ISSN 0974 - 9373

Vol. 18 No. 4 (2014) **Journal of International Academy of Physical Sciences** pp. 279-284

On the Flatness of Weakly Symmetric Kähler Manifolds

B. B. Chaturvedi and Pankaj Pandey

Department of Pure and Applied Mathematics
Guru Ghasidas Vishwavidyalaya Bilaspur (C.G.), India
Email: brajbhushan25@gmail.com, pankaj.anvarat@gmail.com

(Received Oct. 17, 2014)

Abstract: In this paper, we have studied conformally flat weakly symmetric, concircularly flat weakly symmetric and W_2-flat weakly symmetric Kähler manifolds and proved that in such type of manifolds either the scalar curvature vanishes or the manifolds are of recurrent type.

2010 Mathematics Subject Classification: 53C25, 53C55.

Keywords: Weakly symmetric manifold, Kähler manifold, concircularly symmetric manifold, W_2-manifold.

1. Introduction

The idea of weakly symmetric manifold is introduced by L. Tamassy and T. Q. Binh[1]. This idea was further extended by M. Prvanovic[2], F. Malek and M. Samavaki[3], Tamassy, De and Binh[4]. In 2006, P. N. Pandey and B. B. Chaturvedi[5] studied almost Hermitian manifold with semi-symmetric recurrent connection and gave some interesting results. In 2000, Tamassy, De and Binh[4] discussed weakly symmetric and weakly Ricci symmetric Kähler manifolds and showed that if the scalar curvature is non-zero constant then the sum of associated 1-forms is zero.

An n-dimensional Riemannian manifold M is said to be weakly symmetric if the curvature tensor R of M satisfies

$$(1.1)\qquad (\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X),$$

where A, B, C, D, E are simultaneously non-vanishing 1-forms and X, Y, Z, U, V are vector fields. In 1995, Prvanovic[2] proved that if M be a weakly symmetric manifold satisfying (1.1) then the 1-forms B, C, D, and E are equal *i.e.* $B = C = D = E$.

In this paper, we have assumed that $B = C = D = E = \omega$ such that $g(X,\rho) = \omega(X)$ and $g(X,\alpha) = A(X)$, for associated vector fields ρ and α

of the 1-forms ω and A respectively. Therefore, the equation (1.1) can be written as

$$(1.2)\quad \begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) \\ &+\omega(Y)R(X,Z,U,V)+\omega(Z)R(Y,X,U,V) \\ &+\omega(U)R(Y,Z,X,V)+\omega(V)R(Y,Z,U,X).\end{aligned}$$

An n (even)-dimensional manifold is said to be Kähler manifold if the following conditions hold:

$$F^2=-X,\quad g(\bar{X},\bar{Y})=g(X,Y),\quad (\nabla_X F)Y=0,$$

where F is a tensor field of type (1,1) such that $F(X)=\bar{X}$, g is a Riemannian metric and ∇ is Levi-Civita connection.

2. Conformally flat weakly symmetric Kähler manifold

If M be a weakly symmetric Kähler manifold then the curvature tensor R satisfies

$$(2.1)\quad R(\bar{Y},\bar{Z},U,V)=R(Y,Z,U,V).$$

Taking covariant derivative of equation (2.1), we can write

$$(2.2)\quad (\nabla_X R)(\bar{Y},\bar{Z},U,V)=(\nabla_X R)(Y,Z,U,V).$$

Using (1.2) in (2.2), we have

$$(2.3)\quad \begin{aligned}\omega(Y)R(X,Z,U,V)+\omega(Z)R(Y,X,U,V) &= \omega(\bar{Y})R(X,\bar{Z},U,V) \\ &+\omega(\bar{Z})R(\bar{Y},X,U,V).\end{aligned}$$

Putting $Z=U=e_i,\ 1\le i\le n$ in (2.3) and summing over i, we get

$$(2.4)\quad \omega(Y)S(X,V)+R(X,Y,V,\rho)=\omega(\bar{Y})S(X,\bar{V})-R(X,\bar{Y},V,\bar{\rho}).$$

We know that the Weyl conformal curvature tensor C on an n (>3)-dimensional manifold M is given by

$$(2.5)\quad \begin{aligned}C(X,Y,Z,T) &= R(X,Y,Z,T)-\frac{1}{(n-2)}[S(Y,Z)g(X,T) \\ &-S(X,Z)g(Y,T)+S(X,T)g(Y,Z)-S(Y,T)g(X,Z)] \\ &+\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,T)-g(X,Z)g(Y,T)].\end{aligned}$$

If the manifold be conformally flat then from (2.5) the expression of the Riemannian curvature tensor R is given by

$$R(X,Y,Z,T)=\frac{1}{(n-2)}[S(Y,Z)g(X,T)-S(X,Z)g(Y,T)+S(X,T)g(Y,Z)-S(Y,T)g(X,Z)]-\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,T)-g(X,Z)g(Y,T)]. \tag{2.6}$$

Using (2.6) in (2.4), we have

$$\begin{aligned}&\omega(Y)S(X,Z)+\frac{1}{(n-2)}[\omega(X)S(Y,Z)\\&-\omega(Y)S(X,Z)+S(X,\rho)g(Y,Z)-S(Y,\rho)g(X,Z)]\\&-\frac{r}{(n-1)(n-2)}[\omega(X)g(Y,Z)-\omega(Y)g(X,Z)]\\&=\omega(\bar{Y})S(X,\bar{Z})-\frac{1}{(n-2)}[\omega(\bar{X})S(Y,\bar{Z})-\omega(Y)S(X,Z)\\&\quad+S(X,\bar{\rho})g(\bar{Y},Z)-S(Y,\rho)g(X,Z)]\\&\quad+\frac{r}{(n-1)(n-2)}[\omega(\bar{X})g(Y,\bar{Z})-\omega(Y)g(X,Z)].\end{aligned} \tag{2.7}$$

Substituting $X=Z=e_i, 1\leq i\leq n$ in (2.7) and summing over i, we get

$$S(Y,\rho)=\frac{r}{2}\omega(Y). \tag{2.8}$$

Also, equation (2.4) can be written as

$$\omega(Y)S(X,V)+R(X,Y,V,\rho)=\omega(\bar{Y})S(X,\bar{V})+R(X,\bar{Y},\bar{V},\rho). \tag{2.9}$$

Using (2.6) in (2.9), we have

$$\begin{aligned}\omega(Y)S(X,Z)-\omega(\bar{Y})S(X,\bar{Z})=&\frac{1}{(n-2)}[\omega(Y)S(X,Z)\\&+S(Y,\rho)g(X,Z)-\omega(\bar{Y})S(X,\bar{Z})\\&-S(\bar{Y},\rho)g(X,\bar{Z})]\\&-\frac{r}{(n-1)(n-2)}[\omega(Y)g(X,Z)-\omega(\bar{Y})g(X,\bar{Z})].\end{aligned} \tag{2.10}$$

Putting $X = Z = e_i, 1 \le i \le n$ and taking summation over i, equation (2.10) yields

$$S(Y,\rho) = \frac{(n^2 - 3n + 3)}{n(n-1)} r\omega(Y). \tag{2.11}$$

Now, using (2.8) in (2.11), we get

$$(n-2)(n-3)r\omega(Y) = 0. \tag{2.12}$$

Since $n>3$, we have $r.\omega(Y)=0$ which implies either $r=0$ or $\omega(Y)=0$.
But if $\omega(Y)=0$ then equation (1.2) reduces to

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V), \tag{2.13}$$

which shows that M is a recurrent manifold.
Thus we can state:

Theorem 2.1: *Let M be a conformally flat weakly symmetric Kähler manifold then either scalar curvature vanishes or M is a recurrent manifold.*

3. Concircularly flat weakly symmetric Kähler manifold

The concircular curvature tensor H of type (0,4) in an n-dimensional Riemannian manifold M is defined by

$$H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{r}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{3.1}$$

If M be concircularly flat then above equation gives

$$R(X,Y,Z,U) = \frac{r}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{3.2}$$

Using (3.2) in (2.4), we have

$$\omega(Y)S(X,Z) - \omega(\bar{Y})S(X,\bar{Z}) + \frac{r}{n(n-1)}[\omega(X)g(Y,Z) + \omega(\bar{X})g(Y,\bar{Z}) - 2\omega(Y)g(X,Z)] = 0. \tag{3.3}$$

Putting $X = Z = e_i, 1 \le i \le n$ and taking summation over i, equation (3.3) yields

$$(n-2)r\omega(Y) = 0. \tag{3.4}$$

Clearly, above equation implies either *n=2* or *r=0* or *ω(Y)=0.*
But if *ω(Y)=0* then equation (1.2) reduces to

(3.5) $$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V),$$

which shows that *M* is a recurrent manifold.
Hence, we have:

Theorem 3.1: *Let M be a concircularly flat weakly symmetric Kähler manifold then for n>2 either scalar curvature vanishes or M is a recurrent manifold.*

4. W_2-flat weakly symmetric Kähler manifold

In 1970, the W_2 curvature tensor is introduced by G. P. Pokhariyal and R. S. Mishra[8] and for *n*-dimensional Riemannian manifold *M,* defined by

(4.1) $$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{(n-1)}[S(Y,U)g(X,Z) - S(X,U)g(Y,Z)].$$

If *M* be W_2-flat then above equation reduces to

(4.2) $$R(X,Y,Z,U) = \frac{1}{(n-1)}[S(X,U)g(Y,Z) - S(Y,U)g(X,Z)].$$

By using (4.2), equation (2.4) gives

(4.3) $$\omega(Y)S(X,Z) - \omega(\bar{Y})S(X,\bar{Z}) + \frac{1}{(n-1)}[S(X,\rho)g(Y,Z) + S(X,\bar{\rho})g(\bar{Y},Z) - 2S(Y,\rho)g(X,Z)] = 0.$$

Putting $X = Z = e_i, 1 \le i \le n$ and taking summation over *i*, equation (4.3) gives

(4.4) $$S(Y,\rho) = \frac{(n-1)r+2}{2n}\omega(Y).$$

Again, using (4.2) in (2.9), we have

(4.5) $$\omega(Y)S(X,Z) - \omega(\bar{Y})S(X,\bar{Z})$$

$$+\frac{1}{(n-1)}[S(\bar{Y},\rho)g(X,\bar{Z})-S(Y,\rho)g(X,Z)]=0.$$

Putting $X=Z=e_i, 1\le i\le n$ and taking summation over *i*, equation (4.5) gives

(4.6) $$S(Y,\rho)=\frac{(n-1)}{n}r\omega(Y).$$

Equations (4.4) and (4.6) together yields

(4.7) $$[(n-1)r-2]\omega(Y)=0,$$

which implies either $r=2/(n-1)$ or $\omega(Y)=0$.

Now, if $\omega(Y)=0$ then equation (1.2) gives

(4.8) $$(\nabla_X R)(Y,Z,U,V)=A(X)R(Y,Z,U,V),$$

which is the condition of recurrent manifold.

Thus we conclude:

Theorem 4.1: *Let M be a* W_2-flat *weakly symmetric Kähler manifold then for n>1 either scalar curvature r=2/(n-1) or M is recurrent manifold.*

References

1. L. Tamassy and T. Q. Binh, On weakly symmetric and weakly projective symmetric manifolds, *Colloq. Math. J. Bolyai*, **56** (1989) 663-670.
2. M. Prvanovic, On weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen,* **46 (1)** (1995) 19-25.
3. F. Malek and M. Samavaki, On weakly symmetric Riemannian manifolds*, Diff. Geom. Dyn. Syst*., **10** (2008) 215-220.
4. L. Tamassy, U. C. De and T. Q. Binh, On weak symmetries of Kähler manifolds, *Balkan Journal of Geometry and Its Applications*, **5** (2000) 149-155.
5. P. N. Pandey and B. B. Chaturvedi, Almost Hermitian manifold with semi-symmetric recurrent connection, *J. Inter. Acad. Phy. Sci*., **10** (2006) 69-74.
6. G. P. Pokhariyal and R. S. Mishra, *Yokohama Math. Journal*, **18** (1970) 105-108.