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Convergence of Deficient Quintic Spline Interpolation*

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Abstract : In this paper, we have obtained best error bounds of deficient quintic spline interpolation matching the given functional values and spline at two intermediate points between the successive mesh points and also the second derivatives of spline and given function at mesh points.
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1. Introduction

In the method of piecewise polynomial approximation cubic splines and higher degree splines are widely used to represent a function which is generally not analytic. Rana[1] has obtained as asymptotically precise estimate for the error of quadratic spline interpolating the given function at mid points between the successive mesh points. Considering a problem of spline interpolation Dikshit and Rana[2] have shown the convergence properties of cubic spline for a wider choice of points of interpolation for a non uniform mesh. In the direction of more higher degree spline Jian Zhong and Huang[3] have obtained optimal errors bounds for quartic and quintic interpolatory splines, (see also Howell and Verma[4]). Best error bounds for deficient quartic spline interpolation have been studied by Rana and Dubey[5]. For explicit error estimates of quintic and biquintic interpolatory splines reference may be made to Agrawal and Wong[6] and Gmeling-Meyling[7]. In the present paper, we shall obtain the best error bounds for deficient quintic spline interpolation matching the given functional values and spline at two

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intermediate points between the successive mesh points and second derivative at mesh points with appropriate boundary conditions.

2. Existence and Uniqueness

Consider a mesh P of [0, 1] is given by

$$P : 0 = x_o < x_1 < x_2 < .... < x_n = 1,$$

such that $x_{i+1} - x_i = h_i$ for all i. Let $\prod_m$ denote the set of all algebric polynomials of degree not greater than a positive integer m. For a function s defined over [0,1], we denote the restriction of s over $\left[x_i, x_{i+1}\right]$ by s_i. The class S(5,P,1) of deficient quintic splines of deficiency 1 defined over P is given by

$$S(5,P,1) = \left\{s : s \in c^2[0,1],\ s_i \in \prod\nolimits_5 : i = 0,\ 1, ...,\ n-1\right\}$$

where S*(5,P,1) denotes a class of all deficient quintic splines of S (5,P,1) which satisfies the boundary conditions :

(2.1) $\qquad s'(x_o) = f'(x_o),\ s'(x_n) = f'(x_n).$

Writing $\quad \alpha_i = x_i + h_i/3 \quad$ and $\quad \beta_i = x_i + 2h_i/3$,

we introduce following interpolatory conditions :

(2.2) $\qquad s(\alpha_i) = f(\alpha_i),\ s(\beta_i) = f(\beta_i),\ i = 0,1,...,\ n-1,$ $\qquad$ and

(2.3) $\qquad s''(x_i) = f''(x_i),\ i = 0,1,...,\ n.$

In fact, we shall prove the following :

Theorem 2.1: *Suppose* f', f'' *exist over P. Then there exist a unique deficient quintic splines s in S*(5,P,1) which satisfies the interpolatory conditions (2.2) and (2.3) and boundary condition (2.1) .*

Proof : Consider a quintic polynomial P(t) on [0,1], we can easily verify that

$$\begin{aligned} P(t) = {} & P(1/3)Q_1(t) + P(2/3)Q_2(t) + P''(0)Q_3(t) + P''(1)Q_4(t) \\ & + P'(0)Q_5(t) + P'(1)Q_6(t), \end{aligned} \tag{2.4}$$

where,

$$Q_1(t) = \left(64 - 810t^3 + 1215t^4 - 486t^5\right)/47,$$

$$Q_2(t) = \left(-17 + 810t^3 - 1215t^4 - 486t^5\right)/47,$$

$$Q_3(t) = \left(-\frac{74}{81} + \frac{47}{2}t^2 - \frac{383}{6}t^3 + \frac{121}{2}t^4 - \frac{39}{2}t^5\right)\Big/47,$$

$$Q_4(t) = \left(-\frac{20}{81} + \frac{101}{6}t^3 - \frac{111}{3}t^4 + \frac{39}{2}t^5\right)\Big/47,$$

$$Q_5(t) = \left(-\frac{922}{81} + 47t - 182t^3 + 226t^4 - 81t^5\right) \Big/ 47,$$

$$Q_6(t) = \left(\frac{112}{81} - 88t^3 + 179t^4 - 81t^5\right) \Big/ 47.$$

Now, writing $t = (x - x_i)/h_i$, $0 \le t \le 1$, (2.4) may be expressed in terms of the restrictions of s as follows,

$$s_i(x) = f(\alpha_i)\, Q_1(t) + f(\beta_i)\, Q_2(t) + h_i^2 f''(x_i)\, Q_3(t)$$

(2.5) $$+ h_i^2 f''(x_{i+1})\, Q_4(t) + h_i s_i'(x_i)\, Q_5(t) + h_i s_i'(x_{i+1})\, Q_6(t).$$

We see that $s_i(x)$ given by (2.5) is quintic in $[x_i, x_{i+1}]$ for i=0, 1,…, n-1, and clearly satisfies the conditions (2.1) – (2.3). Since $s \in C^2$ [0,1], therefore applying continuity condition of spline s in (2.5), we get

(2.6) $$-56h_{i-1}s'(x_{i-1}) + 461(h_i + h_{i-1})s'(x_i) - 56h_i s'(x_{i+1}) = F_i,$$

where

$$F_i = 81\left[64\left(f(\alpha_i) - f(\beta_{i-1})\right) + 17\left(f(\alpha_{i-1}) - f(\beta_i)\right)\right] \Big/ 2$$
$$-37\left(h_i^2 - h_{i-1}^2\right) f''(x_i) - 10\left[h_i^2 f''(x_{i+1}) - h_{i-1}^2 f''(x_{i-1})\right].$$

Clearly, the coefficient matrix of the system of equations (2.6) is diagonally dominant and hence invertible. Thus, the system of equations (2.6) has a unique solution. This completes the proof of theorem 2.1.

3. Error Bounds

Following the method of Hall and Mayer[8] in this section of the paper, we shall obtain the bounds of the error function

$$e^{(r)}(x) = f^{(r)}(x) - s^{(r)}(x),\ r = 1,\ 2,$$

for the spline interpolant of Theorem 2.1 which are best possible. Let $f \in C^6[0,1]$ and denote the unique quintic by $L_i[f, x]$ which agrees with $f(\alpha_i)$, $f(\beta_i)$, $f'(x_i)$, $f'(x_{i+1})$, $f''(x_i)$ and $f''(x_{i+1})$. Let s(x) be twice continuously differentiable quintic spline function satisfying the conditions of theorem 2.1. Therefore, for $x_i \le x \le x_{i+1}$, we have,

(3.1) $$|f(x) - s(x)| \cong |f(x) - s_i(x)| \le |f(x) - L_i[f, x]| + |L_i[f, x] - s_i(x)|$$

Thus, it is clear from (3.1) that in order to get the bounds of e(x), we have to estimate point wise bounds of both the terms on the right hand side of (3.1). The first terms of (3.1) can be estimated by using well known Cauchy remainder theorem for polynomial interpolation (see Davis[9])

(3.2) $$\left|f(x)-L_i[f,x]\right| \le \frac{h_i^6}{6!}\left|t^2(1-t)^2-(t-1/3)(t-2/3)\right|.F$$

where $t=(x-x_i)/h_i$ and $F=\max\limits_{0\le x\le 1}\left|f^{(6)}(x)\right|$. We now turn our attention to derive a similar bounds for $\left|L_i[f,x]-s_i(x)\right|$. From (2.5), we see that

(3.3) $$\begin{aligned} L_i[f,x]-s_i(x) &= h_i\left[f'(x_i)-s_i'(x_i)\right]Q_5(t) \\ &\quad + h_i\left[f'(x_{i+1})-s_i'(x_{i+1})\right]Q_6(t). \end{aligned}$$

Thus,

(3.4) $$\left|L_i[f,x]-s_i(x)\right| \le h_i\left|e'(x_i)\right|\left|Q_5(t)\right| + h_i\left|e'(x_{i+1})\right|\left|Q_6(t)\right|,$$

as $$Q_5(t)=(t-1/3)(t-2/3)\left(-81t^3+145t^2-19t-\frac{461}{9}\right)\Big/47$$

and

$$Q_6(t)=(t-1/3)(t-2/3)\left(-81t^3+98t^2+28t+\frac{56}{9}\right)\Big/47,\ \text{for } 0\le t\le 1.$$

Thus, we have,

(3.5) $$\left|Q_5(t)+Q_6(t)\right| \le K(t) \le 17/10,$$

where $$K(t)=\max\left\{\left|Q_5(t)\right|+\left|Q_6(t)\right|\right\}.$$

Now, from (3.3) and (3.4), we have,

(3.6) $$\left|L_i[f,x]-s_i(x)\right| \le h.\max\left\{\left|e'(x_i)\right|,\ \left|e'(x_{i+1})\right|\right\}.K(t)$$

where h = max h_i, for all i.

Let the max $\left|\underset{0\le i\le n}{e'(x_i)}\right|$ exists for i=j, then (3.6) becomes,

(3.7) $$\left|L_i[f,x]-s_i(x)\right| \le h\left|e'(x_j)\right|\left|K(t)\right|.$$

Now, we proceed to find the upper bound for $\left|e'(x_j)\right|$. From (2.6), it follows that

(3.8) $$56h_{j-1}e'(x_{j-1})-461(h_j+h_{j-1})e'(x_j)+56h_je'(x_{j+1})=E_o(f),$$

where

$$\begin{aligned} E_o(f) &= \left\{56h_{j-1}f'(x_{j-1})-461(h_j+h_{j-1})f'(x_j)+56h_jf'(x_{j+1})\right. \\ &\quad +81\left[64\left(f(\alpha_j)-f(\beta_{j-1})\right)+17\left(f(\alpha_{j-1})-f(\beta_j)\right)\right]\Big/2 \end{aligned}$$

$$-37\left(h_j^2-h_{j-1}^2\right)f''\left(x_j\right)-10\left[h_j^2 f''\left(x_{j+1}\right)-h_{j-1}^2 f''\left(x_{j-1}\right)\right]\Big\}.$$

In view of the above $E_o(f)$ is a linear functional which is zero for polynomial of degree 5 or less, we can apply the Peano theorem (see Davis[9]) to get,

$$E_o\left(f\right)=\int_{x_{j-1}}^{x_{j+1}} \frac{f^{(6)}\left(y\right)}{5!} E_o\left(x-y\right)_+^5\, dy. \tag{3.9}$$

Now from (3.9) it follows that,

$$\left|E_o\left(f\right)\right| \le \frac{1}{5!} F \int_{x_{j-1}}^{x_{j+1}} E_o\left[\left(x-y\right)_+^5\right] dy. \tag{3.10}$$

Further, it can be observed from (3.9) that for $x_{j-1} \le y \le x_{j+1}$,

$$\begin{aligned}
E_o\left[\left(x-y\right)_+^5\right] &= 280h_j\left(x_{j+1}-y\right)^4-2305\left(h_j+h_{j-1}\right)\left(x_j-y\right)_+^4 \\
&\quad +81\Big[64\left\{\left(\alpha_j-y\right)^5-\left(\beta_{j-1}-y\right)^5\right\} \\
&\quad +17\left\{\left(\alpha_{j-1}-y\right)^5-\left(\beta_j-y\right)^5\right\}\Big/2 \\
&\quad -740\left(h_j^2-h_{j-1}^2\right).\left(x_j-y\right)_+^3-200h_j^2\left(x_{j+1}-y\right)^3.
\end{aligned}$$

In order to evaluate the integral of the right hand side if (3.8), we rewrite the above expression in the following symmetric form:

$$\begin{aligned}
E_o\left[\left(x-y\right)_+^5\right] &= 40h_{j-1}\Big[-7\left(x_j-y\right)^4+23h_{j-1}\left(x_j-y\right)^3-27h_{j-1}^2\left(x_j-y\right)^2 \\
&\quad +13h_{j-1}^3\left(x_j-y\right)-2h_{j-1}^4\Big];\ x_{j-1} \le y \le \alpha_{j-1} \\
&= -\frac{1377}{2}\left(x_j-y\right)^5-2015h_{j-1}\left(x_j-y\right)^4-2140h_{j-1}^2\left(x_j-y\right)^3 \\
&\quad +960h_{j-1}^3\left(x_j-y\right)^2-160h_{j-1}^4\left(x_j-y\right)+\frac{32}{3}h_{j-1}^5;\alpha_{j-1} \le y \le \beta_{j-1} \\
&= \frac{3807}{2}\left(x_j-y\right)^5-2305h_{j-1}\left(x_j-y\right)^4+740h_{j-1}^2\left(x_j-y\right)^3; \\
&\qquad \beta_{j-1} \le y \le x_j \\
&= \frac{3807}{2}\left(x_j-y\right)^5+2305h_j\left(x_j-y\right)^4+740h_j^2\left(x_j-y\right)^3; \\
&\qquad x_j \le y \le \alpha_j \\
&= -\frac{1377}{2}\left(x_j-y\right)^5-2015h_j\left(x_j-y\right)^4-2140h_j^2\left(x_j-y\right)^3
\end{aligned}$$

$$-960h_j^3\left(x_j-y\right)^2-160h_j^4\left(x_j-y\right)-\frac{32}{3}h_j^5;\quad \alpha_j\le y\le\beta_j$$

$$=280h_j\left(x_j-y\right)^4+920h_j^2\left(x_j-y\right)^3+1080h_j^3\left(x_j-y\right)^2$$
$$+520h_j^4\left(x_j-y\right)+80h_j^5\ ;\qquad \beta_j\le y\le x_{j+1}\ .$$

Thus, it is clear from the above expression that $\left|E\left[\left(x-y\right)_+^5\right]\right|$ is non-negative for $x_{j-1}\le y\le x_{j+1}$.

Thus,

(3.11) $$\int_{x_{j-1}}^{x_{j+1}}\left|E_o\left[\left(x-y\right)_+^5\right]\right|dy=\frac{94}{27}\left(h_{j-1}^6+h_j^6\right).$$

Thus, we have the following from (3.11) when we appeal to (3.10)

(3.12) $$\left|E_o\left(f\right)\right|\le\frac{188}{9\left(6!\right)}F\left(h_{j-1}^6+h_j^6\right).$$

Now, from (3.8) and (3.12), we have,

(3.13) $$\left|e'\left(x_j\right)\right|=\left|e_j'\right|\le\frac{188}{6!\left(3645\right)}F\frac{\left(h_{j-1}^6+h_j^6\right)}{h_{j-1}+h_j}.$$

On combining (3.2), (3.7), (3.13) & (3.1), we have,

(3.14) $$\left|e\left(x\right)\right|\le\frac{h^6}{6!}\left|t^2\left(1-t\right)^2\left(t-1/3\right)\left(t-2/3\right)\right|.F+h\left|e_j'\right|K\left(t\right)$$

(3.15) $$\le\frac{h^6}{6!}FC\left(t\right)\ ,$$

where $$C\left(t\right)=\left|t^2\left(1-t\right)^2\left(t-1/3\right)\left(t-2/3\right)+\frac{188}{3645}K\left(t\right)\right|.$$

Thus, we prove the following :

Theorem 3.1: *Let s(x) be the quintic spling interpolant of theorem 2.1 interpolating a function f and* $f\in C^6$ *[0,1], then*

(3.16) $$\left|e\left(x\right)\right|\le K\frac{h^6}{\left(6!\right)}\max_{0\le x\le1}\left|f^6\left(x\right)\right|$$

where

$$K=\max_{0\le t\le1}\left|C\left(t\right)\right|=\left[t^2\left(1-t\right)^2\left(t-1/3\right)\left(t-2/3\right)+\frac{188}{3645}K\left(t\right)\right].$$

Also, we have

$$(3.17)\qquad \left|e'(x_i)\right| \le \frac{h^5}{(6!)}\cdot\frac{188}{3645}\max_{0\le x\le 1}\left|f^6(x)\right|.$$

Here, it may be observed that K in (3.16) can not be improved for an equally spaced partition. Inequality (3.17) is also best possible. Furthermore,

$$(3.18)\qquad \left|e'(x)\right| \le K_1\frac{h^5}{(6!)}\frac{188}{3645}\max_{0\le x\le 1}\left|f^6(x)\right|,$$

where K_1 is some positive constant. Inequality (3.16) is proved, when we appeal (3.15). Further, (3.17) is a direct consequence of (3.13).

Now we shall show that inequality (3.16) is best possible in the limit. Let $f(x)=\frac{x^6}{6!}$, then from the Cauchy formula (see Davis[9]), we have for i = 0,1,…,n-1,

$$(3.19)\qquad \frac{x^6}{6!}-L_i\left[\frac{x^6}{6!},x\right]=-\frac{h^6}{6!}t^2(1-t)^2(t-1/3)(t-2/3).$$

Furthermore, for the function under consideration (3.8) gives the following for equally spaced knots,

$$(3.20)\qquad E_o\left(\frac{x^6}{6!}\right)=\frac{h^5}{6!}\frac{188}{9}=28e'(x_{j-1})-461e'(x_j)+28e'(x_{j+1}).$$

Suppose for a moment that,

$$(3.21)\qquad e'(x_i)=-\frac{h^5}{6!}\frac{188}{(3645)}=e'(x_{i-1})=e'(x_{i+1}),$$

then using (3.4), we have

$$L_i[f,x]-s(x)=-\frac{188}{(3645)}\frac{h^6}{6!}=\left(Q_5(t)+Q_6(t)\right)$$

$$(3.22)\qquad =-\frac{188}{(3645)}\frac{h^6}{6!}=\left(t^4-2t^3+t-\frac{22}{81}\right).$$

Now combining (3.19) with (3.22), we get for $x_i\le x\le x_{i+1}$

$$(3.23)\qquad f(x)-s(x)=-\frac{h^6}{6!}\left[t^2\left(t^4-3t^3+\frac{29}{9}t^2-\frac{13}{9}t+\frac{2}{9}\right)+\frac{188}{3645}\left(t^4-2t^3+t-\frac{22}{81}\right)\right].$$

From (3.23), it follows that (3.15) is best possible provided we could prove that

$$(3.24)\qquad e'(x_{i-1}) = e'(x_{i+1}) = e'(x_i) = -\left(\frac{h^5}{6!}\right)\frac{188}{3645}.$$

It may be mentioned that (3.24) is attained only in the limit. The difficulty will take place in the case of boundary conditions, i.e. $e'(x_o) = e'(x_n) = 0$, we can show however, that as one moves many sub intervals away from the boundaries $e'(x_i) \to -\frac{188}{3645}\left(\frac{h^5}{6!}\right)$. For that we shall apply (3.20) inductively to move away from the end conditions $e'(x_o) = e'(x_n) = 0$. To achieve this we first show that $e'(x_i) \le 0$, for i=0, 1,…,n by assuming a contradictory result.

Let $e'(x_i) > 0$ for some i, i = 1, 2,…n–1. Then on using (3.17), we have

$$28\left(\frac{188}{3645}\right)\frac{h^5}{6!} \ge 28\max\left|e'(x_i)\right| \ge \frac{1}{2}\left[28e'(x_{i-1}) + 28e'(x_{i+1})\right],\quad \text{i =1,2,…,n-1}$$

$$> \frac{1}{2}\left[28e'(x_{i-1}) - 461e'(x_i) + 28e'(x_{i+1})\right] = \frac{94}{9}\frac{h^5}{6!},$$

which is a contradiction. Thus, we have shown that $e'(x_i) \le 0$, for i = 0, 1,.., n .

Now from (3.20), we can get,

$$(3.25)\qquad 461e'(x_i) = -\frac{188}{9}\frac{h^5}{6!} + 28e'(x_{i-1}) + 28e'(x_{i+1}),\quad \text{i = 1, 2,.., n–1.}$$

Since $e'(x_i) \le 0$, therefore, we have

$$(3.26)\qquad e'(x_i) \le -\frac{188}{(4149)}\frac{h^5}{6!}.$$

Next, applying (3.26) in (3.25), we get,

$$(3.27)\qquad e'(x_i) \le -\frac{188}{(4149)}\frac{h^5}{6!}\left[1 + \frac{56}{461}\right].$$

Thus, it is clear that repeated use of (3.25) leads us to

$$(3.28)\qquad e'(x_i) \le -\frac{188}{(4149)}\frac{h^5}{6!}\left[1 + \frac{56}{461} + \left(\frac{56}{461}\right)^2 + \ldots + \left(\frac{56}{461}\right)^{J-1}\right].$$

Now we can verify that right hand side of (3.28) in limiting case tends to $-\frac{188}{(3645)}\frac{h^5}{6!}$ corresponding to $f(x)=\frac{x^6}{6!}$ for equal spaced knots. This completes the proof of Theorem 3.1.

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