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A Note on Fixed Point Theorems in Menger Space

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Abstract: In the present paper we employ the notion of reciprocal continuity to obtain a common fixed point theorem in Menger space in which the fixed point may be a point of discontinuity. We also investigate the relationship between continuity of mappings and reciprocal continuity in the setting of Menger spaces. Our result improves the recent result of Singh and Jain[1] in Menger spaces and extends many known results in metric spaces.
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1. Introduction

Menger K.[2] introduced the notion of probabilistic metric space (or statistical space or Menger space) which is a generalization of metric space and the study of this space was expanded rapidly with the pioneering work of Schweizer and Skalar[3] & Stevens[4]. Bharucha Reid[5] set out the tradition of proving fixed point theorems in Menger space. Since that time a substantial literature has been developed on this topic. In recent years, some interesting fixed point theorems for four self maps or a collection of maps satisfying contractive type condition in Menger space have been reported in the literature e g. D. Xieping[6], S. L. Singh[7], Y. J. Cho[8-9], S. N. Mishra[10], B. Singh[1,11,12,13,14], Kutukchu[15,16]. These theorems invariably require a commutative or compatibility condition and a contractive condition besides assuming continuity of at least one of the mappings and each theorems aims at weakening one or more of these conditions.

The present paper is an attempt to obtain a common fixed point theorem by replacing continuity condition with a weaker condition called reciprocal continuity. We also show by means of an example that in the setting of fixed point theorem of Singh et al[1], the notion of reciprocal continuity is actually weaker then the assumption of continuity of one of the mappings. Using the notion of reciprocal continuity of mappings we can widen the scope of many interesting fixed point theorems on Menger spaces as well as fuzzy metric spaces (e g. Kutukchu[15-16], B. Singh et al[1,11,12,13,14], R. Chug[17], Hong[9], Khan et al[18]).

2. Preliminaries

Definition[5] 1: A mapping F: R → R^+ is called a distribution if it is non-decreasing left continuity with inf {F(t): t ∈R } = 0 and sup {F(t): t ∈ R } = 1. We shall denote by L the set of all distribution function defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Definition[5] 2: A probabilistic metric space (PM-space) is an ordered pair (X, F) where X is an abstract set of elements and F: X × X→ L is defined by (p, q) ∈ Fp,q where L is the set of all distribution function i.e. L = {$F_{p,q}$: p, q ∈ X} where the function $F_{p,q}$ satisfy:

(a) $F_{p,q}(x) = 1$ for all x > 0 iff p = q
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q} = F_{q,p}$;
(d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$,
where x, y ∈ R the set of real numbers.

Definition[5] 3: A mapping t: [0, 1] × [0, 1] → [0, 1] is called a t-norm if

(a) t(a,1) = a, t(0, 0) = 0
(b) t(a, b) = t(b, a)
(c) t(c, d) ≥ t(a, b) for c ≥ a, d ≥ b
(d) t(t(a, b),c) = t(a, t(b, c)).

Definition[5] 4: A Menger space is a triplet (X, F, t) where (X, F) is PM-space and t is a t-norm such that for all p, q, r ∈ X and for all x, y > 0

$$F_{p,q}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y))$$

Proposition[5] 1: *If (X, d) is a metric space then the metric d induces a mapping F: X × X→ L, defined by $F_{p,q}(x) = H(x - d(p, q))$, p, q ∈ X and x ∈ R. Further, if the t-norm t: [0,1] × [0,1] → [0,1] is defined by t(a, b) = min(a, b), then (X, F, t) is a Menger space. It is complete if (X, d) is complete.*

The space (X, F, t) so obtained is called the induced Menger space.

Definition[10] 5: A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \to p$) if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer M (ε, λ) such that $F_{pn, q}(\varepsilon) > 1-\lambda$ for all $n \geq M(\varepsilon, \lambda)$.

The sequence is said to be Cauchy sequence if for each $\varepsilon > 0$ and $\lambda > 0$ there exists an integer M (ε,λ) such that $F_{pn, pm}(\varepsilon) \geq 1 - \lambda$ for all n, m $\geq$ M (ε,λ).

A Menger space is said to be complete if every Cauchy sequence converges to a point of it.

Definition[1] 6: Self-maps A and S of a Menger space (X, F, t) is said to be weakly compatible (or coincidently commuting) if they commute at their coincidence point, i.e. if $A_p = S_p$ for some $p \in X$ then $AS_p = SA_p$.

Definition[10] 7: Self-mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASpn,\, SApn}(x) \to 1$ for all $x > 0$, whenever $\{p_n\}$ is a sequence in X such that $\{A_{pn}\}, \{S_{pn}\} \to u$, for some $u \in X$ as $n \to \infty$.

Proposition 2: *Self-maps A and S of a Menger space (X, F, t) are compatible then they are weakly compatible.*

[However, the converse of the above proposition is need not be true as shown in example 3.2 below]

Definition 8: Let A and S be two self maps of a Menger space (X, F, t), we will call A and S to be reciprocally continuous if $\lim_{n\to\infty} AS_{pn} = Au$ and $\lim_{n\to\infty} SA_{pn} = Su$, whenever $\{p_n\}$ is a sequence in X such that $A_{pn}, S_{pn} \to u$ as $n \to \infty$ for some $u \in X$.

We observe that if A and S both are continuous then they are obviously reciprocally continuous but the converse need not be true as shown in our example 3.1 below.

Lemma[19] 1: *In a Menger space (X, F, t), $t(x, x) = x$, $\forall x \in [0, 1]$ if and only if $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$.*

In view of above and as observed by Xiao & Zhu[20] it is clear that only t-norm satisfying $t(a, a) \geq a$ is min t-norm and so the number of authors (e g. Cho[21,8], Cho[9], Kutucku[16-22], Khan[18], B. Singh et al[1,12,14], Barucha Ried et al[5], Sharma[23]) assuming $t(x, x) \geq x$ to obtain common fixed point in Menger spaces as well as fuzzy metric spaces reduces to the assumptions that $t(a, a) = a$.

Not only this but we have already a lower as well as upper bound for t-norm in the following result:

Lemma[19] 2: *$i_{min}(a, b) \leq i_p(a, b) \leq \min(a, b)$.*

Lemma[10] 3: *Let (X, F, t) be a Menger space if there exists* $k \in (0, 1)$ *such that for* $p, q \in X$, $F_{p, q}(kx) \geq F_{p, q}(x)$, *then* $p = q$.

3. Main Results

Theorem 3.1: *Let A, B, S, T, L and M are self maps on a complete Menger Space(X,F, t) (where t is any continuous t-norm) for all* $a \in [0, 1]$ *satisfying:*

(3.1.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(3.1.2) $AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$

(3.1.3) *(M, ST) is weakly compatible*

(3.1.4) *there exists* $k \in (0, 1)$ *such that*

$$F_{Lp, Mq} \geq \min \{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(x), F_{ABp, Mq}((2-\beta)x), F_{ABp, STq}(x)\}$$

for all $p, q \in X$, $\beta \in (0, 2)$ *and* $x > 0$. *Then the continuity of one of the mappings in compatible pair (L, AB) implies their reciprocal continuity.*

Proof: Suppose that AB is continuous in the compatible pair of mappings L and AB. We claim that (L, AB) are reciprocal continuous. Let $\{x_n\}$ be any sequence in X such that $\lim_{n\to\infty} Lx_n = z$ and $\lim_{n\to\infty} ABx_n = z$ for some $z \in X$. To prove our assertion we shall show that $LABx_n \to Lz$ and $ABLx_n \to ABz$ as $n \to \infty$.

Since AB is continuous we get, $ABABx_n \to ABz$ and $ABLx_n \to ABz$ as $n\to \infty$. Now compatibility of L and AB implies that $\lim_{n\to\infty} F_{LABxn, ABLxn} = 1$, i.e. $LABx_n \to ABz$ as $n\to \infty$. Also since $L(X) \subseteq ST(X)$, for each n, there exists $\{y_n\}$ in X such that $LABx_n = STy_n$. Thus $ABABx_n \to ABz$, $LABx_n \to ABz$, $ABLx_n \to ABz$ and $STy_n \to ABz$ as $n\to \infty$. Now we shall show that $My_n \to ABz$ as $n\to \infty$. For this, from (3.1.5) we have,

$$\begin{aligned} F_{ABz, Myn}(kx) &= F_{LABxn, Myn}(kx) \\ &\geq \min \{F_{ABABxn, LABxn}(x), F_{STyn, Myn}(x), F_{STyn, LABxn}(Bx), F_{ABABxn, Mym}((2-\beta)x), F_{ABABxn, STyn}(x)\} \end{aligned}$$

which implies that $My_n \to ABz$ as $m, n \to \infty$ (by lemma 2 and taking $\beta = 1$). Now the inequality,

$$\begin{aligned} F_{Lz, ABz}(kx) &= F_{Lz, Myn}(kx) \\ &\geq \min \{F_{ABz, Lz}(x), F_{STyn, Myn}(x), F_{STyn, Lz}(Bx), \\ &\quad F_{ABz, Myn}((2-\beta)x), F_{ABz, STyn}(x)\} \end{aligned}$$

which implies, $Lz = ABz$ as $n \to \infty$ (by Lemma 2 and taking $\beta = 1$). Thus $ABLx_n \to ABz$ and $ABLx_n \to ABz = Lz$ as $n \to \infty$ .Therefore, L and AB are reciprocal continuous in (X, F, t).

Suppose that L is continuous in the compatible pair of mappings L and AB. We claim that (L, AB) is reciprocal continuous. Let $\{x_n\}$ be any sequence in X such that $Lx_n \to z$ and $ABx_n \to z$ as $n \to \infty$ for some $z \in X$. To prove our assertion, we shall show that $LABx_n \to Lz$ and $ABLx_n \to ABz$ as $n \to \infty$. Since L is continuous, we get $LLx_n \to Lz$, $LABx_n \to Lz$ as $n \to \infty$. Now compatibility of L and AB gives us $ABLx_n \to Lz$ as $n \to \infty$. Now using step (8) and (9) of the proof of theorem 2.1 of Singh et al[1] we get Lz = ABz which implies that $ABLx_n = Lz = ABz$ as $n \to \infty$. Therefore L and AB are reciprocal continuous in (X, F, t).

In the above theorem we have shown that in the setting of the theorem 2.1 of Singh et al[1] continuity of one of the mappings in compatible pair implies their reciprocal continuity. Therefore the condition (2.1.3) of continuity of one of the mapping in compatible pair (L, AB) can be further replaced by the weaker notion of reciprocal continuity which still assume the existence of common fixed point for maps but does not force the maps to be continuity even at common fixed point.

The following theorem was proved by Singh & Jain[1]

Theorem[1] 3.2: *Let A,B,S,T and M are self maps on a complete Menger space (X, F, t) with $t(a, a) \geq a$ for all $a \in [0,1]$ satisfying:*

(3.2.1) *$L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$*

(3.2.2) *AB = BA, ST = TS, LB = BL, MT = TM*

(3.2.3) *either AB or L is continuous*

(3.2.4) *(L, AB) is compatible and (M, ST) is weakly compatible*

(3.2.5) *there exists $k \in (0, 1)$ such that*

$$F_{LpMq} \geq \min \{F_{ABp,\,Lp}(x),\ F_{STq,\,Mq}(x),\ F_{STq,\,Lp}(x),\ F_{ABp,\,Mq}((2-\beta)x),\ F_{ABp,\,STq}(x)\}$$

for all $p, q \in X$, $\beta \in (0, 2)$ and $x > 0$. Then A, B, S, T and M have a unique common fixed point in X.

Now as an application of the relationship between continuity of the mappings and reciprocal continuity established in the above theorem 3.1, we now prove the following theorem which improves the result of Singh et al[1] and presents an example which demonstrates that the notion of reciprocal continuity of mappings is weaker than the continuous map.

Theorem 3.3: *Let A, B, S, T, L and M are self maps on a complete Menger space (X, F, t) with $t(a, a) = a$ for all $a \in [0, 1]$ satisfying conditions (3.2.1), (3.2.2), (3.2.4) and (3.2.5) of the above theorem 3.1. Suppose that (L, AB) is compatible pair of reciprocal continuous mappings. Then all the maps A, B, S, T, L and M have a unique common fixed point.*

Proof: let $x_0 \in X$, from condition (3.2.1) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+1} = y_{2n+1}$ for n = 0, 1, 2, 3,….. .

Then following the argument by Singh et al[3] we have,

$$F_{yn,\, yn+1}(kx) \geq \min\{F_{yn-1,\, yn}(x), F_{yn,\, yn+1}(x)\}.$$

Since $F_{p,\, q}(.)$ is non-decreasing therefore, we get

(3.3.1) $$F_{yn,\, yn+1}(kx) \geq F_{yn-1,\, yn}(x)$$

To prove $\{y_n\}$ is a Cauchy sequence, we prove (3.3.2) is true for all $n \geq n_0$ and for every $m \in N$,

(3.3.2) $$F_{yn,\, yn+m}(kx) > 1-\lambda \qquad \text{for } t > 0,\ \lambda \in (0, 1)$$

Hence from (3.3.1) we have,

$$F_{yn,\, yn+1}(kx) \geq F_{yn-1,\, yn}(xk^{-1}) \geq F_{yn-2,\, yn-1}(xk^{-2}) \geq \ldots\ldots \geq F_{y0,\, y1}(xk^{-n}) \rightarrow$$

1 as $n \rightarrow \infty$.

Thus (3.3.2) is true for m = 1. Suppose (3.3.2) is true for m then we shall show that this is also true for m+1. For this, using the definition of Menger space, (3.3.1) and (3.3.2) we have,

$$F_{yn,\, yn+m+1}(x) \geq t\,(F_{yn,\, yn+m}(x/2), F_{yn+m,\, yn+m+1}(x/2))$$
$$= \min (F_{yn,\, yn+m}(x/2), F_{yn+m,\, yn+m+1}(x/2)) > 1-\lambda .$$

Hence (3.3.2) is true for m+1. Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete hence $\{y_n\} \rightarrow z$ in X. Also its subsequences converge as follows:

$$\{Mx_{2n+1}\} \rightarrow z \quad \text{and} \quad \{STx_{2n+1}\} \rightarrow z$$
$$\{Lx_{2n}\} \rightarrow z \quad \text{and} \quad \{ABx_{2n+1}\} \rightarrow z$$

Now reciprocal continuity and compatibility of the pair (L, AB) gives us

$$LABx_{2n} \rightarrow Lz \text{ and } ABLx_{2n} \rightarrow ABz \text{ and } \lim_{n\rightarrow\infty} (F_{LAB\, x2n\, ,ABL\, x2n}) = 1$$

i.e. $F_{Lz,ABz}(x) = 1$. Hence Lz = ABz.

Now putting $p = ABx_{2n}$, $q = x_{2n+1}$ with $\beta = 1$ in contractive condition and using lemma 2 we get ABz = z. Thus Lz = ABz = z.

To conclude the proof we can follow step 4 to step 10 of the proof of theorem 2.1 of B. Singh et al[1].

We now give an example, which not only illustrate our theorem 3.2 but also show that the notion of reciprocal continuity is weaker than the continuity condition of maps.

Example 3.1: Let (X, d) be a metric space where X = [0, 3] and (X, F, t) be the induced Menger space with $F_{p,\, q}(\varepsilon) = H(\varepsilon - d(p, q))$, for all $p, q \in X$ and for all $\varepsilon > 0$ and $t(a, b) = \min(a, b)$, for all $a, b \in [0, 1]$. . Define self maps A, B, S, T, L and M on X as follows:

$Lx = 1$ if $0 \leq x < 2$ and $2 < x \leq 3$, $L2 = 0$;

$Mx = 0$ if $0 \leq x < 1$, $1 < x < 2$, $2 < x \leq 3$, $M1 = M2 = 1$;

(A = B) $Ax = 0$ if $0 \leq x < 1$, $1 < x < 2$, $2 < x \leq 3$, $A1 = A3 = 1$, $A2 = 2$;
$Sx = 0$ if $0 < x < 1$, $1 < x < 2$, $2 < x < 3$ $S1 = 1$, $S2 = 0$, $S0 = S3 = 2$;
$Tx = 0$ if $0 \leq x < 1$, $1 < x < 2$, $2 < x \leq 3$ $T1 = 1$, $T2 = 2$.

Then the maps A(= B), S, T, L, M satisfy all the conditions of the above theorem 3.3 with $k \in (1/2, 1)$ and $\beta = 1$ and have a unique common fixed point $x = 1$. It may be noted that in this example $L(X) = \{0,1\} \subseteq ST(X) = \{0,1,2\}$, $M(X) = \{0,1\} \subseteq AB(X) = \{0,1,2\}$and the pair (L, AB) is reciprocally continuous for a sequence $\{x_n\} = \{1\}$ in X. Also (L, AB) is commuting maps and hence compatible. But neither L nor AB is continuous.

Remark1: The maps A (= B), S, T and M are discontinuous even at the common fixed point $x = 1$.

Remark2: The known common fixed point theorems involving a collection of maps in Menger spaces as well as fuzzy metric spaces require one of the maps in compatible pair to be continuous. For example, main theorems of B. Singh et al[1,11,12,13,14] assumes at least one of the maps to be continuous in compatible pair of maps. Likewise, theorem 3.1 of Kutukcu[15] assumes either AB or L to be continuous maps. One more theorems of Kutukcu[16] assume the mappings S to be continuous and (S, T_n) to commuting pair of maps in Menger spaces. Similarly, Hong[9] assumes S and T to be continuous mapping and the main theorem of R.Chug et al[17] assume one of the mappings A, B, S or T to be continuous in fuzzy metric spaces. The present theorem however does not require any of the mappings to be continuous and hence all the results mentioned above can be further improved and generalized in the spirit of our theorem 3.3. Further, since every metric space induces a Menger space. Thus our theorem 3.2 above extends the results of R. P. Pant[24,25,26], Fishrer[27], Jungck[28,29], Jachymski[30] for six mappings in metric spaces.

Remark 3: It is obvious that in most of the fixed point theorems in Menger spaces as well as fuzzy metric spaces to prove the sequence of iterates of a point is a Cauchy sequence a particular class of t-norm is required. In our theorem 3.2 above we have assumed the t-norm as min norm, however, adopting the approach of Liu et al[22] one can easily replace the condition of min norm by a larger class of t-norm called Hadzic type t-norm (in short H type t-norm). The work along this line has been done in our paper[31] recently communicated)

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