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A Batch Arrival Retrial Queue with Bernoulli Vacation Policy and Server Breakdown*

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Abstract: This paper examines the operating characteristics of an $M^X/G/1$ retrial queueing system under Bernoulli vacation schedule with setup times. The server renders first phase of essential service (FPS) to all the arriving units whereas second phase of multi-optional services (SPS) to only some of them who demand for the same. The server may breakdown according to Poisson process in working state i.e. either during FPS or SPS. The broken down server is then repaired by the repairman and becomes as good as before failure. When the server just after a service or repair completion finds no customers waiting to be served, he departs for a single vacation of arbitrary distributed length according to Bernoulli schedule. When the batch of arriving customers finds the server busy or on vacation, then the whole batch either with probability p joins a pool of blocked customers called 'orbit' or with complementary probability $\overline{p}$ (=1-p) leaves the system. On the other hand, if the arriving batch finds the server idle, then one of the customers from the batch starts its service and the rest join the 'orbit'. The service time, repair time, vacation time and setup time are assumed to be general distributed. Using supplementary variable technique and generating function method, the explicit expressions for the average system size, average orbit size and other performance indices have been determined.

Keywords: Retrial queue, Batch arrival, Two-phase service, Bernoulli vacation, Unreliable server, Setup time.
2000 Mathematics Subject Classification No.: 60K25

1. Introduction

Retrial queues are widely used as a mathematical tool to model several systems such as computer systems, packet switching networks, shared bus local area networks operating under the CSMA (Carrier-Sense Multiple Access) protocol and collision avoidance star local area networks, etc.. The first work on the M/G/1 retrial queue with general retrial times was due to Falin[1]. Krishna Kumar and Madheswari[2] investigated $M^x/G/1$ retrial queue

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with multiple vacations and starting failures. Atencia and Moreno[3] considered an M/G/1 retrial queue with general retrial times. An M/G/1 retrial queue with active breakdowns and Bernoulli schedule in the server was developed by Atencia et al.[4]. They have considered both classical and constant retrial policies. Further Atencia et al.[5] have applied the concept of batch arrival in this model. Mokaddis et al.[6] have studied the M/G/1 retrial queue with Bernoulli feedback and single vacation where the server is subjected to starting failure. Aguir et al.[7] modeled a call centre as a continuous time Markov chain with retrial phenomenon. In recent years, there have been several contributions based on Poisson input queueing systems wherein the server delivers a second phase of optional service followed by the first phase of essential service (cf. Choudhury and Deka[8], Wang and Xu[9] and Wang and Li [10]).

In this paper, we analyze the steady state behaviour of a batch arrival repairable retrial queueing system with Bernoulli vacation schedule and two phase service. The rest of the paper is structured as follows. The notations and assumptions to formulate the concerned model have been mentioned in section 2. In section 3, supplementary variable technique and generating function method are used to determine the queue size distribution. Various performance indices are computed in section 4. Numerical results and sensitivity analysis are presented in section 5. In final section 6, conclusion has been drawn.

2. Model Description

Consider an $M^X/G/1$ retrial queue wherein the customers arrive in the system according to Poisson process with rate dependent upon server status as

$$\lambda_n = \begin{cases} \lambda_0, \text{ if the server is idle} \\ \lambda_1, \text{ if the server is busy either during FPS or SPS} \\ \lambda_2, \text{ if the broken down server is under repair} \\ \lambda_3, \text{ if the server is on vacation} \\ \lambda_4, \text{ if the server is under setup} \end{cases}$$

If the server is busy, broken down or on vacation at the arrival epoch, then all the arriving customers join the orbit. On the contrary, if the server is free then he starts the service of the arriving customer from the batch which is on the head of the queue whereas others leave the service area and enter a group of blocked customers (i.e. orbit). Let X be a random variable denoting batch size having probability function defined by Pr $\{X=k\}=c_k$, $k \geq 1$ and

$\sum_{k=1}^{\infty} c_k = 1$. We denote first and second moments of batch size by c_1 and c_2 respectively, so that c_1=E(X) and c_2=E(X^2). When a batch of customers arrives to the service station, the server takes setup time before initiating the service to the customers. When the setup time of the server finishes, he starts providing service to one of the customers of the incoming batch. The FPS is requested by all the arriving customers. As soon as the FPS of a customer finishes, he may opt for any of the l different kinds of optional services with probability q_i ($1 \le i \le l$), otherwise leaves the system with probability $\bar{q}_i$ (=1- q_i). The service is done according to FCFS discipline.

Table 1: Notations

Time	r. v.	DF	pdf	LST	Mean rate	k^{th} moments	Hazard rate
Retrial	A_e	$A_e(t)$	$a_e(t)$	$a_e^*(s)$	a	$a^{(k)}$	$\gamma(w) = \frac{a_e(w)}{\bar{A}_e(w)}$
FPS	S_0	$S_0(t)$	$s_0(t)$	$b_0^*(s)$	μ	$b_0^{(k)}$	$\mu_0(x) = \frac{s_0(x)}{\bar{S}_0(x)}$
SPS	S_i $(1 \le i \le l)$	$S_i(t)$ $(1 \le i \le l)$	$s_i(t)$ $(1 \le i \le l)$	$b_i^*(s)$ $(1 \le i \le l)$	b_i $(1 \le i \le l)$	$b_i^{(k)}$ $(1 \le i \le l)$	$\mu_i(x) = \frac{s_i(x)}{\bar{S}_i(x)}$ $(1 \le i \le l)$
Vacation	V	F(t)	f(t)	$\xi^*(s)$	ν	$\xi^{(k)}$	$\xi(u) = \frac{f(u)}{\bar{F}(u)}$
Repair	G_i	$G_i(t)$	$g_i(t)$	$g_i^*(s)$	β_i	$g_i^{(k)}$	$\eta_i(y) = \frac{g_i(y)}{\bar{G}_i(y)}$ $(1 \le i \le l)$
Setup	U	U(t)	u(t)	$\zeta^*(s)$	ζ	$\zeta^{(k)}$	$\zeta(v) = \frac{u(v)}{\bar{U}(u)}$

The server may breakdown in working state (either in FPS or SPS) and is immediately sent for repair. When server failure occurs, the customer in service waits for the repair of the server to complete its remaining service. We assume that the server's life time is exponentially distributed with mean $1/\alpha_0$ during FPS and $1/\alpha_i$ ($1 \le i \le l$) during SPS of i^{th} type. After each service completion, the server may go for vacation of random length with probability θ ($0 \le \theta \le 1$) or may continue to serve the next customer if any with probability $\bar{\theta} = (1-\theta)$. Table 1 summarizes the notations used for some random variables (r.v.) along with their distribution function (DF), probability density function (pdf), Laplace Steiltjes Transform (LST), k^{th}

($k \geq 1$) moment and hazard rates. $\overline{H}(.)$ denotes the complementary distribution function of H.

The transient probabilities of different states of the system are defined as follows:

$D_n(t)$: Probability that the server is idle during retrial of the customers at time t when there are n customers in the system.

$P_{0,n}(t,u)$: Probability that there are n customers in the system at time t and the server is busy in providing FPS to the customer and the customer is being served with elapsed service time lying between u and u+du.

$P_{i,n}(t,u)$: Probability that there are n customers in the system at time t and the server is busy in providing i^{th} $(1 \leq i \leq l)$ optional SPS to the customer and the customer is being served with elapsed service time lying between u and u+du.

$Q_n(t,u)$: Probability that there are n customers in the system at time t and the server is on vacation with elapsed vacation time lying between u and u+du.

$W_n(t,u)$: Probability that there are n customers in the system at time t and the server is under setup with elapsed setup time lying between u and u+du.

$R_{0,n}(t,u,v)$: Joint probability that there are n customers in the system at time t, the elapsed service time of the customer undergoing FPS is equal to u and the server is undergoing the repair with elapsed repair time lying between v and v+dv .

$R_{i,n}(t,u,v)$: Joint probability that there are n customers in the system at time t, the elapsed service time of the customer undergoing i^{th} $(1 \leq i \leq l)$ optional SPS is equal to u and the server is undergoing the repair with elapsed repair time lying between v and v+dv .

3. Queue Size Distribution at a Random Epoch

In this section, we employ the supplementary variable technique (SVT) to convert non-markovian $M^X/G/1$ queue into markovian one. Let $N_q(t)$ denotes the number of customers in the retrial queue at time t. The state of the system at time t can be described by means of the process X(t)={C(t), $N_q(t)$, ζ (t)} where C(t) is equal to 0, 1, 2, 3 or 4 according to whether the server is idle, busy with FES or busy with i^{th} $(1 \leq i \leq l)$ optional SPS, broken down, under vacation or under setup, respectively. If $C(t) \in \{1, 2, ..., l\}$, then ζ (t) represents the corresponding elapsed time of the process. Let us define the limiting probabilities at the steady state of system as

$D_n = \lim_{t\to\infty} D_n(t)$, $n \geq 0$; $\mathrm{P}_{i,n}(u) = \lim_{t\to\infty} \mathrm{P}_{i,n}(t,u)$, $R_{i,n}(u,v) = \lim_{t\to\infty} R_{i,n}(t,u,v)$; $n \geq 1$, $0 \leq i \leq l$; $Q_n(u) = \lim_{t\to\infty} Q_n(t,u)$, $n \geq 1$; $W_n(u) = \lim_{t\to\infty} W_n(t,u)$, $n \geq 1$.

The steady state equations of different states of the system are as follows:

(3.1) $$\lambda D_0 = \int_0^\infty W_0(v)\zeta(v)dv + \theta q_0 \int_0^\infty P_{0,0}(x)\mu_0(x)dx + \theta \sum_{i=1}^{l} \int_0^\infty P_{0,i}(x)\mu_i(x)dx,$$

(3.2) $$\left[\frac{d}{dw} + \lambda_0 + \gamma(w)\right] D_n(w) = 0;\ \mathrm{n} \geq 1,$$

(3.3) $$\left[\frac{d}{dx} + p\lambda_1 + \mu_0(x) + \alpha_0\right] P_{0,n}(x) = p\lambda_1 \sum_{m=1}^{n} c_m P_{0,n-m}(x) + \int_0^\infty R_{0,n}(x,y)\eta_0(y)dy;\ \mathrm{n} \geq 1,$$

(3.4) $$\left[\frac{d}{dx} + p\lambda_1 + \mu_i(x) + \alpha_i\right] P_{i,n}(x) = p\lambda_1 \sum_{m=1}^{n} c_m P_{i,n-m}(x) + \int_0^\infty R_{i,n}(x,y)\eta_i(y)dy;$$

$\mathrm{n} \geq 0,\ 1 \leq i \leq l$

(3.5) $$\left[\frac{\partial}{\partial y} + p\lambda_2 + \eta_0(y)\right] R_{0,n}(x,y) = p\lambda_2 \sum_{m=1}^{n} c_m R_{0,n-m}(x,y);\ \mathrm{n} \geq 1$$

(3.6) $$\left[\frac{\partial}{\partial y} + p\lambda_2 + \eta_i(y)\right] R_{i,n}(x,y) = p\lambda_2 \sum_{m=1}^{n} c_m R_{i,n-m}(x,y);\ \mathrm{n} \geq 1;\ 1 \leq i \leq l$$

(3.7) $$\left[\frac{d}{du} + p\lambda_3 + \xi(u)\right] Q_n(u) = p\lambda_3 \sum_{m=1}^{n} c_m Q_{n-m}(u);\ \mathrm{n} \geq 1$$

(3.8) $$\left[\frac{d}{dv} + p\lambda_4 + \zeta(v)\right] W_n(u) = p\lambda_4 \sum_{m=1}^{n} c_m W_{n-m}(v);\ \mathrm{n} \geq 1$$

The boundary conditions under steady state are

(3.9) $$D_n(0) = \theta q_0 \int_0^\infty P_{0,n}(x)\mu_0(x)dx + \theta \sum_{i=1}^{l} \int_0^\infty P_{i,n}(x)\mu_i(x)dx + \int_0^\infty W_n(v)\zeta(v)dv;\ \mathrm{n} \geq 1$$

(3.10) $$P_{0,0}(0) = \int_0^\infty D_1(w)\gamma(w)dw + p\lambda \int_0^\infty D_1(w)dw$$

(3.11) $$P_{0,n}(0) = \int_0^\infty D_{n+1}(w)\gamma(w)dw + p\lambda \int_0^\infty D_{n+1}(w)dw + \bar{p}\lambda \int_0^\infty D_n(w)dw;\ \mathrm{n} \geq 0$$

(3.12) $$P_{i,n}(0) = q_i \int_0^\infty P_{0,n}(x)\mu_0(x)dx\,;\ \mathrm{n} \ge 1;\ 1 \le i \le l$$

(3.13) $$R_{0,n}(x,0) = \alpha_0 P_{0,n}(x)\,;\ \mathrm{n} \ge 0$$

(3.14) $$R_{i,n}(x,0) = \alpha_i P_{i,n}(x)\,;\ \mathrm{n} \ge 0;\ 1 \le i \le l$$

(3.15) $$Q_n(0) = (1-\theta)q_0 \int_0^\infty P_{0,n}(x)\mu_0(x)dx + (1-\theta)\sum_{i=1}^{l}\int_0^\infty P_{i,n}(x)\mu_i(x)dx\,;\ \mathrm{n} \ge 0$$

(3.16) $$W_n(0) = \int_0^\infty Q_n(u)\xi(u)du\,;\ \mathrm{n} \ge 0$$

We define probability generating functions as follows:

$$D(w,z) = \sum_{n=0}^{\infty} D_n(w)z^n\,;\quad \mathrm{P}_i(x,z) = \sum_{n=1}^{\infty} \mathrm{P}_{i,n}(x)z^n\,,\quad R_i(z,x,y) = \sum_{n=1}^{\infty} R_{i,n}(x,y)z^n\,;$$

$$0 \le i \le l;\ Q(u,z) = \sum_{n=0}^{\infty} Q_n(u)z^n\,;\ C(z) = \sum_{x=1}^{\infty} c_x z^x\,;\ W(v,z) = \sum_{n=0}^{\infty} W_n(v)z^n\,.$$

Theorem 1: *The marginal probability generating functions at random epochs when the server is idle during retrial time, busy with i^{th} $(1 \le i \le l)$ phase service, on vacation state, under setup state and under repair state while broken down during the i^{th} phase service, respectively, are given by*

(3.17) $$D(z) = \int_0^\infty Q(z,w)dw = \frac{D_0\left[X(z)\{\theta+(1-\theta)Y(z)\}-1\right]\left[1-a_e^*(\lambda)\right]}{\left[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}\right]},$$

(3.18) $$P_0(z) = \int_0^\infty \mathrm{P}_0(z,x)dx = \frac{\lambda_0 D_0\left[1-N(z)\right]}{\left[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}\right]} \times \frac{\left[1-b_0^*(\psi_0(z))\right]}{\psi_0(z)},$$

(3.19) $$P_i(z) = \int_0^\infty \mathrm{P}_i(z,x)dx = \frac{\lambda_0 q_i D_0\left[1-N(z)\right]b_0^*(\psi_0(z))}{\left[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}\right]} \times \frac{\left[1-b_i^*(\psi_i(z))\right]}{\psi_i(z)}\,;\ 1 \le i \le l$$

(3.20) $$R_0(z) = \int_0^\infty\int_0^\infty R_0(z,x,y)dxdy = \frac{\lambda_0\alpha_0 D_0\left[1-N(z)\right]}{\left[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}\right]} \times \frac{\left[1-b_0^*(\psi_0(z))\right]}{\psi_0(z)} \times \frac{\left[1-g_0^*(p\lambda_2\bar{C}(z))\right]}{p\lambda_2\bar{C}(z)}$$

(3.21)
$$R_i(z)=\int_0^\infty\int_0^\infty R_i(z,x,y)\,dxdy=\frac{\lambda_0\alpha_0 q_i D_0[1-N(z)]b_0^*(\psi_0(z))}{[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}]}\times\frac{[1-b_i^*(\psi_i(z))]}{\psi_i(z)}\times\frac{[1-g_i^*(p\lambda_2\bar{C}(z))]}{p\lambda_2\bar{C}(z)}\quad ;1\le i\le l$$

(3.22)
$$Q(z)=\int_0^\infty Q(z,u)du=\frac{\lambda_0(1-\theta)D_0[1-N(z)]X(z)}{[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}]}\times\frac{[1-\xi^*(p\lambda_3\bar{C}(z))]}{p\lambda_3\bar{C}(z)},$$

(3.23)
$$W(z)=\int_0^\infty W(z,v)dv=\frac{\lambda_0(1-\theta)D_0[1-N(z)]X(z)\xi^*(p\lambda_3\bar{C}(z))}{[1-N(z)X(z)\{\theta+(1-\theta)Y(z)\}]}\times\frac{[1-\zeta^*(p\lambda_4\bar{C}(z))]}{p\lambda_4\bar{C}(z)}$$

where

$$N(z)=\frac{[1+\bar{p}(z-1)\{1-a_e^*(\lambda_0)\}]}{z};\ X(z)=q_0b_0^*(\psi_0(z))+b_0^*(\psi_0(z))\sum_{i=1}^{l}q_ib_i^*(\psi_i(z));$$

$$Y(z)=\xi^*(p\lambda_3\bar{C}(z))\zeta^*(p\lambda_4\bar{C}(z));\quad \psi_i(z)=p\lambda_1\bar{C}(z)+\alpha_i\{1-g_i^*(p\lambda_2\bar{C}(z))\};\quad 0\le i\le l;$$

$$\bar{C}(z)=1-C(z),\qquad \rho=\lambda_1b_0^{(1)}+\alpha_0\lambda_2b_0^{(1)}g_0^{(1)}+\sum_{i=1}^{l}q_i\{\lambda_1b_i^{(1)}+\alpha_0\lambda_2b_i^{(1)}g_i^{(1)}\}.$$

Proof: Multiplying eqs (3.2)-(3.16) by the appropriate powers of z and summing over n=0,1,2,3,……., and then solving, we get the partial probability generating functions at random epochs for different states of the system which is then used for obtaining the marginal probability generating function for different states of the server.

4. Performance Indices

In this section, we establish some queueing measures as follows:

Theorem 2: *The probability that the server and system are all idle $P(I_s)$, server is idle but system is non-empty $P(I)$, server is busy $P(B)$, server is under repair $P(R)$, server is on vacation $P(V)$ and server is in setup period $P(S)$, respectively are given by*

(3.24)
$$\mathrm{P}(\mathrm{I}_s)=D_0,$$

(3.25)
$$P(I)=\lim_{z\to1}D(z)=\frac{D_0[1-a_e^*(\lambda_0)][(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}+pc_1\rho]}{[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}]},$$

(3.26) $$P(B_0)=\lim_{z\to 1}P_0(z)=\frac{D_0\left[p+\bar{p}a_e^*(\lambda_0)\right]\rho_0}{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]},$$

(3.27) $$P(B_i)=\lim_{z\to 1}P_i(z)=\frac{D_0\left[p+\bar{p}a_e^*(\lambda_0)\right]q_i\rho_i}{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]} \quad ;1\le i\le l$$

(3.28) $$P(R_0)=\lim_{z\to 1}R_0(z)=\frac{D_0\left[p+\bar{p}a_e^*(\lambda_0)\right]\rho_0\alpha_0 g_0^{(1)}}{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]},$$

(3.29) $$P(R_i)=\lim_{z\to 1}R_i(z)=\frac{D_0\left[p+\bar{p}a_e^*(\lambda_0)\right]q_i\rho_i\alpha_i g_i^{(1)}}{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]} \quad ;1\le i\le l$$

(3.30) $$P(V)=\lim_{z\to 1}Q(z)=\frac{\lambda\xi^{(1)}(1-\theta)D_0\left[p+\bar{p}a_e^*(\lambda_0)\right]}{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]},$$

(3.31) $$P(S)=\lim_{z\to 1}W(z)=\frac{\lambda\zeta^{(1)}(1-\theta)D_0\left[p+\bar{p}a_e^*(\lambda)\right]}{\left[p+\bar{p}a_e^*(\lambda)-pc_1\rho-(1-\theta)pc_1\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]},$$

where $\rho_0=\lambda_0 b_0^{(1)}$; $\rho_i=\lambda_0 b_i^{(1)}$, $1\le i\le l$;

$$\rho=\lambda_1 b_0^{(1)}+\alpha_0\lambda_2 b_0^{(1)}g_0^{(1)}+\sum_{i=1}^{l}q_i\{\lambda_1 b_i^{(1)}+\alpha_0\lambda_2 b_i^{(1)}g_i^{(1)}\};$$

$$D_0=\frac{\left[p+\bar{p}a_e^*(\lambda_0)-pc_1\rho-pc_1(1-\theta)\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]}{\left[p+\bar{p}a_e^*(\lambda_0)\right]\left[1+\rho'+\lambda(1-\theta)(\zeta^{(1)}+\xi^{(1)})\right]-pc_1a_e^*(\lambda_0)\left[\rho+(1-\theta)\{\lambda_4\zeta^{(1)}+\lambda_3\xi^{(1)}\}\right]}$$

$$;\ \rho'=\rho_0\left(1+\alpha_0 g_0^{(1)}\right)+\sum_{i=1}^{l}q_i\rho_i\left(1+\alpha_i g_i^{(1)}\right).$$

Proof: The long run probabilities for different states of the system can be obtained by using the marginal probability generating functions for different states of the system. The factor D_0 can be obtained by using the normalizing condition given by

(3.32) $$D_0+\lim_{z\to 1}\left[D(z)+P_0(z)+\sum_{i=1}^{l}P_i(z)+R_0(z)+\sum_{i=1}^{l}R_i(z)+Q(z)+W(z)\right]=1.$$

Theorem 3: *The number of customers in the retrial queue and in the system in case of multi-optional phase service at random epoch with* $\lambda_1 = \lambda_2$ *is given by*

(3.33) $$L_R = \lim_{z\to 1} L_R'(z) = \frac{D_0\left[N_1 + N_2 + N_3\right]}{D}$$ and

(3.34) $$L_S = \lim_{z\to 1} L_S'(z) = \frac{D_0\left[N_1 + N_2 + N_4\right]}{D}$$

where $T_1 = \left[1 - q\left\{1 - a_e^*(\lambda_0)\right\}\right]$; $T_2 = \left[1 - a_e^*(\lambda_0)\right]$; $T_3 = \left[\lambda_3\xi^{(1)} + \lambda_4\zeta^{(1)}\right]$;

$T_4 = \left[\xi^{(1)} + \zeta^{(1)}\right]$; $T_5 = \left[\lambda_3{}^2\xi^{(2)} + \lambda_4{}^2\zeta^{(2)} + 2\lambda_3\lambda_4\xi^{(1)}\zeta^{(1)}\right]$;

$T_6 = \left[\lambda_0\lambda_3\xi^{(2)} + \lambda\lambda_4\zeta^{(2)} + 2\lambda\lambda_3\xi^{(1)}\zeta^{(1)}\right]$;

$T_7 = c_2\rho + pc_1{}^2\eta + 2c_1\rho - 2pc_1{}^2\rho^2$;

$T_8 = pc_1{}^2\lambda_3\xi^{(2)}\left(\lambda_3 - \lambda_4\right) - c_2\lambda_4\zeta^{(1)} - pc_1{}^2T_5$;

$$T_9 = 2pc_1(1-\theta)T_3 - 2p^2c_1{}^2(1-\theta)^2T_3{}^2 + pc_2\rho + p^2c_1{}^2\eta - 2p^2c_1{}^2(1-\theta)\rho T_3 - 2p^2c_1{}^2\rho^2 + 2pc_1\rho,$$

$T_{10} = -2pc_1{}^2\rho^2 + 2c_1\rho + pc_1{}^2\eta + c_2\rho$;

$N_1 = T_1T_2T_9 + T_2\left\{pc_2(1-\theta)T_3 + p^2c_1{}^2(1-\theta)T_5\right\}$;

$$N_2 = 2pqc_1{}^3\lambda_0(1-\theta)^2T_1T_2T_3T_4 + pc_1{}^2\lambda_0(1-\theta)^2T_1T_4\left\{2c_2T_3 + pc_1{}^2T_5\right\} + pc_1{}^2\lambda_0(1-\theta)T_1T_4T_{10} + pc_1{}^2\lambda_0\lambda_3\Lambda(1-\theta)^2T_1T_3T_8 + pc_1{}^3(1-\theta)T_1{}^2T_6 - p^2c_1{}^4(1-\theta)\rho T_1T_6$$

$$N_3 = -\lambda_0p^2c_1{}^4(1-\theta)\Lambda_1\eta T_1T_3 + \lambda_0pc_1{}^3\Lambda_1\eta T_1{}^2 + 2\lambda_0pqc_1{}^3(1-\theta)\rho\Lambda_1T_1T_2T_3 + 2\lambda_0\Lambda_1pqc_1{}^3\rho^2T_1T_2 + 2\lambda_0\Lambda_1p^2c_1{}^4\rho^2(1-\theta)T_1T_3 + \lambda_0\Lambda_1p^2c_1{}^4\rho(1-\theta)T_1T_5 + \lambda_0\Lambda_1pc_1{}^2c_2\rho(1-\theta)T_1T_3 + \lambda_0\Lambda_1pc_1{}^2c_2\rho^2T_1$$

$$N_4 = -\lambda_0p^2c_1{}^4(1-\theta)\Lambda_1\eta T_1T_3 + \lambda_0pc_1{}^3\Lambda_1\eta T_1{}^2 + 2\lambda_0pqc_1{}^3(1-\theta)\rho\Lambda_1T_1T_2T_3 - 2\lambda_0\Lambda_1pc_1{}^3\rho^2T_1{}^2 + 2\lambda_0\Lambda_1p^2c_1{}^4\rho^2(1-\theta)T_1T_3 + \lambda_0\Lambda_1p^2c_1{}^4\rho(1-\theta)T_1T_5 + \lambda_0\Lambda_1pc_1{}^2c_2\rho(1-\theta)T_1T_3 + \lambda_0\Lambda_1pc_1{}^2c_2\rho^2T_1 - 2pc_1{}^3\rho(1-\theta)T_1T_3 + 2c_1{}^3\rho\lambda_0\Lambda_1T_1{}^2$$

$$D = 2\left[pc_1{}^2(1-\theta)T_3 - c_1T_1 + pc_1{}^2\rho\right]^2.$$

Proof: $L_R(z)=D_0+D(z)+P_0(z)+\sum_{i=1}^{l}P_i(z)+R_0(z)+\sum_{i=1}^{l}R_i(z)+Q(z)+W(z)$ and

$$L_S(z)=D_0+D(z)+z\left[P_0(z)+\sum_{i=1}^{l}P_i(z)+R_0(z)+\sum_{i=1}^{l}R_i(z)\right]+Q(z)+W(z).$$

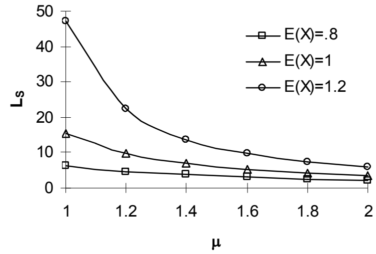
5. Numerical Results

In this section, we obtain numerical results using MATLAB software to explore the effect of some sensitive parameters on various performance indices. For computational purpose, we assume the batch size to be geometric distributed. The retrial time, repair time, vacation time and setup time are taken as exponential distributed. The results are summarized for $M^X/\gamma/1$ model in table 1 and fig. 1.

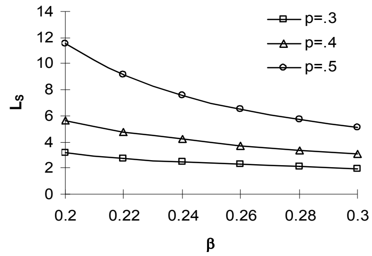
Table 1 displays the effect of λ_1, α_0, μ and p on the long run probabilities of different states of the server. The default parameters for these tables are chosen as $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$, $\alpha_0=\alpha_1=\alpha_2=0.3$, $\mu=5$, $p=0.6$, $\theta=0.9$, $\beta=0.25$, $\zeta=\nu=1$, $q_1=q_2=0.8$ and $E(X)=1$. It is noted from table 1 that P(B), P(V), P(S) and P(R) increase while P(I) decrease on increasing λ_1, α_0 and p. The reverse effect is found on these performance measures with the increase in service rate μ which is quite obvious. The trend of average system size (L_S) with respect to different parameters such as μ, β, ν, E(X) and p can be visualized from fig. 1. The default parameters for fig. 1 are chosen as $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$, $\alpha_0=\alpha_1=\alpha_2=0.3$, $\mu=5$, $p=0.6$, $\theta=0.9$, $\beta=0.25$, $\zeta=\nu=1$, $q_1=q_2=0.8$ and $E(X)=1$. Fig. 1 shows that L_S first decreases sharply on increasing either μ, β or ν and then becomes almost constant which can be seen in many real life congestion situations. Fig. 1(a-c) shows that L_S increases sharply on increasing either E(X) or p.

6. Conclusion

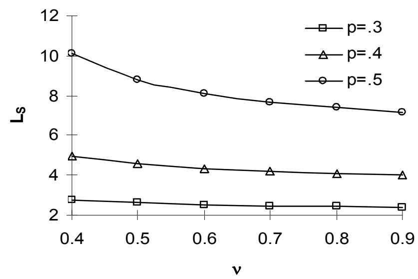
In this paper, we have examined $M^X/G/1$ retrial queueing system with Bernoulli vacation schedule, unreliable server and multi-optional services. Our results can be treated as performance evaluation tool for the concerned system which may be suited to many congestion situations arising in many practical applications encountered in computer and communication systems, distribution and service sectors, production and manufacturing systems, etc.. The concept of optional service included can be realized as a complement to a set of primitives in various queueing models, which throw light on the behaviour of the customers such as abandonment, retrials and returns.



(a)



(b)



(c)

Fig. 1: Effect of (a) μ (b) β and (c) ν on average system size

Table 1: Effects of arrival rate (λ_1), breakdown rate (α_0), service rate (μ) and retrial probability (p) on the long run probabilities.

λ_1	α_0	μ	p	P(I)	P(B)	P(V)	P(R)	P(S)
1.2	0.30	5	0.6	0.1071	0.3454	0.0664	0.4145	0.0664
1.4	0.30	5	0.6	0.0932	0.3508	0.0674	0.4109	0.0674
1.6	0.30	5	0.6	0.0825	0.3549	0.0682	0.4259	0.0682
1	0.30	5	0.6	0.1259	0.3381	0.0650	0.4058	0.0650
1	0.35	5	0.6	0.0865	0.3431	0.0659	0.4382	0.0659
1	0.4	5	0.6	0.0460	0.3483	0.0669	0.4715	0.0669
1	0.30	5	0.6	0.1259	0.3381	0.0650	0.4058	0.0650
1	0.30	6	0.6	0.2107	0.2965	0.0684	0.3558	0.0684
1	0.30	7	0.6	0.2770	0.2640	0.0710	0.3168	0.0710
1	0.30	5	0.4	0.2559	0.2878	0.0553	0.3454	0.0553
1	0.30	5	0.5	0.1961	0.3110	0.0598	0.3732	0.0598
1	0.30	5	0.6	0.1259	0.3381	0.0650	0.4058	0.0650

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