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Coupled Fixed Point Theorems In G-Metric Spaces with Application in Solving Integral Equation

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Abstract: In this paper, we have proved coupled coincidence point results for pair ofmappings in partially ordered G-metric spaces. We have defined weak compatibility in this context to ensure the uniqueness of the coupled common fixed point. There are several corollaries which extend some existing results of coupled coincidence points and coupled fixed points. The main theorem is illustrated with an example. An application is established to solve some integral equation. The example demonstrates that our main result is an actual improvement over the results which are generalized.

Keywords: Partially ordered set; Coupled coincidence point; Mixed g-monotone property; Compatible mappings; Weakly compatible mappings.

MSC: 47H10, 54H25

1. Introduction

The Banach contraction principle is most celebrated fixed point theorem. Mustafa and Sims[1, 2] introduced a improved version of the generalized metric space structure, which they called it as G-metric spaces and establish Banach contraction principle in this work. For more details on G-metric spaces, one can refer to the papers[1-11]. Fixed point theorems in partially ordered G-metric spaces have been considered in[8].

Studies on coupled fixed point problems in partially ordered metric spaces have received considerable attention in recent years. One of the reason of this interest is their potential applicability. Specifically, Bhaskar and Lakshmikanthan[12] established coupled fixed point for mixed monotone operator in partially rdered metric spaces. Afterward, Lakshmikanthan and

Ciric[13] extended the results[12] of by furnishing coupled coincidence and coupled fixed point theorem for two commuting mappings having mixed g-monotone property. In a subsequent series, B. S. Choudhary and A. kundu[14] introduced the concept of compatibility and proved the result of[13] under different set of condition.

Recently, Choudhary and Maity[15] published coupled fixed point results in partially ordered G-metric spaces. Motivated by[12-15] we introduce the notion of weak compatibility in partially ordered G-metric spaces and utilize this to prove a coupled fixed point result for mixed g-monotone mapping. An illustrative example is discussed which shows that the above mentioned improvements are actual.

2. Mathematical Preliminaries

Let $(X, \preccurlyeq)$ be partially ordered set and $F: X \to X$ be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1) \leq F(x_2)$ and non-increasing, if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1) \geq F(x_2)$.

In 2004, Mustafa and Sims[2] introduced the concept of G-metric spaces as follows:

Definition 2.1: Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following axioms[2]:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, a, z \in X$ (rectangle inequality),

then the function G is called a generalized metric on X and the pair (X,G) is called a G-metric space.

Definition 2.2: Let (X,G) be a G-metric space and let $\{x_n\}$ a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$, $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, that if $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$ in a G-metric space (X, G), then if for each $\varepsilon > 0$, there exists a positive integer ℕ such that $G(x, x_n, x_m) < \varepsilon$ for all *m, n* $\geq \mathbb{N}$.[2]

Proposition 2.1:*Let (X, G) be a G-metric space. Then the following are equivalent*[2]*:*

(1) $\{x_n\}$ *is G -convergent to x,*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Definition 2.3: [4]If (X,G) and (X_1,G_1) be two G-metric space and let $f : (X,G) \to (X_1,G_1)$ be a function, then f is said to be G-continuous at a point $x_0 \in X$ if given ε > 0, there exists δ > 0, such that for $x, y \in X$ and $G(x_0, x, y) < \delta$ implies $G_1(f(x_0), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $x_0 \in X$ or function f is said to be G-continuous at a point $x_0 \in X$ if and only if it is G-sequentially continuous at x_0, that is, whenever $\{x_n\}$ is G-convergent to x_0, $\{f(x_n)\}$ is G-convergent to $f(x_0)$.

Definition 2.4: [2]Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq \mathbb{N}$; *i. e. if* $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.2: [2]*If (X,G) is a G-metric space then the following are equivalent:*

(1) The sequence $\{x_n\}$ *is G−Cauchy,*

(2) for each $\varepsilon > 0$, $\exists$ *a positive integer N such that* $G(x_n, x_m, x_l) < \varepsilon$ *for all* $n, m, l \geq \mathbb{N}$.

Proposition 2.3: [2]*Let (X,G) be a G-metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.*

Definition 2.5: [2]A G-metric space (X,G) is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \in X.$$

Proposition 2.4: [2]*Every G-metric space (X,G) will defines a metric space (X,d_G) by*

(i) $$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \text{ in } X.$$

If (X,G) is a symmetric G-metric space, then

(ii) $$d_G(x, y) = 2G(x, y, y) \text{ for all } x, y \text{ in } X.$$

However, if (X,G) is not symmetric, then it follows from the G-metric properties that

(iii) $$^3/_2 G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \text{ in } X.$$

Definition 2.6: [12]A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Proposition 2.5: [12]*A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.*

Definition 2.7: [12]Let (X, $\preccurlyeq$) be partially ordered set and $F: X \times X \to X$ be a mapping. The mapping $F: X \times X \to X$ is said to have mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $F(x_1, y) \leq F(x_2, y)$ for $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \leq y_2$ implies $F(x, y_1) \geq F(x, y_2)$ for $x \in X$.

Definition 2.8: (Mixed g-monotone property[13]) Let (X, $\preccurlyeq$) be partially ordered set and $F: X \times X \to X$ and $g: X \to X$ be two mappings. F has mixed g-monotone property if F is monotone g-non-decreasing in its fist argument and is monotone g-non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $g x_1 \leq g x_2$ implies $F(x_1, y) \leq F(x_2, y)$ for $y \in X$ and for all $y_1, y_2 \in X$, $g y_1 \leq g y_2$ implies $F(x, y_1) \geq F(x, y_2)$ for $x \in X$.

Definition 2.9: [12]An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F: X \times X \to X$ if $F(x,y)=x$ and $F(y, x) = y$.

Definition 2.10: [12]An element $(x, y) \in X \times X$ is called a coupled coincident point of mapping $F: X \times X \to X$ and $g: X \to X$ if $F(x,y)=gx$ and $F(y, x) = gy$.

Definition 2.10: [15]Let (X, G) be a G-metric space. A mapping $F: X \times X \to X$ is said to be continuous if for any two G-convergent sequence $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

Using the concept of continuous, mixed monotone property and coupled fixed point, Choudhary and Maity[15] introduce the following theorem:

Theorem 2.1: *Let (X, ≼) be partially ordered set and let G be a G-metric on X such that (X, G) is a complete G-metric space. Let F : X × X → X be a continuous mapping having mixed monotone property. Assume that there exist k ∈ [0, 1) such that for x, y, u, v, w, z ∈ X, the following holds:*

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \frac{k}{2}\big[G(x, u, w) + G(y, v, z)\big] \textit{for all } x \succcurlyeq u \succcurlyeq$$

w and y ≼ v ≼ z where either u ≠ w or v ≠ z.

If there exist x_0 and $y_0 \in X$, such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then F has coupled coincidence in X, that is, there exist x, y ∈ X such that $x=F(x,y)$ and $y=F(y, x)$.

In a sequel, very recently Aydi et al. generalized the above theorem using commutative mappings and g-mixed monotone property in the following way:

Theorem 2.2: *Let (X, ≼) be partially ordered set and G-be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F: X \times X \to X$* and $g: X \to X$ *such that*

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \varphi\left(\frac{G(gx, gu, gw)+G(gy, gv, gz)}{2}\right)$$

for all x, y, u, v, w, z $\in X$, *with* $gw \preccurlyeq gu \preccurlyeq gx$ *and* $gy \preccurlyeq gv \preccurlyeq gz$. *Suppose also that F is continuous and has the mixed g-monotone property,* $F(X\times X)\subseteq g(X)$ *and g is continuous and commute with F. If there exist* x_0 *and* $y_0 \in X$ *such that* $g(x_0) \preccurlyeq F(x_0, y_0)$ *and* $F(y_0, x_0) \preccurlyeq g(x_0)$, *then F and g have a coupled coincidence points, that is, there exist* $(x, y)\in X \times X$ *such that* $g(x)=F(x,y)$ and $g(y)=F(y, x)$.

We define a notion of compatibility in the following:

Definition 2.12: The mappings F and g where $F: X\times X \to X$ and $g: X \to X$, are said to be compatible if

$$\lim_{n\to\infty} G\big(g(F(x_n, y_n)), F(gx_n, gy_n), F(gx_n, gy_n)\big)= 0$$

and

$$\lim_{n\to\infty} G\big(g(F(y_n, x_n)), F(gy_n, gx_n), F(gy_n, gx_n)\big)= 0.$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n)=x$ and $\lim_{n\to\infty} F(y_n, x_n)=\lim_{n\to\infty} g(y_n)=y$, for all $x, y\in X$ are satisfied.

Definition 2.13: [13]We say that mappings $F: X\times X \to X$ and $g: X\to X$ are commutative if $g\big(F(x, y)\big) = F\big(gx, gy\big)$ for all x, y ∈ X.

We denote by Φ the set of function $\phi: [0, +\infty) \to [0, +\infty)$ satisfying

(a) $\phi^{-1}(\{0\}) = \{0\}$,

(b) $\phi(t)<t$ for all $t>0$,

(c) $\lim_{r\to t^+} \phi(r)<t$ for all $t>0$.

Lemma 2.1: *([20]) Let* $\phi\in \Phi$. *For all* $t > 0$, *we have* $\lim_{n\to\infty} \phi^n(t)=0$.

Definition 2.14: The mappings F and g where $F: X\times X \to X$ and $g: X\to X$, are said to be weakly compatible if they commute at there coincidence points, that is if

$$F(x, y) = g\,x \text{ for some x} \in \text{X, then } F(gx, gy) = g\,F(x, y)$$

and

$F(y,x) = gy$ for some x ∈ X, then $F(gy,gx) = gF(y,x)$.

3. Main Results

Theorem 3.1*: Let (X, $\preccurlyeq$) be partially ordered set and G be a G-metric on X such that (X, G) is a G-metric space. Also suppose that $F: X\times X \to X$ and $g: X \to X$ are such that F has mixed g-monotone property on X satisfying*

$$(3.1) \quad \varphi(G(F(x,y), F(u,v), F(w,z)) \leq \varphi(max(G(gx, gu, gw)$$
$$G(gy, gv, gz))) - \phi(max(G(gx, gu, gw), G(gy, gv, gz)))$$

for all x, y, u, v, w, z ∈X, with gw $\preccurlyeq$ gu $\preccurlyeq$ gx and gy $\preccurlyeq$ gv $\preccurlyeq$ gz, where either u ≠ w or v ≠ z and φ and ϕ are altering distance functions.Suppose $F(X\times X) \subseteq g(X)$ *and g(X) is complete subset of X. Also suppose X satisfy the following property:*

(i) *if* $\{x_n\}$ *is a non−decreasing sequence such that* $x_n \to x$, *then* $x_n \preceq x$ *for all n,*

(ii) *if* $\{y_n\}$ *is a non−increasing sequence such that* $y_n \to y$, *then* $y_n \succeq y$ *for all n.*

If there exist x_0, y_0∈ X such thatg(x_0) $\preccurlyeq$F(x_0, y_0) and g(y_0) $\succcurlyeq$ F(y_0, x_0),then there exist x, y ∈X such that $g(x)=F(x,y)$ and $g(y) = F(y,x)$ *,i. e. F and g have a coupled coincidence points.*

Proof: Let x_0, y_0 be such that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$. Since $F(X\times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing like this we can construct two sequences $\{gx_n\}$ and $\{gy_n\}$ such that

(3.2) $\quad gx_n = F(x_{n-1}, y_{n-1})$ and $gy_n = F(y_{n-1}, x_{n-1})$ for all $n \geq 0$.

We shall prove that for all $n \geq 1$,

(3.3) $\quad gx_n \preceq gx_{n+1}$,

and

(3.4) $\qquad g\,y_n \succeq g\,y_{n+1}$.

Since $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$ and $g\,x_1 = F(x_0, y_0)$ and $g\,y_1 = F(y_0, x_0)$, we have $gx_0 \preccurlyeq gx_1$ and $gy_0 \succcurlyeq gy_1$; that is, (3.3) and (3.4) hold for n = 0.

We presume that (3.3) and (3.4) holds for some n > 0. As F has mixed g-monotone property and $g\,x_n \preceq g\,x_{n+1}$, $g\,y_n \succeq g\,y_{n+1}$, from (3.2), we have

(3.5) $\qquad g\,x_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n)$ and $g\,y_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n)$.

also for the same reason, we have

(3.6) $\qquad F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = g\,x_{n+2}$ and $F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g\,y_{n+2}$.

From (3.3) and (3.4), we have that $g\,x_{n+1} \preceq g\,x_{n+2}$ *and* $g\,y_{n+1} \succeq g\,y_{n+2}$.
Then by mathematical induction it follows that (3.3) and (3.4) holds for $n \geq 0$. Therefore

(3.7) $\qquad g\,x_0 \preceq g\,x_1 \preceq g\,x_2 \preceq g\,x_3 \preceq \dots g\,x_n \preceq g\,x_{n+1} \preceq \dots$

(3.8) $\qquad g\,y_0 \succeq g\,y_1 \succeq g\,y_2 \succeq g\,y_3 \succeq \dots g\,y_n \succeq g\,y_{n+1} \succeq \dots$

If for some n, we have $\left(g\,x_{n+1}, g\,y_{n+1}\right) = \left(g\,x_n, g\,y_n\right)$, then $g\,x_n = F\left(x_n, y_n\right)$ and $gy_n = F\left(y_n, x_n\right)$, that is, F and g have a coincidence point. So from now we assume $\left(g\,x_{n+1}, g\,y_{n+1}\right) \neq \left(g\,x_n, g\,y_n\right)$ for all n ∈ N, that is we assume that either $g\,x_{n+1} = F\left(x_n, y_n\right) \neq g\,x_n$ or $g\,y_{n+1} = F\left(y_n, x_n\right) \neq g\,y_n$.
From (3.1), we have

(3.9)
$$\begin{aligned}\varphi\big(G(g\,x_{n+1}, g\,x_{n+1}, g\,x_n)\big) &= \varphi\big(G\big(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\big)\\ &\leq \varphi(max(G(g\,x_n, g\,x_n, g\,x_{n-1}), G(g\,y_n, g\,y_n, g\,y_{n-1})))\\ &\quad -\phi(max(G(g\,x_n, g\,x_n, g\,x_{n-1}), G(g\,y_n, g\,y_n, g\,y_{n-1}))).\end{aligned}$$

As $\phi \geq 0$,

$$\varphi\big(G(g\,x_{n+1}, g\,x_{n+1}, g\,x_n)\big) \leq \varphi(max(G(g\,x_n, g\,x_n, g\,x_{n-1}), G(gy_n, gy_n, gy_{n-1}))),$$

and, using the fact that φ is non-decreasing, we have

(3.10) $G(gx_{n+1}, gx_{n+1}, gx_n) \le max(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}))$.

Repeating the same argument ones obtain

(3.11) $G(gy_{n+1}, gy_{n+1}, gy_n) \le max(G(gy_n, gy_n, gy_{n-1}), G(gx_n, gx_n, gx_{n-1}))$.

By (3.10) and (3.11), we have

$$\begin{aligned} &max(G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)) \\ &\qquad \le max(G(gy_n, gy_n, gy_{n-1}), G(gx_n, gx_n, gx_{n-1})), \end{aligned}$$

and, thus, the sequence $max(G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n))$ is non-negative decreasing. This implies that there exists $r \ge 0$ such that

(3.12) $\lim_{n\to\infty} max(G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)) = r$.

It is easily seen that if $\varphi: [0, \infty) \to [0. \infty)$ is non-decreasing, $\varphi(max(a, b)) = max(\varphi(a), \varphi(b))$ for $a, b \in [0, \infty)$. Taking in to account this and (3.9) - (3.12), we get

$$\begin{aligned} &max\big(\varphi(G(gx_{n+1}, gx_{n+1}, gx_n)), \varphi(G(gy_{n+1}, gy_{n+1}, gy_n))\big) \\ &\quad = \varphi\big(max(G(gx_{n+1}, gx_{n+1}, gx_n)), (G(gy_{n+1}, gy_{n+1}, gy_n))\big) \\ &\quad \le \varphi(max(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}))) \\ &\qquad\qquad - \phi(max(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}))) \end{aligned}$$

Letting n $\to\infty$ in the above inequality and using (3.12), we have

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r)$$

this implies that $\phi(r) = 0$. Sinceϕ is an altering distance function, $r = 0$ and, consequently

$$\lim_{n\to\infty} max\ (G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)) = 0,$$

or

(3.13) $\lim_{n\to\infty} G(gx_{n+1}, gx_{n+1}, gx_n) = G(gy_{n+1}, gy_{n+1}, gy_n) = 0$.

Next, we show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in G-metric spaces (X, G).Assume to the contrary that at least one of above sequence is not a Cauchy sequence. This gives that $\lim_{m,n\to\infty} G(gx_n, gx_m, gx_m) \nrightarrow 0$ or $\lim_{m,n\to\infty} G(gy_n, gy_m, gy_m) \nrightarrow 0$, and consequently

$$\lim_{m,n\to\infty} max(G(gx_n, gx_m, gx_m), G(gy_n, gy_m, gy_m)) \nrightarrow 0.$$

This means that there exist $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) \geq k$ such that

$$max\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right) \geq \varepsilon. \tag{3.14}$$

Further, we can choose $n(k)$ corresponding to $m(k)$ in such a way that it is smallest integer with $n(k) > m(k)$ and satisfying (3.14).

$$max\left(G\left(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right) < \varepsilon. \tag{3.15}$$

By contractive condition, (3.1), we get

$$\begin{aligned} \varphi\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right)\right) &= \varphi\Big(G\Big(F\left(x_{n(k)-1}, x_{n(k)-1}\right), \\ &\qquad F\left(x_{m(k)-1}, x_{m(k)-1}\right), F\left(x_{m(k)-1}, x_{m(k)-1}\right)\Big)\Big) \\ &\leq \varphi\Big(max\Big(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}), \\ &\qquad G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\Big)\Big) \\ &\quad -\phi\Big(max\Big(G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}), \\ &\qquad G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1})\Big)\Big) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \varphi\left(G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right) &= \varphi\Big(G\Big(F\left(y_{n(k)-1}, x_{n(k)-1}\right), \\ &\qquad F\left(y_{m(k)-1}, x_{m(k)-1}\right), F\left(y_{m(k)-1}, x_{m(k)-1}\right)\Big)\Big) \\ &\leq \varphi(max(G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}), G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}))) \\ &\quad -\phi(max(G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}), G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}))) \end{aligned} \tag{3.17}$$

By (3.17) and (3.18), we get

$$(3.18)\qquad \varphi\Big(max\Big(G\big(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\big), G\big(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\big)\Big)\Big)$$
$$\leq \varphi(max(G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}), G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1})))$$
$$-\phi(max(G(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}), G(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1})))$$

On the other hand, the rectangle inequality and (3.15) give us

$$(3.19)\qquad \begin{aligned} G\big(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\big) &\leq G\big(gx_{n(k)}, gx_{m(k)-1}, gx_{m(k)-1}\big) \\ &\quad + G\big(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\big) \\ &< G\big(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\big) + \varepsilon \end{aligned}$$

$$(3.20)\qquad \begin{aligned} G\big(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\big) &\leq G\big(gy_{n(k)}, gy_{m(k)-1}, gy_{m(k)-1}\big) \\ &\quad + G\big(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\big) \\ &< G\big(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\big) + \varepsilon \end{aligned}$$

From (3.14), (315), (3.19) and (3.20), we get

$$\varepsilon \leq max\Big(G\big(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\big), G\big(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\big)\Big)$$
$$\leq max\Big(G\big(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\big), G\big(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\big)\Big) + \varepsilon$$

Letting $k \to \infty$ in the inequality and using (3.13), we have

$$(3.21)\qquad \lim_{k\to\infty} max\Big(G\big(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\big), G\big(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\big)\Big) = \varepsilon.$$

Again, using rectangular inequality and (3.15), we obtain

$$(3.22)\qquad \begin{aligned} G\big(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\big) &\leq G\big(gx_{n(k)-1}, gx_{m(k)}, gx_{m(k)}\big) \\ &\quad + G\big(gx_{m(k)}, gx_{m(k)-1}, gx_{m(k)-1}\big) \\ &< G\big(gx_{m(k)}, gx_{m(k)-1}, gx_{m(k)-1}\big) + \varepsilon \end{aligned}$$

and

$$(3.23)\qquad \begin{aligned} G\big(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\big) &\leq G\big(gy_{n(k)-1}, gy_{m(k)}, gy_{m(k)}\big) \\ &\quad + G\big(gy_{m(k)}, gy_{m(k)-1}, gy_{m(k)-1}\big) \\ &< G\big(gy_{m(k)}, gy_{m(k)-1}, gy_{m(k)-1}\big) + \varepsilon. \end{aligned}$$

By (3.22) and (3.23), we get

$$(3.24)\qquad \max\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right)$$
$$\leq \max\left(G\left(gx_{m(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{m(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right)+\varepsilon.$$

Using the rectangular inequality we have

$$\begin{aligned} G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right) \leq\ & G\left(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}\right) \\ & + G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right) \\ & + G\left(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\right) \end{aligned}$$

and

$$\begin{aligned} G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right) \leq\ & G\left(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}\right) \\ & + G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right) \\ & + G\left(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\right) \end{aligned}$$

and by last two inequalities and

$$(3.25)\qquad \varepsilon \leq \max\left(G\left(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{n(k)}, gy_{m(k)}, gy_{m(k)}\right)\right)$$
$$\leq \max\left(G\left(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}\right), G\left(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}\right)\right)$$
$$+\max\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right)$$
$$+\max\left(G\left(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\right)\right)$$

By (3.24) and (3.25) we have

$$\begin{aligned} & \varepsilon - \max\left(G\left(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}\right), G\left(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}\right)\right) \\ & - \max\left(G\left(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)}\right), G\left(gy_{m(k)-1}, gy_{m(k)}, gy_{m(k)}\right)\right) \\ & \leq \max\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right) \\ & \leq \max\left(G\left(gx_{m(k)}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{m(k)}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right) \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (3.13), we obtain

$$(3.26)\quad \lim_{n\to\infty} \max\left(G\left(gx_{n(k)-1}, gx_{m(k)-1}, gx_{m(k)-1}\right), G\left(gy_{n(k)-1}, gy_{m(k)-1}, gy_{m(k)-1}\right)\right) = \varepsilon.$$

Finally, letting $k \to \infty$ in (3.18) and using (3.21), (3.26) and the continuity of φ and ϕ, we get

$$\varphi(\varepsilon) \le \varphi(\varepsilon) - \phi(\varepsilon) \le \varphi(\varepsilon)$$

and, consequently, $\varphi(\varepsilon) = 0$. Since ϕ is altering distance function, $\varepsilon = 0$, and this is a contradiction.Therefore, both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in g(X). From the completeness of g(X),there exist $x, y \in X$ such that

(3.27) $$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_{n+1} = gx \text{ and } \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_{n+1} = gy,$$

By (3.7), (3.8) and (3.12), we have

$$gx_n \preceq gx \text{ and } gy_n \succeq gy.$$

For all $n \ge 0$, we get

$$\begin{aligned} G(F(x,y),F(x,y),F(x_{n-1},y_{n-1})) \le \varphi\big(&max(G(gx,gx,gx_{n-1}),\\ &G(gy,gy,gy_{n-1}))\big)\\ &-\phi(max(G(gx,gx,gx_{n-1}),G(gy,gy,gy_{n-1})))\end{aligned}$$

Letting the limit as n $\to\infty$ in the above inequality, using (3.27), we have

$$G(F(x,y),F(x,y),gx) = 0; \text{that is } gx = F(x,y).$$

Again, we have

$$\begin{aligned} G(F(y,x),F(y,x),F(y_{n-1},x_{n-1})) \le \varphi\big(&max(G(gy,gy,gy_{n-1}),\\ &G(gx,gx,gx_{n-1}))\big)\\ &-\phi\big(max\big(G(gy,gy,gy_{n-1}),G(gx,gx,gx_{n-1})\big)\big)\end{aligned}$$

Letting the limit as n $\to\infty$ in the above inequality, using (3.27), we have

$$G(F(y,x),F(y,x),gy) = 0; \text{that is } gy = F(y,x).$$

Hence the element $(x, y) \in X\times X$, is coupled coincidence point of mapping $F: X\times X \to X$ and $g: X \to X$.

By setting $\varphi(t)=$ Identity mapping in theorem 3.1, we have following corollary.

Corollary 3.2: *Let (X, $\preccurlyeq$) be partially ordered set and G-be a G-metric on X such that (X, G) is a G-metric space. Also suppose that $F: X\times X\to X$ and $g: X\to X$ are such that F has mixed g-monotone property on X such that*

$$G(F(x,y), F(u,v), F(w,z)\le max\big(G(gx, gu, gw), G(gy, gv, gz)\big) - \phi\big(max\big(G(gx, gu, gw), G(gy, gv, gz)\big)\big) \tag{3.28}$$

for all x, y, u, v, w, z $\in X$, with $gw\preccurlyeq gu\preccurlyeq gx$ and $gy\preccurlyeq gv\preccurlyeq gz$, where either $u\neq w$ and $v\neq z$ and ϕ are altering distance functions. Suppose $(X\times X)\subseteq g(X)$ and g(X) is complete subset of X. Also suppose X satisfy the following property:

(i) if $\{x_n\}$ is a non-decreasing sequence such that $x_n\to x$, then $x_n\preccurlyeq x$ for all n,

(ii) if $\{y_n\}$ is a non-increasing sequence such that $y_n\to y$, then $y_n\succcurlyeq y$ for all n.

If there exist $x_0, y_0\in X$ such that $g(x_0)\preccurlyeq F(x_0, y_0)$ and $g(y_0)\succcurlyeq F(y_0, x_0)$, then there exist x, y $\in X$ such that g(x) = F(x, y) and g(y) = F(y, x), that is F and g have a coupled coincidence points.

Again by setting $\phi(t)$ = *1 –kt*, in theorem 3.1, we have following corollary.

Corollary 3.3: *Let (X, $\preccurlyeq$) be partially ordered set and G-be a G-metric on X such that (X, G) is a G-metric space. Also suppose that $F: X\times X\to X$ and $g: X\to X$ are such that F has mixed g-monotone property on X such that*

$$G\big(F(x,y), F(u,v), F(w,z)\big)\le k\big(max\big(G\big(gx, gu, gw\big), G\big(gy, gv, gz\big)\big)\big) \tag{3.29}$$

for all x, y, u, v, w, z $\in X$, with $gw\preccurlyeq gu\preccurlyeq gx$ and $gy\preccurlyeq gv\preccurlyeq gz$, where either $u\neq w$ and $v\neq z$.Suppose $(X\times X)\subseteq g(X)$ and g(X) is complete subset of X. Also suppose X satisfy the following property:

(i) if $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$, then $x_n \preccurlyeq x$ for all n,

(ii) if $\{y_n\}$ is a non-increasing sequence such that $y_n \to y$, then $y_n \succcurlyeq y$ for all n.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0)$ and $g(y_0) \succcurlyeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is F and g have a coupled coincidence points.

Remark 3.4: Notice that theorem 3.1 of Choudhary and Maity[27] which is stated here as theorem 2.1is a consequence of corollary 3.3. In fact, the contractive condition appearing in theorem 2.1

$$G\big(F(x, y), F(u, v), F(w, z)\big) \le \frac{k}{2}\big[G(x, u, w) + G(y, v, z)\big]$$

for all $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \neq w$ or $v \neq z$, with $k \in [0, 1)$ implies

$$\begin{aligned} G\big(F(x, y), F(u, v), F(w, z)\big) &\le \frac{k}{2}\big[G(gx, gu, gw) + G(gy, gv, gz)\big] \\ &\le \frac{k}{2}\times 2\Big(max\big(G(gx, gu, gw), G(gy, gv, gz)\big)\Big) \\ &= k\Big(max\big(G(gx, gu, gw), G(gy, gv, gz)\big)\Big) \end{aligned}$$

and, therefore, setting g = Identity function and applying the corollary 3.3 we can get the desired result.

Theorem 3.5:*Let $(X, \preccurlyeq)$ be partially ordered set and G-be a G-metric on X such that (X, G) is a G-metric space. Also suppose that $F: X \times X \to X$ and $g: X \to X$ are such that F has mixed g-monotone property on X satisfying*

$$G(F(x, y), F(u, v), F(w, z) \le \phi\Big(max\big(G(gx, gu, gw), G(gy, gv, gz)\big)\Big) \qquad (3.30)$$

for all $x, y, u, v, w, z \in X$, with $gw \preccurlyeq gu \preccurlyeq gx$ and $gy \preccurlyeq g\,v \preccurlyeq gz$, where either $u \neq w$ and $v \neq z$ and $\phi$is altering distance functions. Suppose $F(X \times X) \subseteq g(X)$ and g(X) is complete subset of X. Also suppose X satisfy the following property:

(i) if $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$, then $x_n \preccurlyeq x$ for all n,

(ii) if $\{y_n\}$ is a non-increasing sequence such that $y_n \to y$, then $y_n \succcurlyeq y$ for all n.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0)$ and $g(y_0) \succcurlyeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is F and g have a coupled coincidence points.

Proof: Repeating the similar procedure as in the proof of theorem 3.1, the result is obvious.

Remark 3.6 Notice that theorem 3.1 of Aydi et al. [28] which is stated here as theorem 2.2 is a consequence of theorem 3.5. In fact, the contractive condition appearing in theorem 2.2

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \varphi\left(\frac{G(gx, gu, gw)+G(gy, gv, gz)}{2}\right)$$

for all x, y, u, v, w, z ∈ X, with $gw \preccurlyeq gu \preccurlyeq gx$ and $gy \preccurlyeq gv \preccurlyeq gz$, implies

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \varphi\left(\frac{G(gx, gu, gw)+G(gy, gv, gz)}{2}\right)$$

$$\leq \varphi\big(max\big(G(gx, gu, gw), G(gy, gv, gz)\big)\big)$$

and, therefore, applying the theorem 3.5 we can get the desired result.

Next our aim is to prove the uniqueness of coupled fixed pointin the above theorem. For this, note that if $(X, \preccurlyeq)$ is partially ordered set, then we endow the product space X × X with following partial order:

for $(x, y), (u, v) \in X \times X$, $(u, v) \preceq (x, y) \Leftrightarrow x \succeq u,\ y \preceq v$.

Theorem 3.7: *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exist a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), (y^*, x^*))$ and, also F and g are weakly compatible. Then F and g have a unique coupled common fixed point; that is, there exist a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.*

Proof. From theorem 3.1, the set of coupled coincidence point is non-empty. Suppose (x, y) and (x^*, y^*) are coupled coincidence points of F and g; that is, $gx = F(x, y)$, $gy = F(y, x)$ and $gx^* = F(x^*, y^*)$, $gy^* = F(y^*, x^*)$. Now we show

(3.31) $\quad gx = gx^*$ and $gy = gy^*$.

By the assumption, there exist $(u, v) \in X \times X$ such that $\big(F(u, v), F(v, u)\big)$ is comparable to $\big(F(x, y), F(y, x)\big)$ and $(F(x^*, y^*), (y^*, x^*))$.
Put $u_0 = u, v_0 = v$, and choose $u_1, v_1 \in X$, so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$.
Then, repeating the same argument as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ where

$$gu_n = F(u_{n-1}, v_{n-1}) \text{ and } gv_n = F(v_{n-1}, u_{n-1}), \text{ for all } n \in N.$$

Hence $\big(F(x, y), F(y, x)\big) = (gx, gy)$ and $\big(F(u, v), F(v, u)\big) = (gu_1, gv_1)$ are comparable.

Suppose that $(gx, gy) \succeq (gu_1, gv_1)$ (the proof is similar in other case).

We claim that $(gx, gy) \succeq (gu_n, gv_n)$, for each $n \in N$.

In fact, we will use mathematical induction.

Since, we have $(gx, gy) \succeq (gu_1, gv_1)$. Our claim is true for $n = 1$. We presume that $(gx, gy) \succeq (gu_n, gv_n)$ holds for $n > 1$. Then, we have $gx \preccurlyeq gu_n$ and $gy \succeq gv_n$. Using the mixed g-monotone property of F, we get

$$gu_{n+1} = F(u_n, v_n) \succeq F(x, v_n) \succeq F(x, y) = gx$$

and

$$gv_{n+1} = F(v_n, u_n) \preccurlyeq F(y, u_n) \preccurlyeq F(y, x) = gy$$

and this proves our claim.

Since $gx \preccurlyeq gu_n$ and $gy \succeq gv_n$, using the contractive condition (3.1), we have

$$\varphi G(gx, gx, gu_n) = \varphi G\big(F(x, y), F(x, y), F(u_{n-1}, v_{n-1})\big)$$

$$\leq \varphi(max(G(gx, gx, gu_{n-1}), G(gy, gy, gv_{n-1}))) \\ -\phi(max(G(gx, gx, gu_{n-1}), G(gy, gy, gv_{n-1})))$$

As $\phi \geq 0$,

$$\varphi G\big(F(x, y), F(x, y), F(u_{n-1}, v_{n-1})\big) \leq \varphi\big(max\big(G(gx, gx, gu_{n-1}), \\ G(gy, gy, gv_{n-1})\big)\big)$$

and, using the fact that φ is non-decreasing, we have

$$G\big(F(x, y), F(x, y), F(u_{n-1}, v_{n-1})\big) \leq max\big(G(gx, gx, gu_{n-1}), \\ G(gy, gy, gv_{n-1})\big) \tag{3.32}$$

Repeating the same reasoning we obtain

$$G\big(F(y, x), F(y, x), F(v_{n-1}, u_{n-1})\big) \leq max\big(G(gy, gy, gv_{n-1}), \\ G(gx, gx, gu_{n-1})\big) \tag{3.33}$$

Using (3.32) and (3.33), we have

$$max\big(G(gx, gx, gu_n), G(gy, gy, gv_n)\big) \leq max\big(G(gx, gx, gu_{n-1}), \\ G(gy, gy, gv_{n-1})\big) \tag{3.34}$$

and thus, the sequence $max(G(gx,gx,gu_n), G(gy,gy,gv_n))$ is nonnegative decreasing. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} max(G(gx, gx, gu_n), G(gy, gy, gv_n)) = r. \tag{3.35}$$

It is easily seen that if $\varphi : [0, \infty) \to [0, \infty)$ is non-decreasing, $\varphi(max(a, b)) = max(\varphi(a), \varphi(b))$ for $a, b \in [0, \infty)$. Taking into account this and (3.32) - (3.35), we get

$$max\big(\varphi G(gx, gx, gu_n), \varphi G(gy, gy, gv_n)\big) = \varphi\big(max\big(G(gx, gx, gu_n), \\ G(gy, gy, gv_n)\big)\big)$$

$$\leq \varphi\big(max\big(G(gx, gx, gu_{n-1}), \varphi G(gy, gy, gv_{n-1})\big)\big)$$

$$-\phi\big(max\big(G(gx, gx, gu_{n-1}), \varphi G(gy, gy, gv_{n-1})\big)\big)$$

Letting $n \to \infty$ in the above inequality and using (3.35), we have

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r)$$

this implies that $\phi(r) = 0$. Since ϕ is an altering distance function, r ≥ 0 and, consequently

$$\lim_{n\to\infty} max\ (G(gx,gx,gu_n),\ G(gy,gy,gv_n)) = 0.$$

Or

(3.36) $\lim_{n\to\infty} G(gx,gx,gu_n) = \lim_{n\to\infty} G(gy,gy,gv_n) = 0.$

Repeating the similar argument, we show that

(3.37) $\lim_{n\to\infty} G(gx^*,gx^*,\ gu_n) = \lim_{n\to\infty} G(gy^*,\ gy^*,\ gv_n) = 0.$

Using, G_5, $\left(G(x,\ x,\ y) \le 2G(x,\ y,\ y)\right)$, (3.21) and (3.22), we have

$$\begin{aligned} G(gx,\ gx^*,\ gx^*) &\le G(gx,\ gu_n,\ gu_n) + G(gu_n,\ gx^*,\ gx^*) \\ &\le [2G(gx,\ gx,\ gu_n) + G(gu_n,\ gx^*,\ gx^*)] \to 0 \ as\ n \to \infty. \end{aligned}$$

and

$$\begin{aligned} G(gy,\ gy^*,\ gy^*) &\le G(gy,\ gv_n,\ gv_n) + G(gv_n,\ gy^*,\ gy^*) \\ &\le [2G(gy,\ gy,\ gv_n) + G(gv_n,\ gy^*,\ gy^*)] \to 0 \ as\ n \to \infty. \end{aligned}$$

Hence $gx = gx^*$ and $gy = gy^*$. Thus we proved (3.31).

Since $gx = F(x,\ y)$ and $gy = F(y,\ x)$, by weak compatibility of F and g we have

(3.38) $ggx = gF(x,\ y) = F(gx,\ gy)$ and $ggy = gF(y,\ x) = gF(gy,\ gx).$

Denote $gx = z$ and $gy = w$. Then from (3.38),

(3.39) $gz = F(z,\ w)$ and $gw = F(w,\ z).$

Then $(z,\ w)$ is a coupled coincidence point. Then from (3.38) with $x^* = z$ and $y^* = w$ it follows $gz = gx$ and $gw = gy$, that is,

(3.40) $gz = z$ and $gw = w$.

From (3.39) and (3. 40),

$$z = gz = F(z, w) \text{ and } w = gw = F(w, z).$$

Therefore (z, w) is a coupled common fixed point of F and g. To prove uniqueness, assume that (r, s) is another coupled common fixed point. Then by (3.38) we have

$$r = gr = gz = z \text{ and } s = gs = gw = w.$$

Corollary 3.8: *In addition to the hypotheses of corollary 3.2, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exist a* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u))$ *is comparable to* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), (y^*, x^*))$ *and, also F and g are weakly compatible. Then F and g have a unique coupled common fixed point; that is, there exist a unique* $(x, y) \in X \times X$ *such that* $x = gx = F(x, y)$ *and* $y = gy = F(y, x)$.

Corollary 3.9: *In addition to the hypotheses of corollary 3.3, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exist a* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u))$ *is comparable to* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), (y^*, x^*))$ *and, also F and g are weakly compatible. Then F and g have a unique coupled common fixed point; that is, there exist a unique* $(x, y) \in X \times X$ *such that* $x = gx = F(x, y)$ *and* $y = gy = F(y, x)$.

Example 3.10: Let $X = [0, 2]$ be endowed with Euclidean metric $G(x, y, z) = (|x - y| + |y - z| + |z - x|)$, for all $x, y \in X$.

Then, $(X, \leq)$ is a partial ordered set with natural ordering of real numbers. Let $F: X \times X \to X$ and $g: X \to X$ defined as $g(x) = x/2$ for all $x \in X$ and

$$F(x, y) = \begin{cases} \left(\dfrac{x-y}{2}\right)^2, & \text{if} \quad x, y \in [0,1],\ x \geq y \\ 0, & \text{if} \quad x < y \end{cases}, \text{ respectively.}$$

Clearly, $F(X \times X) \subseteq g(X)$, also F obeys mixed g-monotone property.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X definedrespectively by $x_n = 2 + 1/n$ and $y_n = 1 - 1/n$.

then we have by letting $n \to \infty$

$$g\big(F(x_n, y_n)\big) = g(F(2 + 1/n, 1 - 1/n) = g\left\{\left(\frac{1+2/n}{2}\right)^2\right\}$$
$$= g(1/\ 4) = 1/8 \neq 1/16 = F(gx_n, gy_n).$$

Also,

$$gF(1, 0) = g(1/4) = 1/8 \neq 1/16 = F(g1, g0),$$

which shows that the mappings F and g are neither compatible and nor commutative. But it is obvious that the mappings are weakly compatible.
So we can not use the theorem 2.16 for mappings F and g.Also is oblivious that, (0, 0) is the coupled fixed point of F and g.

Remark 3.11: It is obvious that if the mapping F and G neither compatible and nor commutative, then this example will not be applicable, which proves the generality of our result.

4.Applications

Theorem 4.1: *Let* $\Omega = [0, 1]$ *be bounded open set in* $\mathbb{R}$, $L^2(\Omega)$, *the set of function on* Ω *whose square in integrable on* Ω. *Consider the integral equation*

(4.1) $p(t, (x(t), y(t))) = \int q(t, s, (x(s), y(s)))ds$, *integration is taken over* Ω,

where $P: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $q: \Omega \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be two mappings. Define* $G: X \times X \times X \to R^+$ *by*

$$G(x, y, z) = \sup_{t\in\Omega}|x-y| + \sup_{t\in\Omega}|y-z| + \sup_{t\in\Omega}|z-x|.$$

Then X is a complete G-metric space. Suppose that there exist a function $G: \Omega \times \mathbb{R} \times \mathbb{R} \to R^+$ *satisfying,*

(i) $p(s, (u(t),v(t)) \geq \int q(t, s,(w(s), z(s)))ds \geq G(s,(u(t),v(t)))$ *for each* $s, t \in \Omega$..

(ii) $p(s, (u(t),v(t)) - G(s,(u(t),v(t))) \leq \alpha\big(|p(s, (u(t),v(t))) - (u(t),v(t))|\big)$.

Then the integral equation (4.1) has solution in $L^2(\Omega)$.

Proof: Define $F\left((x(t),\, y(t)),t\right)=p\left(t,(x(t),\, y(t))\right)$ and $g\,x(t)=\int q\left(t,\, s,\, (x(s),\, y(s))\right) ds$, then it is obvious that condition of corollary 3.2 are satisfied. Now we can apply corollary 3.2 to obtain the solution of integral equation 4.1

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