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Well-Posedness of a Class of Parabolic Integro-Differential Equations

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Abstract: In this paper we study the following parabolic integro-differential equation,

$$\begin{aligned} p(x)w_t(x,t) - \Delta w(x,t) &= f(x,t) \\ &\quad - \int_0^t k(t-s)\Delta w(x,s)ds, \qquad x \in \Omega,\ t \in (0,T], \\ w(x,t) &= 0, \quad x \in \partial\Omega,\ t \in [0,T], \\ w(x,0) &= \varphi(x),\ \ x \in \Omega, \end{aligned}$$

where $0 < t < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $p : \Omega \to \mathbb{R}$ with $p(x) > 0$ for a.e. $x \in \Omega$ and $p, u \in L^2(\Omega)$ for $u \in L^2(\Omega), \Delta$ denotes the n-dimensional Laplacian, $f : \Omega \times [0,T] \to \mathbb{R}$ is such that $f(.,t) \in L^2(\Omega)$ for each $t \in [0,T]$, the function $\varphi \in L^2(\Omega)$ and the kernel $k : [0,T] \to \mathbb{R}$ is a continuously differentiable function such that $k(0) \geq 0$ .We reformulate the problem mentioned above as an ordinary differential equation in a Hilbert space and apply the method of semi-groups of bounded linear operators in Hilbert space to study the well-posedness of the problem.

Keywords: Well-posedness, Cauchy Problems, Integro-differential Equation, Semi-groups.

2000 MathematicsSubject Classification: 34G10, 34G20, 34K06, 34K30.

1. Introduction

In this paper we deal with the following parabolic integro-differential equation,

$$(1.1)\quad p(x)w_t(x,t)-\Delta w(x,t)=f(x,t)$$
$$-\int_0^t k(t-s)\Delta w(x,s)ds,\qquad x\in\Omega,\ t\in(0,T],$$

$$(1.2)\quad w(x,t)=0,\quad x\in\partial\Omega,\ t\in[0,T],$$

$$(1.3)\quad w(x,0)=\varphi(x),\ x\in\Omega,$$

where $0<T<\infty$, $\Omega\subset\mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $p:\Omega\to\mathbb{R}$ with $p(x)>0$ for a.e. $x\in\Omega$ and $p,u\in L^2(\Omega)$ for $u\in L^2(\Omega)$, Δ denotes the n-dimensional Laplacian, $f:\Omega\times[0,T]\to\mathbb{R}$ is such that $f(.,t)\in L^2(\Omega)$ for each $t\in[0,T]$, the function $\varphi\in L^2(\Omega)$ and the kernel $k:[0,T]\to\mathbb{R}$ is a continuously differentiable function. The initial boundary value problem for a parabolic integro-differential equation of the type (1.1) with conditions (1.2) and (1.3) arise in the mathematical modeling of many physical problems. For instance, certain materials have the property that heat conduction or the wave propagation in them not only depends on the instantaneous values of certain quantities but the values at the previous instants also. Such materials are said to have the memory effects. Some of these materials are viscoelastic materials (1.1, 1.2, 1.3).

2. Abstract Formulations

In this section we rewrite the problem as an ordinary integro-differential equation in the Hilbert space of all real square integrable functions with the usual inner product and the norm generated by the inner product:

$$(u,v)=\int_\Omega h(x)v(x)dx,\quad \|u\|^2=\int_\Omega |u(x)|^2\,dx,\quad u,v\in H.$$

We define $u:[0,T]\to H$, by $u(t)(x)=w(x,t)$ for $x\in\Omega$ and $t\in[0,T]$ assuming that $w(.,t)\in H$ for $t\in[0,T]$. Further we define the operator $A:D(A)\subset H\to H$ given by

$$D(A)=W^{2,2}(\Omega),\ Au=-\Delta u,\ for\ \ u\in D(A).$$

Here $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are the Sobolev spaces for $m = 0, 1, 2, \ldots$ and $1 < p < \infty$ (cf. Pazy[1] for the definitions). Let $B : D(B) \subset H \to H$ be given by

$$D(B) = \{u \in H : pu \in H\}, \ Bu = pu, \ \textit{for} \ \ u \in D(B).$$

Then the problem (1.1) with conditions (1.2) and (1.3), may be reformulated as,

$$B\frac{du(t)}{dt} + Au(t) = F(t) - \int_0^t k(t-s)Au(s)ds, \tag{2.4}$$

$$u(0) = \varphi, \tag{2.5}$$

where $F : [0,T] \to H$, is given by $F(t)(x) = f(x,t)$. Our aim is to establish the well-posedness of the abstract initial value problem for the integro-differential equation (2.4) - (2.5).

3. Well-Posedness of the Problem

In this section we extend the results of Zaidman[2] established for the Cauchy problem

$$B\frac{du(t)}{dt} = Au(t), \ \ 0 < t \le T < \infty, \tag{3.6}$$

$$u(0) = \varphi. \tag{3.7}$$

Definition 1. By a solution to (2.4) – (2.5) we mean a function $u : [0,T] \to D(A)$ such that the strong derivative $u'(t)$ exists and $u'(t) \in D(B)$ for all $t \in [0,T]$, the equality (2.4) holds for $t \in [0,T]$.

Definition 2. By the well-posedness of (2.4) – (2.5) on [0, T] we mean that for any $\varphi \in D(A)$ it has a unique solution and this solution depends continuously on the initial data in the sense that if $u_n(0) \to 0$ then $u_n(t) \to 0$ on $[0,T]$ as $n \to \infty$.

Proposition 3.1. *Suppose* $\mathrm{B}\mathrm{x} = 0 \Rightarrow \mathrm{x} = 0$*, i.e., B is invertible. Let $F : [0,2T] \to H$ and $k : [0,2T] \to \mathbb{R}$. If (2.4) – (2.5) is well-posed on $[0,T]$ then it is well-posed on $[0,2T]$.*

Proof. Suppose that (2.4) – (2.5) is well-posed on $[0,T]$. Let $u(t)$ be the solution of (2.4) – (2.5) for $\varphi \in D(A)$. Then $u(t_0+T) \in D(A)$. Let $v:[0,T] \to D(A)$ be the unique solution of

$$B\frac{dv(t)}{dt} + Av(t) = [F(t+T) - \int_0^T k(t+T-s)Au(s)ds] - \int_0^t k(t-s)Av(s)ds, \tag{3.8}$$

$$u(0) = u(T). \tag{3.9}$$

Define

$$w(t) = \begin{cases} u(t) & t \in [0,T] \\ v(t-T), & t \in [T,2T] \end{cases}.$$

Clearly, $w(t) \in D(A)$ for $t \in [0,2T]$. Also, $w(t)$ exists separately on $[0,T]$ and $[T,2T]$ (where at the end points we mean one sided derivatives). We need to show that at $t = T$ one sided derivatives match. Let $\delta > 0$. Then

$$\frac{w(T+\delta) - w(T)}{\delta} = \frac{v(\delta) - v(0)}{\delta} \to v'(0) \quad \text{as } \delta \downarrow 0.$$

Similarly, for $\delta < 0$, we have

$$\frac{w(T+\delta) - w(T)}{\delta} = \frac{u(T+\delta) - u(T))}{\delta} \to u'(T) \quad \text{as } \delta \uparrow 0.$$

Now

$$Bv'(0) = F(T) - Av(0) = F(T) - Au(T).$$

Also

$$Bu'(T) = F(T) - Au(T).$$

Hence $Bv'(0) = Bu'(T)$. The invertibility of B implies that $v'(0) = u'(T)$. Therefore $w'(T)$ exists. Also, it is easy to see that $w'(t) \in D(B)$ for $t \in [0,2T]$ and the equation (2.4) is satisfied on $[0,2T]$. Now we need to prove the uniqueness. Let $w_1(t)$ be another solution of (2.4) on $[0,2T]$. Then because of the well-posedness on $[0,T]$, we have $w(t) = w_1(t)$ on $[0,T]$. Let $v_1(t) = w_1(t+T)$ for $[T,2T]$. Then u and v_1 are two solutions of (3.8) - (3.9) on $[0,T]$. Since the problem (3.8) - (3.9) is a well-posed problem with F and k replaced by similar functions $F(T+.)$ and $k(T+.)$, $v=v_1$ on $[0,T]$. This implies that $w=w_1$ on the whole $[0,2T]$.

Now we show the continuous dependence of solution of (2.4) - (2.5) on $[0,2T]$. Let u_n be a solution of (2.4) - (2.5) such that $u_n(0)\to 0$ as $n\to\infty$. Since (2.4) – (2.5) is well-posed on [0, T], $u_n(t)\to 0$ on $[0,T]$ and hence $u_n(T)\to 0$ as $n\to\infty$, i.e., if $v_n(t)$ are the solutions of (3.8) - (3.9) with $v_n(0)=u_n(T)$, then the well-posedness of (3.8) - (3.9) on [0, T] implies that $v_n(t)\to 0$ as $n\to\infty$. Thus $w_n(t)$, defined in a similar manner as w (t) above, we have $w_n(t)\to 0$ on $[0,2T]$ as $n\to\infty$.

Theorem 3.1. Let B be invertible and let $F:[0,\infty)\to H$, $k:[0,\infty)\to\mathbb{R}$. If (2.4) - (2.5) is well-posed on a finite interval $[0,T]$, then it is well-posed on $[0,\infty)$.

Proof: Repeated application of the Proposition 3.1 gives the desired result.

References

1. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

2. S. Zaidman, *Topics in Abstract Differential Equations*, Pitman Research Notes in Mathematics Series - 304, Longman Sci. Tech., UK, 1994.

3. Salim A. Messaoudi, Tatar, Nasser-eddine: Exponential decay for a quasilinear viscoelastic equation., *Math. Nachr.*, **282 (10)** (2009) 1443-1450.

4. Abdessamad Zouhair, Kostin Ilya, Panasenko Grigory and Valery P. Smyshlyaev, Memory effect in homogenization of a viscoelastic Kelvin-Voigt model with time-dependent coefficients, *Math. Models Methods Appl. Sci.,* **19 (9)** (2009) 1603-1630.

5. Xiaosen Han and Mingxin Wang, Global existence and blow-up of solution for a system of nonlinear viscoelastic wave equations with damping and source., *Nonlinear Analysis.*, **71 (11)** (2009) 5427-5450.