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Modelling Effects of Eutrophication on the Survival of Fish Population Incorporating Nutrient Recycling

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Abstract: It is well known that the aquatic bodies undergo eutrophication process due to the nutrient enrichment caused by waste disposal from industries and discharge of chemicals from the agricultural fields consequently, increasing the concentration of phytoplankton and algae thereby decreasing the dissolved oxygen and transparency of the aquatic environment. Due to oxygen deficit and loss of transparency, growth of much aquatic population such as fish is adversely affected. In view of this, therefore, in this paper mathematical models are proposed to study the survival or extinction of fish populations incorporating the impact of direct and indirect recycling of the nutrients under the adverse effects of eutrophication .The mathematical models have been formulated in the form of system of non-linear partial differential equations by taking the phenomenon of diffusion (dispersal) into account. Models have been analysed by obtaining the criteria for local stability and global stability of three feasible equilibrium points. It is found that the two positive equilibrium states which are unstable without diffusion may become stable by introducing diffusion. On the basis of this analysis, conditions for the survival or extinction of the fish populations have been derived.
Keywords: Mathematical model, eutrophication, diffusion, stability, habitat.

1. Introduction

Since long both our atmosphere and the aquatic environment are getting polluted by various kinds of industrial discharges and wastes, causing damage to our ecosystem which adversely affects the natural resources, thereby influencing the growth of other biological populations which may be depending upon these resources. As the aquatic bodies undergo eutrophiication process due to nutrient enrichment caused by waste disposal from industries and the discharge of chemicals from the agricultural fields so the nutrients increases the concentration of phytoplankton and algae thereby decreasing the dissolved oxygen and transparency of the aquatic bodies. Due to the oxygen deficit, growth rate of many aquatic populations decrease and the habitat also deteriorates due to the decrease in the level of transparency of the aquatic population. The mathematical modelling is proved quite useful in the study and analysis of the effects of toxicants on populations. Some mathematical models have been studied and discussed in this direction by several researchers[1 - 10].

In view of this, in this paper, we have studied the effect of eutrophication on the existence of aquatic population with direct and indirect recycling of the nutrients through detritus. In the models, amount of chlorophyll, phosphorus present in the water and transparency of the water are taken to be the noteworthy components and the model is studied with and without diffusion. In the model growth rate of fish population is considered to be a function of oxygen deficit which is decreasing with the amount of oxygen deficit. It is also assumed that the concentration of dissolved oxygen is changing with transparency and the dissolved oxygen is going down with the addition of phosphates in the water. In the main model it is also assumed that the amount of soluble phosphorus is increasing as a result of the presence of the detritus biomass in the aquatic environment. The mathematical model proposed in this paper is analyzed using stability theory.

2. Mathematical Model: Recycling of the Nutrients through Detritus

The mathematical model is given by the following system of non linear partial differential equations in a linear habitat at $0 \le z \le a$,

(2.1) $$\frac{\partial H}{\partial t} = \beta P - \alpha_h H,$$

(2.2) $$\frac{\partial F}{\partial t} = r(D)F - \frac{r_0 F^2}{k(N)} + D_0 \frac{\partial^2 F}{\partial z^2},$$

(2.3) $$\frac{\partial C}{\partial t} = -k_1 CP - d_B(N) + k_2(m)(C_s - C) + D_1 \frac{\partial^2 C}{\partial z^2},$$

(2.4) $$\frac{\partial P}{\partial t} = I - rP - \frac{d_1 PN}{b+P} + \alpha_1 m_d D_N + D_2 \frac{\partial^2 P}{\partial z^2},$$

(2.5) $$\frac{\partial N}{\partial t} = \frac{d_1 PN}{b+P} - aN - gN^2 + D_3 \frac{\partial^2 N}{\partial z^2},$$

(2.6) $$\frac{\partial D_N}{\partial t} = I_N + \alpha a N - m_d D_N + D_3 \frac{\partial^2 N}{\partial z^2}.$$

With the initial conditions which are given as follows

$$H(z,0) = f_1(z) \geq 0, F(z,0) = f_2(z) \geq 0, C(z,0) = f_3(z) \geq 0,$$
$$P(z,0) = f_4(z) \geq 0, H(z,0) = f_5(z) \geq 0, D_N(z,0) = f_6(z) \geq 0..$$

The model is associated with the following boundary conditions z=*0, a.* $H = H^*, F = F^*, C = C^*, P = P^*, N = N^*, D_N = D_N^*$.

$H^*, F^*, C^*, P^*, N^*, D_N^*$ are the equilibrium values (or steady states).

For the analysis of the model given by (2.1) to (2.6), we assume the following forms for the functions $r(D), k(N), d_B(N) \,\&\, k_2(m)$.$D$ is assumed to be $D = C_s - C$.

$$m(t) = \frac{m_0}{1+H(t)}, \qquad k_2(m) = \frac{k_{20} m}{k_{22} + m}, \qquad d_B(N) = d_{B0} + d_{B1} N,$$

$$k(N) = k_0 - k_{11} N, \qquad r(D) = \frac{r_0}{1+D}$$

where, *F*=Fish population, *C*=Concentration of dissolved oxygen, *D*=Oxygen deficit, *m*=Transparency, *H*=Concentration of chlorophyll, *N*=Phytoplankton(Algae and vascular plants), *P*=Concentration of soluble phosphorus, $k_2(m)$ = Reaeration rate, $d_B(N)$= Removal rate of DO due to respiration by organisms ,*a*=Death rate of phytoplankton, *r* = Death rate of fish, *r(D)* = Growth rate of fish, *K(N)* = Carrying capacity of fish, C_s = Saturated concentration of dissolved oxygen, *I* = Input rate of *P*, D_N = Detritus biomass in terms of the limiting nutrient, D_i = Diffusion (dispersal) coefficients where, *i*= 0, 1, 2, 3; I_N = Input of organic nutrient in the form of plant detritus, m_d = Plant detritus mineralisation rate, α=Fraction of nutrient released by plants that stays with the system and goes to the herbivore detritus, *a* = Rate of plant detritus production and

$\alpha_h, g, r_0, k_1, \alpha_1, m_d, d_1, b, \alpha, \beta$ are positive constants and z is the space variable.

3. Mathematical Model without Diffusion

(3.1) $$\frac{dH}{dt} = \beta P - \alpha_h H,$$

(3.2) $$\frac{dF}{dt} = r(D)F - \frac{r_0 F^2}{k(N)},$$

(3.3) $$\frac{dC}{dt} = -k_1 CP - d_B(N) + k_2(m)(C_s - C),$$

(3.4) $$\frac{dP}{dt} = I - rP - \frac{d_1 PN}{b+P} + \alpha_1 m_d D_N,$$

(3.5) $$\frac{dN}{dt} = \frac{d_1 PN}{b+P} - aN - gN^2,$$

(3.6) $$\frac{dD_N}{dt} = I_N + \alpha a N - m_d D_N.$$

4. Uniform Equilibrium Points

The uniform equilibrium points $E_i\ (i = 1,2,3)$ of the model given by (2.1)-(2.6) are obtained as follows:

(i) The equilibrium point $E_1(\overline{H}, \overline{F}, \overline{C}, \overline{P}, \overline{N}, \overline{D}_N)$ is given by,

where, $\overline{F} = 0, \overline{N} = 0, \overline{H} = \dfrac{\beta \overline{P}}{\alpha_h}$, $\quad \overline{C} = \dfrac{\dfrac{k_{20} m_0 C_s}{k_{22}(1+\overline{H}) + m_0} - d_{B0}}{k_1 \overline{P} + \dfrac{k_{20} m_0}{k_{22}(1+\overline{H}) + m_0}}$ >0 , provided ,

$\dfrac{k_{20} m_0 C_s}{k_{22}(1+\overline{H}) + m_0} > d_{B0}$, $\overline{P} = \dfrac{I + \alpha_1 m_d \overline{D}}{r}$, $\overline{D}_N = \dfrac{I_N}{m_d}$.

(ii) The equilibrium point $E_2(\tilde{H}, \tilde{F}, \tilde{C}, \tilde{P}, \tilde{N}, \tilde{D}_N)$ is given by,

$\tilde{F} = 0, \tilde{N} \neq 0, \tilde{H} = \dfrac{\beta \tilde{P}}{\alpha_h}, \tilde{C} = \overline{C}$, $\tilde{P} = \dfrac{b(g\tilde{N} + \alpha)}{d_1 - (g\tilde{N} + \alpha)}$

and $\tilde{N} > 0$ is one positive root of the cubic equation:

$$gN^3 + \left(2ag - d_1 g - ga\alpha\alpha_1\right)N^2 + (a^2 + \alpha a\alpha_1 d_1 - Ig - rbg$$
$$- d_1 a - \alpha_1 g I_N - a^2\alpha)N - \left(Ia + rab + a\alpha_1 I_N - Id_1 - \alpha_1 I_1 d_1\right) = 0.$$

Which exists provided,

$2ag > d_1 g + ga\alpha\alpha_1$, $a^2 + \alpha a\alpha_1 d_1 > Ig + rbg + d_1 a + \alpha_1 g I_N + a^2\alpha$

and $Ia + rab + a\alpha_1 I_N > Id_1 + \alpha_1 I_1 d_1$

(iii) The equilibrium point $E_3\left(\hat{H}, \hat{F}, \hat{C}, \hat{P}, \hat{N}, \hat{D}_N\right)$ is given by,

$\hat{H} = \tilde{H}$, $\hat{C} = \tilde{C}$, $\hat{P} = \tilde{P}$, $\hat{N} = \tilde{N}$, $\hat{D}_N = \tilde{D}_N$,

$$\hat{F} = \frac{k_0 - k_{11}\hat{N}}{1 + C_s - \hat{C}} > 0, \quad if, k_0 > k_{11}\hat{N} \,\&\, C_s > \hat{C} .$$

5. Linear Stability Analysis

The local or linear stability analysis of the equilibrium points E_i $(i = 1,2,3)$ can be carried out with the help of the variational matrix about the equlibria E_i of the mathematical model given by (3.1) to (3.6) .The variational matrix of the model given by (3.1) to (3.6) is

$$V^T = \begin{bmatrix} -\alpha_h & 0 & 0 & \beta & 0 & 0 \\ 0 & r(D) - \dfrac{2r_0 F}{k(N)} & \dfrac{Fr_0}{(1 + C_s - C)^2} & 0 & \dfrac{-r_0 F^2 k_{11}}{(k_0 - k_{11}N)^2} & 0 \\ 0 & 0 & -(k_1 P + k_2 m) & k_1 C & -d_{B1} & 0 \\ 0 & 0 & 0 & -r - \dfrac{d_1 N}{(b+P)^2} & \dfrac{-d_1 P}{b+P} & \alpha_1 m_d \\ 0 & 0 & 0 & \dfrac{d_1 Nb}{(b+P)^2} & \dfrac{d_1 P}{b+P} - 2gN - a & 0 \\ 0 & 0 & 0 & 0 & -\alpha a & -m_d \end{bmatrix}$$

After the simplification using mathematica, characteristics equations $\left|V_1^T - \lambda I\right| = 0$ and $\left|V_2^T - \lambda I\right| = 0$ about the equilibrium points E_1 and E_2 have positive roots and thus it is concluded that both the equilibrium points E_1 and E_2 are unstable.

For the linear stability analysis of the equilibrium point E_3, we first linearize the system (3.1) to (3.6) by using the following linear transformations or small perturbations in the equilibrium as–

$$H = \hat{H} + n_1, F = \hat{F} + n_2, C = \hat{C} + n_3, P = \hat{P} + n_4, N = \hat{N} + n_5, D_N = \hat{D}_N + n_6.$$

Considering the above small perturbations in the system of equations (3.1)-(3.6) and then neglecting the higher powers and the products of the perturbations, the system (3.1) to (3.6) reduces to the following linear system,

(5.1) $$\frac{dn_1}{dt} = \beta n_4 - \alpha_h n_1,$$

(5.2) $$\frac{dn_2}{dt} = -\frac{r_0 \hat{F}}{k\left(\hat{N}\right)} n_2 - \frac{r_0 \hat{F}}{1 + C_s - \hat{C}} n_3 + \frac{r_0 k_{11} \hat{F}^2}{k\left(\hat{N}\right)} n_5,$$

(5.3) $$\frac{dn_3}{dt} = -k_1 \hat{P} n_3 - k_1 \hat{C} n_4 - d_{B1} n_5 - \frac{k_{20} m_0}{k_{22}\left(1 + \overline{H}\right) + m_0} n_3 - \frac{k_{20} m_0 k_{22}\left(C_s - \hat{C}\right)}{\{k_{22}\left(1 + \overline{H}\right) + m_0\}^2} n_1,$$

(5.4) $$\frac{dn_4}{dt} = -rn_4 - \frac{d_1\left(\hat{N} n_4 + \hat{P} n_5\right)}{b + \hat{P}} + \alpha_1 m_d n_6 - \frac{d_1 \hat{P} \hat{N}}{\left(b + \hat{P}\right)^2} n_4,$$

(5.5) $$\frac{dn_5}{dt} = \frac{d_1 b \hat{N}}{(b + \hat{P})^2} - g n_5,$$

(5.6) $$\frac{dn_6}{dt} = \alpha a n_5 - m_d n_6.$$

Consider the Liapunov function X as

(5.7) $$X = \frac{1}{2}\left(n_1^2 + A_1 n_2^2 + A_2 n_3^2 + A_3 n_4^2 + A_4 n_5^2 + A_5 n_6^2\right),$$

where $A_i > 0, \forall i = 1,2,3,4$ are arbitrary constants.

Now differentiating (5.7) w. r. t. "t" and using (5.1) to (5.6) and the inequality $a^2 + b^2 \geq \pm 2ab$ and also choosing $A_2 = 2\left(\alpha_h - \frac{\beta}{2}\right)\frac{\{k_{22}\left(1 + \overline{H}\right) + m_0\}^2}{k_{20} m_0 k_{22}\left(C_s - \hat{C}\right)}$,the $\frac{dX}{dt}$ is approximated as:

(5.8) $$\frac{dX}{dt} \leq -\left(S_1 n_1^2 + S_2 n_2^2 + S_3 n_3^2 + S_4 n_4^2 + S_5 n_5^2 + S_6 n_6^2\right).$$

Where , $\alpha_h > \frac{\beta}{2}$.

Here $\frac{dX}{dt}$ is negative definite only when $S_i > 0$, $\forall i = 1,2,3,4,5,6$.

Therefore, the equilibrium point is locally stable under the conditions given by-

(5.9) $S_i > 0\ ,\ \ \forall i = 1,2,3,4,5,6$

where,

$$S_1 = \alpha_h - \frac{\beta}{2} - \frac{A_2}{2}\frac{k_{20}m_0k_{22}\left(C_s - \hat{C}\right)}{\{k_{22}\left(1+\overline{H}\right)+m_0\}^2}, \tag{5.10}$$

$$S_2 = A_1\left(\frac{r_0\hat{F}}{k\left(\hat{N}\right)} - \frac{1}{2}\frac{r_0\hat{F}}{1+C_s-\hat{C}} - \frac{1}{2}\frac{r_0k_{11}\hat{F}^2}{k\left(\hat{N}\right)}\right) \tag{5.11}$$

$$S_3 = A_2k_1\hat{P} + \frac{k_{20}m_0A_2}{k_{22}\left(1+\overline{H}\right)+m_0} - \frac{A_1}{2}\frac{r_0\hat{F}}{1+C_s-\hat{C}} - \frac{A_2d_{B1}}{2} - \frac{A_2}{2}\frac{k_{20}m_0k_{22}\left(C_s-\hat{C}\right)}{\{k_{22}\left(1+\overline{H}\right)+m_0\}^2}. \tag{5.12}$$

$$S_4 = A_3\left(r + \frac{d_1b\hat{N}}{(b+\hat{P})} + \frac{d_1\hat{P}\hat{N}}{\left(b+\hat{P}\right)^2} - \frac{\beta_1}{2} - \frac{1}{2}\frac{d_1b\hat{P}}{(b+\hat{P})} - \frac{\alpha_1m_d}{2}\right) - \frac{A_4}{2}\frac{d_1\hat{P}\hat{N}}{\left(b+\hat{P}\right)^2}, \tag{5.13}$$

$$S_5 = A_4g_1\hat{N} - \frac{A_1}{2}\frac{r_0k_{11}\hat{F}^2}{k\left(\hat{N}\right)} - \frac{A_2d_{B1}}{2} - \frac{d_1b\hat{N}A_3}{(b+\hat{P})} - \frac{A_4}{2}\frac{d_1\hat{P}\hat{N}}{\left(b+\hat{P}\right)^2} - \frac{\alpha gA_5}{2}, \tag{5.14}$$

$$S_6 = m_dA_5 - \frac{\alpha_1m_dA_3}{2} - \frac{\alpha gA_5}{2}. \tag{5.15}$$

6. Non-linear Stability Analysis

For non-linear stability analysis of equilibrium point E_3 we assume the region Ω as:

$$\Omega = \left\{\begin{aligned}&\left(H,F,C,P,N,D_N\right): 0 < C^l \le C \le C^u, 0 < P^l \le P \le P^u, 0 < H^l \le H \le H^u,\\ &0 < N^l \le N \le N^u, 0 < F^l \le F \le F^u, 0 < {D_N}^l \le D_N \le {D_N}^u.\end{aligned}\right\}$$

Theorem 1. *If the following inequalities hold,*

$$3\beta^2 < 2\alpha_h\left\{\frac{bd_1\hat{N}}{(b+P^u)(b+\hat{P})}\right\},$$

$$\left\{\begin{array}{c} 2\left\{\dfrac{r_0}{(1+C_s-\hat{C})(1+C_s-C^u)}\right\}^2 < \dfrac{1}{k(N^l)}\left(k_1\hat{P}+\dfrac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right) \\ 3\left\{\dfrac{r_0\hat{F}k_{11}}{k(N^l)}\right\}^2 < 2\dfrac{r_0 g}{k(N^l)}, \\ 3(k_1\hat{C})^2 < \left(k_1\hat{P}+\dfrac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right)\left(r+\dfrac{bd_1\hat{N}}{(b+P^u)(b+\hat{P})}\right) \end{array}\right.$$

$$3d_{B1}^2 < g\left(k_1\hat{P}+\frac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right),\ 3\alpha_1^2 m_d < 2\left\{r+\frac{bd_1\hat{N}}{(b+P^u)(b+P)}\right\},$$

$$3\alpha^2 g < m_d$$

$$2\left\{\frac{k_{20}k_{22}m_0\hat{D}}{(k_{22}+m_0+k_{22}H^u)(k_{22}+m_0+k_{22}\hat{H})}\right\}^2 < \alpha\left(k_1\hat{P}+\frac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right)$$

then equilibrium point E_3 *is nonlinearly asymptotically stable in the region* Ω.

Proof : Using the transformations

$H=\hat{H}+n_1, F=\hat{F}+n_2, C=\hat{C}+n_3, P=\hat{P}+n_4, N=\hat{N}+n_5, D_N=\hat{D}_N+n_6.$

the system (3.1) to (3.6) reduces to

$$\text{(6.1)}\qquad \frac{dn_1}{dt}=\beta n_4-\alpha_h n_1,$$

$$\text{(6.2)}\qquad \frac{1}{F}\frac{dn_2}{dt}=-\frac{r_0\hat{F}}{k(N)}n_2+\frac{r_0}{(1+C_s-\hat{C})(1+C_s-C)}n_3+\frac{r_0k_{11}\hat{F}}{k(\hat{N})k(N)}n_5,$$

$$\text{(6.3)}\qquad \frac{dn_3}{dt}=-k_1\hat{P}n_3-k_1\hat{C}n_4-d_{B1}n_5-\frac{k_{20}m_0}{k_{22}(1+H)+m_0}n_3$$

$$-\frac{k_{20}m_0k_{22}(C_s-\hat{C})}{\{k_{22}(1+\overline{H})+m_0\}\{k_{22}(1+H)+m_0\}}n_1.$$

(6.4) $$\frac{dn_4}{dt} = -rn_4 - \frac{d_1\left(\hat{N}n_4 + \hat{P}n_5\right)}{b+\hat{P}} + \alpha_1 m_d n_6 - \frac{d_1 b\hat{N}}{\left(b+\hat{P}\right)\left(b+P\right)} n_4,$$

(6.5) $$\frac{1}{N}\frac{dn_5}{dt} = \frac{b\hat{P}}{(b+\hat{P})(b+P)} n_4 - gn_5,$$

(6.6) $$\frac{dn_6}{dt} = -\alpha g n_5 - m_d n_6.$$

Consider the positive definite function:

(6.7) $$Y = \frac{1}{2}n_1^2 + \left(n_2 - \hat{F}\log\frac{\hat{F}+n_2}{\hat{F}}\right) + \frac{1}{2}n_3^2 + \frac{1}{2}n_4^2 + \left(n_5 - \hat{N}\log\frac{\hat{N}+n_5}{\hat{N}}\right) + \frac{1}{2}n_6^2.$$

Differentiating (6.7) w.r.t. t and using (6.1) to (6.6), the $\frac{dY}{dt}$ in the region Ω after using the inequality $a^2 + b^2 \geq 2ab$ is obtained as

$$\frac{dY}{dt} \leq -\left[\begin{array}{l} \left\{\frac{S_1}{2}n_1^2 - S_2 n_1 n_4 + \frac{S_{10}}{3}n_4^2\right\} + \left\{\frac{S_5}{2}n_2^2 - S_3 n_2 n_3 + \frac{S_6}{4}n_3^2\right\} + \left\{\frac{S_5}{2}n_2^2 + S_4 n_2 n_5 + \frac{S_{13}}{3}n_5^2\right\} \\ + \left\{\frac{S_{10}}{3}n_4^2 + S_7 n_4 n_3 + \frac{S_6}{4}n_3^2\right\} + \left\{\frac{S_{13}}{3}n_5^2 + S_8 n_5 n_3 + \frac{S_6}{4}n_3^2\right\} + \left\{\frac{S_6}{4}n_3^2 + S_9 n_1 n_3 + \frac{S_1}{2}n_1^2\right\} \\ + \left\{\frac{S_{10}}{3}n_4^2 + S_{12} n_4 n_6 + \frac{S_{14}}{2}n_6^2\right\} + \left\{\frac{S_{13}}{3}n_5^2 + S_{15} n_5 n_6 + \frac{S_{14}}{2}n_6^2\right\} \end{array}\right]$$

By using Sylvester's criterion in the quadratic forms in the R.H.S. of the above expression we obtain the following conditions for $\frac{dY}{dt}$ to be negative definite:

$3S_2^2 < 2S_1S_{10}$, $2S_3^2 < S_5S_6$, $3S_4^2 < 2S_5S_{13}$, $3S_7^2 < S_6S_{10}$,
$3S_8^2 < S_{13}S_6$, $2S_9^2 < S_1S_6$, $3S_{12}^2 < 2S_{14}S_{10}$, $3S_{15}^2 < S_{14}S_{13}$,

where, $S_1 = \alpha_h$, $S_2 = \beta$, $S_3 = \frac{r_0}{\left(1+C_s-\hat{C}\right)\left(1+C_s-C^u\right)}$, $S_4 = \frac{r_0 k_{11}\hat{F}}{k\left(\hat{N}\right)k\left(N^l\right)}$,

$$S_5 = \frac{r_0\hat{F}}{k\left(N^l\right)},\ S_6 = k_1\hat{P} + \frac{k_{20}m_0}{k_{22}\left(1+H^u\right)+m_0},\ S_7 = k_1\hat{C},\ S_8 = d_{B1},$$

$$S_9 = \frac{k_{20}m_0k_{22}\hat{D}}{\{k_{22}\left(1+\overline{H}\right)+m_0\}\{k_{22}\left(1+H^u\right)+m_0\}},\ S_{10} = r + \frac{d_1 b\hat{N}}{\left(b+\hat{P}\right)\left(b+P^u\right)}$$

$$S_{11} = \frac{d_1\hat{P}}{(b+P^u)} - \frac{b\hat{P}}{(b+\hat{P})(b+P^l)},\quad S_{12} = \alpha_1 m_d,\quad S_{13} = g,\ S_{14} = m_d,$$

$$S_{15} = \alpha a.$$

Hence it is proved that E_3 is nonlinearly asymptotically stable in the region Ω.

Remark 1-By taking diffusion (dispersal) into consideration it can be seen that the unstable equilibrium points E_1 and E_2 will tend to stable situation. Also, the equilibrium point E_3 is conditionally stable in both the cases.

7. Submodel- Direct Recycling of the Nutrients

The mathematical model is given by the following system of non linear partial differential equations in a linear habitat, $0 \le z \le a$,

(7.1) $$\frac{\partial H}{\partial t} = \beta P - \alpha_h H,$$

(7.2) $$\frac{\partial F}{\partial t} = r\left(D\right)F - \frac{r_0F^2}{k\left(N\right)} + D_0\frac{\partial^2 F}{\partial z^2},$$

(7.3) $$\frac{\partial C}{\partial t} = -k_1CP - d_B\left(N\right) + k_2\left(m\right)\left(C_s - C\right) + D_1\frac{\partial^2 C}{\partial z^2},$$

(7.4) $$\frac{\partial P}{\partial t} = I - rP - \frac{d_1PN}{b+P} + \beta_1 aN + D_2\frac{\partial^2 P}{\partial z^2},$$

(7.5) $$\frac{\partial N}{\partial t} = \frac{d_1PN}{b+P} - aN - gN^2 + D_3\frac{\partial^2 N}{\partial z^2}.$$

8. Uniform Equilibrium Points

(i) The uniform equilibrium point $E_1\left(\overline{H},\overline{F},\overline{C},\overline{P},\overline{N}\right)$ of the sub model given by (7.1) - (7.5) is given by,

$$\overline{F}=0,\overline{N}=0,\overline{H}=\frac{\beta\overline{P}}{\alpha_h},\quad \overline{C}=\frac{\dfrac{k_{20}m_0C_s}{k_{22}\left(1+\overline{H}\right)+m_0}-d_{B0}}{k_1\overline{P}+\dfrac{k_{20}m_0}{k_{22}\left(1+\overline{H}\right)+m_0}}>0 \text{ ,provided,}$$

$$\frac{k_{20}m_0C_s}{k_{22}\left(1+\overline{H}\right)+m_0}>d_{B0},\ \overline{P}=\frac{I}{r}\ .$$

(ii) The uniform equilibrium point $E_2\left(\tilde{H},\tilde{F},\tilde{C},\tilde{P},\tilde{N}\right)$ of the sub model is given by,

$$\tilde{F}=0,\tilde{N}\neq 0,\tilde{H}=\frac{\beta\tilde{P}}{\alpha_h},\tilde{C}=\overline{C},\ \tilde{P}=\frac{b\left(g\tilde{N}+\alpha\right)}{d_1-\left(g\tilde{N}+\alpha\right)}$$

and $\tilde{N}>0$ is one positive root of the equation ,
$g\tilde{N}^2-\beta_1 a\tilde{N}-\left(I-r\tilde{P}\right)=0$ which exists , provided $I>r\tilde{P}$.

(iii) The uniform equilibrium point $E_3\left(\hat{H},\hat{F},\hat{C},\hat{P},\hat{N}\right)$ of the sub model is given by,

$\hat{H}=\tilde{H}$, $\hat{C}=\tilde{C}$, $\hat{P}=\tilde{P}$, $\hat{N}=\tilde{N}$,

$$\hat{F}=\frac{k_0-k_{11}\hat{N}}{1+C_s-\hat{C}}>0,\quad if, k_0>k_{11}\hat{N}\ .$$

After carrying out the linear stability analysis, it is found that the equilibrium points E_1 and E_2 remain locally unstable without diffusion and become stable when diffusion is taken into account. Equilibrium point E_3 is conditionally stable in both the cases (with and without diffusion).System is nonlinearly stable under the following conditions:

$$(8.1)\qquad 3\beta^2<2\alpha_h\left\{\frac{d_1 b\hat{N}}{\left(b+\hat{P}\right)\left(b+P^u\right)}\right\},$$

$$(8.2)\ 2\left\{\frac{r_0}{\left(1+C_s-\hat{C}\right)\left(1+C_s-C^u\right)}\right\}^2<\frac{1}{k\left(N^l\right)}\left\{k_1\hat{P}+\frac{k_{20}m_0}{k_{22}\left(1+H^u\right)+m_0}\right\},$$

(8.3) $$3\left\{\frac{r_0 k_{11}\hat{F}}{k\left(\hat{N}\right)k\left(N^l\right)}\right\}^2 < 2\frac{r_0\hat{F}}{k\left(N^l\right)},$$

(8.4) $$3\left(k_1\hat{C}\right)^2 < \left(k_1\hat{P} + \frac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right)\left(r + \frac{bd_1\hat{N}}{\left(b+P^u\right)\left(b+\hat{P}\right)}\right),$$

(8.5) $$3d_{B1}^2 < g\left(k_1\hat{P} + \frac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right),$$

(8.6) $$2\left\{\frac{k_{20}k_{22}m_0\hat{D}}{\left(k_{22}+m_0+k_{22}H^u\right)\left(k_{22}+m_0+k_{22}\hat{H}\right)}\right\}^2 < \alpha\left(k_1\hat{P} + \frac{k_{22}m_0}{k_{22}+m_0+k_{22}H^u}\right)$$

(8.7) $$3\alpha^2 g < m_d$$

Remark 2-In case of diffusion it has been observed that the equilibrium points E_1 and E_2 become stable and the equilibrium point E_3 again remains conditionally stable in both the cases(with and without diffusion).

9. Conclusion

In this paper the model has three feasible steady states (equilibria) E_1, E_2 and E_3. From the stability analysis of the equilibrium points E_1, E_2 and E_3, it has been observed that the equilibrium points E_1 and E_2 are locally unstable in the absence of diffusion. The equilibrium point E_3 is linearly as well as nonlinearly asymptotically stable, showing the co-existence of fish and phytoplankton populations. In the presence of diffusion the equilibrium points E_1 and E_2 become linearly stable and the equilibrium point E_3 is again linearly and nonlinearly asymptotically stable. It is noted here that the otherwise unstable equilibrium point E_1 tend to stability with diffusion. Also, the equilibrium point E_2 which is unstable without diffusion becomes stable, if diffusion is considered. From the stability analysis of the equilibrium point E_3, it has been concluded that both the populations of fish and phytoplankton will coexist but the equilibrium level of fish population decreases if the density of the phytoplankton population and oxygen deficit simultaneously increases. It is noted from the analysis that if the dispersal (diffusion) is allowed in the system then the equilibrium point E_1 is stable which shows that the fish population and phytoplankton both die out. Whereas, in the case of equilibrium point E_2 which is stable if dispersal (diffusion) is introduced in the system, the fish population again dies out but phytoplankton exists.

From the stability analysis of the sub-model we found that the model has three non-negative equilibria and from the stability analysis of the equilibrium points E_1, E_2 and E_3 it has been derived that the uniform equilibrium points E_1 and E_2 are locally unstable in the absence of diffusion but the equilibrium point E_3 is locally as well as nonlinearly asymptotically stable involving system parameters. From the stability of the equilibrium point E_3 it may be concluded that the fish population would exist but the equilibrium level of fish population decreases if the density of phytoplankton increases. Further, the equilibrium level of fish population decreases as the level of oxygen deficit increases. It may be noted here that when the equilibrium density of phytoplankton and level of oxygen deficit both increases then the equilibrium value of fish population decreases further showing the synergistic effect of both the stresses. In the presence of diffusion the equilibrium points E_1 and E_2 both are locally stable and the equilibrium point E_3 is linearly as well as nonlinearly asymptotically stable. Thus, in this case also, it is concluded that the fish population will exist but at lower equilibrium level because of eutrophication of the aquatic system. Further, it may also be noted that the otherwise unstable equilibrium points E_1 and E_2 become stable with diffusion .Also, it may be concluded from the stability of the equilibrium point E_1 that both phytoplankton and fish population will tend to extinction, but in the case of the equilibrium point E_2 only the fish population would tend to extinction.

References

1. T. G. Hallam, C. E. Clark and G. S. Jordan, Effects of Toxicants on Populations – A Qualitative approach II. First Order Kinetics, *J. Math. Biol.*, **18** (**a)** (1983) 25-37.
2. T. G. Hallam, C. E. Clark and R. R. Lassister, Effects of Toxicants on Populations– A Qualitative approach I. Equilibrium Environmental Exposure,Ecol.Model., (**b)** (1983) 291-304.
3. T. G. Hallam and J. T. De Luna, Effects of Toxicants on Populations– A Qualitative approach III. Equilibrium Environmental and Food Chain Pathways, *J. Theor. Biol*., (1984)141-411-219.
4. J. T. De Luna and T. G. Hallam, On the Dynamics of a Toxicant-Individual System, *J. Theor . Biol*., **141** (1987) 65-72.

5. O. P. Misra and V. P. Saxena, Effect of Environmental Pollution on The Growth and Existence of Biological Populations: Modelling and Stability Analysis, *Indian J. Pure Appl. Math.,* **22(10)** (1991) 805-817.
6. H. I. Freedman and J. B. Shukla, Models for the Effect of Toxicant in Single Species and Predator-Prey Systems, *Math. Biol.,* **30** (1991) 15-30.
7. N. L. Miller, Stability Analysis of Toxic Substances Within Aquatic Ecosystems and their Effect on Aquatic Populations , *Ecol. Modelling* , **60** (1992) 151-165 .
8. A. K. Misra Peeyush Chandra and J. B. Shukla, *Modelling the depletion of Dissolved oxygen in a water body due to organic pollutants and their effect on survival of species,Mathematical Biology-Recent trends* edited by P. Chandra and B. V. Ratish Kumar,Anamaya Publishers, New Delhi, 2006.
9. Poonam Sinha and O.P. Misra, Effect of a diffusive predator species on the dynamics of two species-system: A model, *Proceeding of Mathematical Society*, B. H. U., **21**(2005).
10. Swati Khare, *Modelling Effects of pollutants on interacting species system with nutrient cycling*, 2010.