}{n}} = K\lambda^{1+\frac{1}{n}}\theta^{n+1},$$

where K and n are constants; n is usually called the polytropic index; K depends on the entropy per nucleon and chemical composition but does not depend on radius or on central density and λ defines the central density atθ=1. Equation (2.1) can be expressed in (ξp , P) ,-(ξρ ,ρ) - and (ξθ ,θ) - planes in the form

(2.3) $$\frac{1}{\xi_P^N}\frac{d}{d\xi_P}\left(\xi_P^N P^{-\frac{n}{n+1}}\frac{dP}{d\xi_P}\right) = -P^{\frac{n}{n+1}},$$

(2.4) $$\frac{1}{\xi_\rho^N}\frac{d}{d\xi_\rho}\left(\xi_\rho^N \rho^{\frac{1-n}{n}}\frac{d\rho}{d\xi_\rho}\right) = -\rho,$$

(2.5) $$\frac{1}{\xi_\theta^N}\frac{d}{d\xi_\theta}\left(\xi\theta^N\frac{d\theta}{d\xi_\theta}\right) = -\theta^n,$$

respectively which are all equivalent to equation (2.1),
where the dimensionless variables ξP, ξρ, and ξθ are connected with the natural variable r by

(2.6) $$r = \alpha_P\xi_P;\quad \alpha_P = \left[\frac{K^{\frac{2n}{n+1}}}{4\pi G}\right]^{\frac{1}{2}},$$

(2.7) $$r = \alpha_\rho\xi_\rho;\quad \alpha_\rho = \left[\frac{K(n+1)}{4\pi Gn}\right]^{\frac{1}{2}},$$

(2.8) $$r=\alpha_\theta\xi_\theta;\quad \alpha_\theta=\left[\frac{K(n+1)\lambda^{\frac{1}{n}-1}}{4\pi G}\right]^{\frac{1}{2}},$$

respectively, the suffixes "P", "ρ", and "θ" attached with the physical variables mean the variables in which the fundamental equation of hydrostatic equilibrium (2.1) has been expressed. In view of the homology theorem1, it is possible to derive the following first-order differential equations equivalent to the generalized LEE in (u_p,v_p)–,(u_ρ,v_ρ)-and (u_θ,v_θ)-planes:

(2.9) $$\frac{u_P}{v_P}\frac{dv_P}{du_P}=\frac{(n+1)u_P+v_P+(n+1)}{(1+N)(n+1)-(n+1)u_P-nv},$$

(2.10) $$\frac{u_\rho}{v_\rho}\frac{dv_\rho}{du_\rho}=\frac{n(1-N)+v_\rho+nu_\rho}{n((N+1)-v_\rho-u_\rho)},$$

and

(2.11) $$\frac{u_\theta}{v_\theta}\frac{dv_\theta}{du_\theta}=\frac{1-N+u_\theta+v_\theta}{1+N-u_\theta-nv_\theta}.$$

The locus of points at which the solution-curves have horizontal tangents is given by

(2.12) $$(\mathrm{n}+1)u_\mathrm{P}+v_\mathrm{P}=(n+1)(n-1).$$

The locus of points at which the solution-curves have vertical tangents is given by

(2.13) $$(\mathrm{n}+1)u_\mathrm{P}+\mathrm{n}\,v_\mathrm{P}=(1+\mathrm{N})\,(\mathrm{n}+1).$$

The two loci (2.12) and (2.13) intersect at points

$$u_\mathrm{P}=\frac{[n(N-1)-(N+1)]}{n-1},$$

(2.14) $$v_\mathrm{P}=\frac{2(1+n)}{n-1},\{(n-1)\neq 0\}.$$

Similar interpretations follow for equations (2.10) and (2.11).

3. Nucleus of Polytropic Stars

In order that some progress could be made towards the study of immediate neighborhood of the origin, we shall give below the expressions representing the extent of the magnitude of the immediate neighborhood of the origin in which the solutions for n tending to zero (n→0) and minus one (n→ -1) hold for N=0, 1, and 2 configurations.

(a) Solutions for n→ -1 for N=0, 1 and 2 Configurations- Substituting n= -1 in above equation (2.9), we obtain an unique closed-form solution

$$(3.1) \qquad v_{\mathrm{P}} = Cu_{\mathrm{P}} \quad \text{(C=constant).}$$

for N= 0, 1, and 2 configurations. Obviously, this equation represents a straight line passing through the origin, and makes an angle tan-1C with u_{P}-axis. The $(u_{\mathrm{P}}, v_{\mathrm{P}})$-variables are of immense importance for the positive quadrant ($u_{\mathrm{P}} \geq 0, v_{\mathrm{P}} \geq 0$) contains only such parts of (ξ_{P}, P)-solutions which are of astrophysical interest.

(b) Solutions for n→ 0 for N = 0, 1 and 2 Configurations.

Case 1. Putting N= 0, n= 0 in equation (2.9), we obtain closed-form solution:

$$(3.2) \qquad v_{\mathrm{P}} = u_{\mathrm{P}}\left[C^{/}\left(1-u_{P}\right)^{-2}-2\right] \text{(C/=constant)}$$

Case 2. If we put N= 1, n= 0 in equation (2.9), then we have the following explicit solution:

$$(3.3) \qquad v_{\mathrm{P}} = \frac{4}{\left(u_{\mathrm{P}}-2\right)} \frac{1}{\left\{\frac{\log u_{\mathrm{P}}}{u_{\mathrm{P}}-2}+4C^{//}\right\}-2}$$

(C//=constant)

Case 3. For N= 2, n= 0, we have the following approximate analytical solution of equation (2.9):

$$(3.4) \qquad v_{\mathrm{P}} \approx 2\left(\frac{u_{\mathrm{P}}}{3}\right)^{\frac{1}{3}}\left[\left\{1-\frac{4\left(u_{\mathrm{P}}-3\right)}{3.5}+\frac{5\left(u_{\mathrm{P}}-3\right)^{2}}{3^{4}.4}-\frac{80\left(u_{\mathrm{P}}-3\right)^{3}}{3^{7}.11}\right\}+C^{///}u_{\mathrm{P}}^{1/3}\left(u_{\mathrm{P}}-3\right)^{\frac{2}{3}}\right]^{-1}$$

(C///=constant).

(c) The Extent of the Immediate Neighborhood of the Origin- We would like now to obtain an expression giving the range of the immediate neighborhood of the origin for which the solution in (3.1) holds.

The generalized form of the series solution of equation (2.1), near the origin (ξP=0), satisfying the boundary conditions

$$(3.5) \qquad \mathrm{P}(0)=1; \qquad \frac{d\mathrm{P}}{d\xi_{\mathrm{P}}}=0 \text{ at } \xi_{\mathrm{P}}=0$$

is given by (retaining terms up to ξ_{P}^{8} only).

$$(3.6)\qquad \mathrm{P}(\xi_{\mathrm{P}})=1+\sum_{m=1}^{\infty}a_m\xi_{\mathrm{P}}^{2m}=1+a_1\xi_{\mathrm{P}}^2+a_2\xi_{\mathrm{P}}^4+a_3\xi_{\mathrm{P}}^6+a_4\xi_{\mathrm{P}}^8.$$

The constants a_1, a_2, a_3 and a_4 are given by

$$a_1=-\frac{1}{2(N+1)},\qquad a_2=\frac{n(2+N)}{4(n+1)(3+N)(N+1)^2},$$

$$(3.7)\qquad a_3=-\frac{\left[2n^2\{3N^2+16N+17\}-n\{2N^2+12N+18\}\right]}{48(n+1)^2(N+1)^3(N+3)(5+N)},$$

$$a_4=\frac{n^3I-n^2J+nK}{384(N+1)^4(N+3)^2(n+1)^3(5+N)(7+N)}$$

$$I=(12N^7+228N^6+1656N^5+5988N^4+11896N^3$$
$$+13224N^2+7556N+1488,$$
$$J=20N^4+280N^3+1388N^2+2856N+2016,$$
$$K=4N^4+60N^3+332N^2+804N+720).$$

With the help of the series solution (3.6) we can express physical variables u_{P} and v_{P} as function of ξP and thus there will be just one curve in $(u_{\mathrm{P}}, v_{\mathrm{P}})$-plane. This curve (not drawn) corresponds to E-solutions which are included in the homology formula $\{\mathrm{P}_n(\xi_{\mathrm{P}})\}$. Let there are $u_{\mathrm{P}}(n,\xi_{\mathrm{P}})$ and $v_{\mathrm{P}}(n,\xi_{\mathrm{P}})$ "E-curves". The points at which "E-curve" and the solution given by equation (3.1) intersect are given by

$$(3.8)\qquad \xi_{\mathrm{P}}^2=\frac{2c(n+1)}{c(n-1)-(n+1)}\qquad (\mathrm{N}=0),$$

$$(3.9)\qquad \xi_{\mathrm{P}}^2=\frac{4c'\left(1+\frac{1}{n}\right)}{c'\left(3+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)}\qquad (\mathrm{N}=1),$$

$$(3.10)\qquad \xi_{\mathrm{P}}^2=\frac{18c''\left(1+\frac{1}{n}\right)}{(2+9c'')\left(2+\frac{3c''}{9}\right)}\qquad (\mathrm{N}=2).$$

As in previous work (eqn. 3.8)25, the expression in equation (3.9) or in

(3.10) gives the values of ξP for different values of n up to which the solution in equation (3.1) is relevant. For given C or C``, ξ_P increases, and for n→∞ , its values from eqs. (3.9) and (3.10) are

$$(3.11) \qquad \xi_P = 2\left(\frac{c'}{3c'+1}\right)^{\frac{1}{2}}$$

and

$$\xi_P = \frac{18c''}{\left(2+9c''\right)\left(2+\frac{c''}{3}\right)},$$

respectively. For n→-1; ξP→0 which establishes the fact that for all polytropes in (ξP,P)-plane, the immediate neighborhood of the origin is governed by the solution for n→-1. The pressure, density and temperature distributions are given by the equations (2.9), (2.10) and (2.11) respectively.

(d) Numerical Results, Interpretations and Implications

Follow the usual procedure, as also applied elsewhere, we may obtain Pade/(2,2) approximation for the series equation (3.6) in the form of rational function, given by

$$(3.12) \qquad P = P_{22}\left(\xi_P\right) = \frac{1+A'\xi_p^2+B'\xi_p^4}{1+C'\xi_p^2+D'\xi_p^4},$$

equating the expression (3.6) and (3.12), it is easy to compute the numerical values of the constant coefficients

A/, B/, C/ and D/

$$A' = a_1 + C', \quad B' = a_2 + a_1 C' + D',$$

$$C' = \frac{\left(a_2 a_3 - a_1 a_4\right)}{\Omega}, \quad D' = \frac{\left(a_2 a_4 - a_3^2\right)}{\Omega};$$

$$(3.13) \qquad \Omega = a_1 a_3 - a_2^2 .$$

Vanishing of the numerator of the expression in equation (3.12) (since $P(\xi_p) \to o$ at $\xi_p = \xi_{p1}$), would define the geometrical size ξ_{p1} (boundary value), that is,

(3.14) B/η2+A/η+1=0 ($\eta \equiv \xi_{P_1}^2$).

would lead to give

$$(3.15)\qquad \xi_{P_1} = \begin{cases} 3.867989962 & (\gamma = 4/3), \\ 3.396011484 & (\gamma = 5/3), \\ 3.202300313 & (\gamma = 2). \end{cases}$$

(e) E-Solutions Near the Origin $\xi_P \sim 0$ - From series expansion in (3.6), we have, after neglecting higher order terms than ξ_P^2,

$$(3.16)\qquad P(\xi_P) = 1 - \frac{\xi_P^2}{6} \quad (\xi_P \to 0).$$

and

$$(3.17)\qquad u_{PE} = -\frac{\xi_P P^{\frac{2n}{n+1}}}{P'} \sim \frac{3\left(1+3n(N+2)-2n(3+N)(1+N)^2\right)\xi_P^2}{(n+1)(3+N)(1+N)^2} \sim 3 \qquad (\xi P \to 0).$$

$$(3.18)\qquad v_{PE} = -\frac{\xi_P P'}{P} \sim \frac{\xi_P^2}{3} \quad (\xi P \to 0)$$

Hence, Ep-curve passes through the point $u_{PE}=3$, $v_{PE}=0$ (Ep→0) for all n, and N. At this point, the slope of the UPE-curve is given by

$$(3.19)\qquad \frac{(1+n)(3+N)(1+N)^2}{9}\left\{3n(N+2) - 2n(3+N)(N+1)^2\right\} .$$

It is also clear from equations (3.17) and (3.18) that $u_{PE}(\xi_P) = u_{PE}(-\xi P)$; $v_P(\xi P) = v_P(-\xi P)$ which means that the solution of astrophysical interest lies in the positive quadrant $u_{PE}>0$, $v_P>0$. The physical meaning of the two homology-invariant variables u_{PE} and v_{PE} is very much clear from the following equations:

The mass M(ξP) interior to ξP can be expressed as

$$(3.20)\qquad M(\xi_P) = \int_0^{\alpha_P \xi_P} 4\pi\rho r^2 dr = 4\pi\alpha_P^3 \int_0^{\xi_P} \xi_P^N P^{\frac{n}{n+1}} \frac{d\xi_P}{K^{\frac{n}{n+1}}},$$

which, on using (2.3), can be expressed in the form

$$(3.21)\qquad M(\xi_P) = -\frac{4\pi\alpha_P^3}{K^{\frac{n}{n+1}}} \int \frac{d}{d\xi_P}\left(\xi_P^N P^{-\frac{n}{n+1}} \frac{dP}{d\xi_P}\right) d\xi_P$$

$$= -4\pi\alpha_P^3 P^{-\frac{n}{n+1}} \xi_p^N \frac{dP}{d\xi_P}$$

The mean density is given by

(3.22) $$\overline{\rho}\left(\xi_{\mathrm{P}}\right)=\frac{3\mathrm{M}\left(\xi_{\mathrm{P}}\right)}{4\pi\alpha_{\mathrm{P}}^{3}\xi_{\mathrm{P}}^{\mathrm{N}}}$$

which, with the help of foregoing equation (3.21)
can be re-written as

(3.23) $$\rho\left(\xi\right)=-3\mathrm{P}^{\frac{n}{n+1}}\mathrm{K}^{\frac{n}{n+1}}\frac{d\mathrm{P}}{d\xi_{\mathrm{P}}}.$$

Further, we can express he variable u_P in terms of density $\rho\left(\xi\right)$ (or mean density $\overline{\rho}$ (ξ) and the radial (dimensionless) distance ξP) of the form

(3.24) $$u_{\mathrm{PE}}=\frac{3\mathrm{K}\rho^{2}\left(\xi\right)\xi_{\mathrm{P}}\mathrm{P}^{-\frac{n}{n+1}}}{\rho\left(\xi\right)}=\frac{3\mathrm{K}^{\frac{n}{n+1}}\rho^{2}\left(\xi\right)\xi_{\mathrm{P}}\left(1+a_{1}\xi_{\mathrm{P}}^{2}+a_{2}\xi_{\mathrm{P}}^{4}+..\right)^{-\frac{n}{n+1}}}{\rho\left(\xi_{\mathrm{P}}\right)}$$
$$\sim\frac{3\mathrm{K}^{\frac{n}{n+1}}\rho^{2}\left(\xi\right)\xi_{\mathrm{P}}}{\rho\left(\xi\right)}.$$

Equations (3.6) and (3.21) enable us to write v_{PE} (equation (3.18)) in the form

(3.25) $$v_{\mathrm{PE}}\sim\frac{\rho\left(\xi\right)\xi_{\mathrm{P}}}{3\mathrm{K}^{\frac{1}{1+n}}\rho\left(\xi\right)^{1+\frac{1}{n}}}\left(\xi_{\mathrm{P}}\to 0\right).$$

Proceeding as above we may find that

(3.26) $$u_{\rho\mathrm{E}}\left(\xi\right)=-\frac{\xi_{\rho}\rho^{2-\frac{1}{n}}}{\rho^{/}}=\frac{3\xi_{\rho}\rho\left(\xi\right)}{\rho\left(\xi\right)}$$

and

(3.27) $$v_{\rho\mathrm{E}}\left(\xi\right)=-\frac{\xi_{\rho}\rho^{/}}{\rho}=\frac{\xi_{\rho}\rho\left(\xi\right)}{3\rho\left(\xi\right)^{\frac{1}{n}}}.$$

using the expression

(3.28) $$\rho^{/}\left(\xi\right)=-\frac{\rho\left(\xi\right)}{3\rho^{\frac{1}{n-1}}},$$

We may find from (3.24) and (3.26), (3.28) and (3.29)

(3.29) $$u_{\mathrm{PE}} = \left(\frac{nk^{\frac{n-1}{n+1}}}{n+1} \right)^{\frac{1}{2}} u_{\rho\mathrm{E}}$$

and

(3.30) $$v_{\mathrm{PE}} = \left\{ \left(n + \frac{1}{n} \right)^{3} .k^{\frac{1-n}{1+n}} \right\}^{\frac{1}{2}} v_{\rho\mathrm{E}} .$$

This shows that we cannot express u_{PE} as function of $u_{\rho\,\mathrm{E}}$ and v_{PE} as function of $v_{\rho\,\mathrm{E}}$ for n=0 and n=-1. Further, we find that when n=-1, α_{P}=0 or ∞ depending on whether K<1 or>1; for $k=1, \alpha_{\mathrm{P}}$ is indeterminate (eqn. (2.6)). The pressure distribution in the immediate neighborhood of the center is given by solution for n tending to minus (eqns (2.9)) and the density distribution is given by the solution for n tending to zero (eqn. (2.10))

4. Implications and Advantages of the Present Approach

As another example of the application of this modern developed and efficient technique, it is suggested that one should at least refer to the Sharma's recent contributions which include the study of (i) thermodynamical equilibrium of stars clusters embedded in an isothermal configuration 25, (ii) relativistic stellar structures and X-ray transients in Ni's theory of gravity31 , (iii) very massive stellar models in Ni's theory of gravity32 (iv) general relativistic neutron stars33

Till quite recently, Tiwari and Shalini in their dissertations have studied the problems related with structural features of plane- symmetric configurations 34 , neutron stars, nebulae, etc, which are fundamentally based on the applications of polytropic theory and Pade`(2,2) approximation technique. She applies Pade` (2, 2) approximation technique to the Lane Emden equation (LEE) of index 1.5 in ($(\xi_\theta;\theta)$ plane and obtains the density distribution function $\theta(\xi_\theta) = \theta_{22}(\xi_0)$ in simple, compact and rational form analogous to the equation (3.12). Values of some important physical quantities, such as, $\theta^{1.5}(\propto \text{density}\,\rho)$, $\theta^{n+1}(\propto$ pressure P) and potential energy have been tabulated showing the physical characteristics of the spherically symmetric nebulae. She further describes the basic physical parameters; the radius R, the mass radius (M-R) relation, the central condensatation $\overline{\rho}/\ \rho_c$ (ratio of mean to central density in terms of the Lane

–Emden function $\theta(\xi_\theta)$ [eqn. (3.16)] .Most importantly, she obtains the geometrical size $\xi_{\theta 1}$=5.99071 which would have otherwise been a very difficult task by using known standard numerical methods

The main advantages of the present technique are:

(i) It offers a reasonable approximation to the solution, (ii) it is suitable for the use on small computer program, (iii) physical quantities can be numerically evaluated with the help of an electronic pocket calculator even without use of computer program , and, if however the latter is used, it consumes minimum time, hence it is economical too over previously known methods, (iv) the geometrical size [eqn. (3.15)] of the configuration is immediately determined from equation (3.12) which , obviously, avoids lengthy and cumbersome process of numerical integrations, and , (v) it can be readily applied to the interdisciplinary research areas (agricultural or pure engineering, biomathematical modellings, environmental and biological systems which may quite often involve complicated forms of linear or non-linear differential equation

5. Astrophysical Applications

Eddington2, Stoner36 Kothari37,38 Srivastava10, the authors12,15,16 and many others used the results of a sphere of uniform density to study the minimal problems of stars , and other problems19-35 of congnate interest , using Pade `(2,2) approximation technique, for example, search of magnetic field in solar convection envelope39 (in which solution of MHD flow equations could be indicative of the occurrence of differential rotation or uniform rotation), pulsation in subdwarf B stars 40-42 (which determine the internal chemical structure, rotational properties, etc.) and the cluster Abell 1367 system 43 (this would provide information about the star formation, chemical evolutionary processes , etc

6. Conclusions

1. An unified analytic study of the nucleus of polytropes N=0 (plane-symmetric), N=1(cylindrical), and N=2 (spherical) has been made following the concept of a sphere of uniform density defined by the polytropic index n tending to zero ($n\to 0$) and minus one ($n\to -1$).
2. A unique, explicit solution for n tending to minus one for N=0, 1 and 2 configurations which govern the center of polytrope, is given by equation (3.1). Equations (3.2), (3.3) and (3.4) represent, respectively, the solutions for n tending to zero for N=0, 1 and 2 configurations.

3. The extents of the immediate neighborhood of the center to which the solution (3.1) is valid are given by equations (3.8), (3.9) and (3.10), respectively, for N=0, 1 and 2 configurations.
4. Using Pade` (2, 2) approximation technique the geometrical size ξ_{P_1} of N=2 configuration has been determined, and is given by ξ_{P_1}=3.867989962 (γ=4/3), 3.3960114844 (γ=5/3) and 3.201230031 (γ=2).
5. The present analytical Technique is useful for very short computer program, and the physical quantities of interest can be numerically calculated with the help of an electronic pocket calculator even without the use of computer. It can be applied to interdisciplinary field of researches which might involve complicated types of linear or non linear differential equation
6. Unlike Harrison and Lake11, it is found that the homology invariant variables have an important role in the study of nucleus of polytropes.

Reference

1. V. R Emden, Gaskugeln, Leipzig, and B.G Teubner, (1965) 250-259.
2. A. S. Eddington, *The internal constitution of the stars cambridge*, Cambridge University Press, (1926) 79-96.
3. E. A. Miline, Structure of self-gravitating polytropic sphere, *Hdb D Ap*, **3** (1930) 65-71.
4. S. Chandrasekhar, *An Introduction to the study of stellar structures*, Chicago, University of Chicago Press, 1967, 84-182.
5. L. Spitzer, Self-gravitating polytropic and isothermal sheets, *Ap. J*, **95 (**1942) 329-335.
6. P. Ledoux, Structural features of saturn ring system, *Ann. d'ap*, **14** (1952) 438-447.
7. P. Goldreich, and D. Bell Lynden, On the Saturn ring system and Laplacian -disk cosmogonies, MNRAS, **130** (1965) 97-103.
8. J. Ostriker, On internal structure of polytropes, *Ap J*, **140** (1964) 1065-1066.
9. J. Ostriker, On Internal structure of polytropes, *Ap J, Suppl,* **11** (1965) 167-176.
10. S. Srivastava, On Polytropic cylindrical configurations, *Indian J Pure Appl Maths,* **3** (1972) 443-453
11. E. R. Harrison and RG Lake, Rotating disk systems *Ap J* , **171** (1971)323-329.
12. J. P. Sharma, D. Sc. Thesis, University of Gorakhpur, Gorakhpur (U.P.) India, (1981).
13. E. Hopf, On solutions of Lane-Emden Equation for $1 < n \leq 3$, MNRAS, **91** (1932**)** 635- 662.

14. R. H. Fowler and Grid properties of E-solutions as $\xi \to 0$,MNRAS, **91 (**1930) 63-71.
15. J. P. Sharma, Limiting density in white dwarfs, *Indian J Phys*, **48 (**1974) 66-70.
16. J. P. Sharma, On the maximum value of the mass of a star, *Trans New York Acad Sc,* **34 (**1972) 691- 694.
17. J. P. Sharma, Planetary structures in general relativity,PAGEOPH, **97** (1972)14-24
18. R. A. Lyttleton, On the origin of mountains, *Proc Roy Soc,* **275 (**1963) 1-11.
19. J. P. Sharma, Lane-Emden equation of index 5 and Pade'approximation ,Experientia (Schweiz), **35** (1979); 293-299
20. J. P. Sharma, Transformations of the generalized isothermal equation (GIE) and Pade' approximations, *Indian J Pure Appl Maths,* **12** (1981) 119-123
21. J. P. Sharma, Planetary structures: An Analyaltical approach, *Indian J Pure Appl maths,* **15** (1984) 709-714
22. J. P. Sharma, Self –consistent fields in atoms, *Acta physica Hungarica* ,**37** (1984) 63-77; also 15th LPSC, 8024, Houstan, Texas (USA) (Abstract) (1984)
23. J. P. Sharma, and RB. Yadav, Structure of uniformly rotating isothermal cylinder, *Jour. PAS*, **1** (1990) 88-98.
24. J. P. Sharma, Equilibrium of uniformly slowly rotating polytropes, *Acta physica hungarica* ,**72** (1993) 63-76
25 J. P. Sharma, Thermodynamically equilibrium of star clusters, *Proc. Nat Acad Sc India* , **76** (2006) 305-307
26 J. P. Sharma, Transformations of the equilibrium for isothermal plane- symmetric configurations, *India Math Soc*,74th Ann Conference.
27 J. P. Sharma, Relativistic structures of white dwarfs, *Proc Math Soc BHU* , **16** (2000) 13-16
28 J. P. Sharma, Neutron star models*, Astrophysy space Sc,* **409** (1987) 183-289
29 J. P. Sharma, Relativistic features of stellar structures ,*Ap J* , **329** (1988) 232-240
30 J. P. Sharma, Relativistic features of non- rotating isothermal gas cylinder, *Astrophys Space Sc,* **163** (1983) 109-113.
31 J. P. Sharma, Relativistic stellar structures and X-ray Transients in Ni's theory of gravity, *Proc Nat. Acad. Sc. India,* **79** (2009) 29-34.
32 J. P. Sharma, D. P Jayapandian and Sanish Thomas, Very massive stellar models in Ni's theory of gravity, *Proc Nat. Acad Sc India*., **79** (2009) 411-416.
33 Sanish Thomas and J. P. Sharma, General relativistic neutron stars, *Praman* (communicated),(2009)
34 A. K. Tewari, A study of minimal properties of plane- symmetric configurations and the generalized Lane-Emden equation of index n . M. Phil thesis, Allahabad Agricultural Institute- Deemed University, Allahabad, 2007-08
35 Shalini, Applications of the theory of polytropes to neutron stars, nebulae and the plane-symmetric configurations M.Phil . Thesis, Allahabad Agricultural Institute– Deemed University, Allahabad, 2008-2009
36 E. C. Stoner ,On the minimal properties of a star, Phil Mag, **11** (1931) 986-992

37 D. S. Kothari, Special relativistic features of white dwarfs, Phil Mag ,**12** (1930) 665-672

38 D. S. Kothari, Minimal properties of relativistic white dwarfs star, Phil , Mag, **11** (1930) 1130-1140

39 K. M. Hiremath, Steady part of rotation and torodoil component of the field in convective envelope, *Bull .Astr. Soc India,* **29** (2001) 169-197.

40 D. Kilkenny, Pulsating subdwarf B stars, ASP Conference Ser, **259** (2002) 356-366.

41 C. S. Jeffery, Non- radial modes of pulsations, *JAA* ,**26** (2005) 661-667

42 P. E. Tremblay**,** G. Fountaine , P. Prassard, P. Bergeron and S.K. Randall, Detection of short and long period pulsations, *Ap J,* **165** (2006) 551-557

43 L. Cortese, G. Gavazzi, A. Boselli , P. Franzatti, R.C. Kennicutt, K. O' Neil and S. Salai, H1 gas with higher sensitivity in irregular galaxies, *A&A,* **453** (2006) 847-857.