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# On Hyperbolic F- Structure Manifold

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**Abstract:** In this paper, we have defined Hyperbolic F-structure manifold and Hyperbolic metric F-structure manifold with some theorems related to projection and rank. The integrability conditions for the manifold have been obtained and some related results are derived. We have defined Pseudo Projective curvature tensor W, Pseudo conformal curvature tensor C, Pseudo conharmonic curvature tensor L, Pseudo concircular curvature tensor V, Pseudo H-Projective curvature tensor P, Pseudo H-conharmonic curvature tensor S and Pseudo Bochner curvature tensor B in Hyperbolic F-structure manifold. Recurrence and Recurrence symmetry of different kinds have been defined and theorems of recurrent and recurrent symmetry of this manifold have been proved.

**Keywords:** Hyperbolic F-structure manifold, Pseudo Projective curvature tensor, Recurrence and Recurrence symmetry.
**2000 AMS Classification:** 53C15

## 1. Introduction

Let us consider an n-dimensional differentiable manifold $V_n$ of differentiability class $C^\infty$ and let there be given on $V_n$ a tensor field F of the type (1,1)of class $C^\infty$ and rank r $(1 \le r \le n)$ everywhere ,such that

(1.1a) $F^3 - F = 0$,

or

(1.1b) $\overline{\overline{\overline{X}}} - \overline{X} = 0$, for an arbitrary vector field X,

where

(1.1c) $\overline{X} = FX$.

Then $\{F\}$ is called a Hyperbolic F-structure and $V_n$ is called Hyperbolic F structure manifold[1]. Let the Riemannian metric tensor g be such that

(1.2a) $\quad 'F(X,Y) \overset{def}{=} g(\overline{X},Y),$

and

(1.2b) $\quad g(\overline{X},Y) - g(X,\overline{Y}) = 0.$

A Hyperbolic F-structure manifold, on which the metric tensor g satisfies (1.2b) is called Hyperbolic metric F-Structure manifold and the structure $\{F,g\}$ is called Hyperbolic metric $F$-structure.
We know that

(1.3) $\quad 'F\ (X,Y) = g(\overline{X},Y).$

Then the following hold:

(1.4a) $\quad 'F\ (X,Y) = -'F(Y,X)$

i.e $'F$ is skew-symmetric in $X$ and $Y$.

(1.4b) $\quad 'F(\overline{X},\overline{Y}) + 'F(X,Y) = 0.$

A bilinear function A in Hyperbolic F-structure manifold is said to be pure in the two slots, if

(1.5) $\quad A(\overline{X},\overline{Y}) = A(X,Y).$

It is said to be hybrid in two slots, if

(1.6) $\quad A(\overline{X},\overline{Y}) + A(X,Y) = 0.$

Hence $'F$ is hybrid in $X$ and $Y$ as given in (1.4b).

## 2. Theorems Related to Projection and Rank

**Theorem2.1.** *The operators l and m given by*

*(2.1a)* $\quad l \overset{def}{=} F^2,$

*(2.1b)* $\quad m \overset{def}{=} F^2 - I_n,$

*applied to the tangent manifold at a point p are complementary projection operators . Then there are two complementary tangent bundles $l^*$ and $m^*$ corresponding to l and m.*

**Proof.** In consequence of (1.1) and (2.1), we have

(2.2) $\quad l - m = I_n,$

(2.3a) $\quad l^2 = l \circ F^2 = F^4 = F^2 = l,$

(2.3b) $m^2 = moF^2 - m = -m$,

(2.3c) $lm = ml = 0$.

(2.2) and (2.3) prove the first part of the statement. Remaining part of the proof is follow similarly.

**Theorem2.2.** *we have*

*(2.4a)* $loF = Fol = F$,

*(2.4b)* $moF = Fom = 0$,

*(2.4c)* $loF^2 = F^2ol = l$,

*(2.4d)* $moF^2 = F^2om = 0$.

*that is ,F acts on l as an almost product structure operator and on m as null operator .*

**Proof.** From equation (2.1a), we have

$$l = F^2 \quad \Rightarrow \quad loF = F^3$$

then $loF = F$.

Hence the equation (2.4a) .
Remaining part of the proof is obvious.

**Theorem2.3.** *If rank ((F)) = n, then*

*(2.5a)* $l = I_n$ ,

*(2.5b)* $m = 0$,

*(2.5c)* *n is even*

*and Hyperbolic F-structure is an almost product structure.*

**Proof.** The equation (1.1) is equivalent to F $(F^2 - I_n) = 0$. Since rank (F) = n, inverse $^{-1}$ F of F exists .Hence we have

$$F^2 - I_n = 0.$$

Thus {F}is an almost product structure. Remaining part of the proof is follows similarly.

**Theorem2.4.** *We have*

*(2.6)* $(m+F)(m+F) = I_n$.

**Proof.** Using (2.3b) and (2.4b) and (2.1b) in left hand side of (2.6),we have

$$\begin{aligned}(m+F)(m+F) &= m^2 + Fom + Fom + F^2\\ &= m^2 + F^2\\ &= I_n .\end{aligned}$$

Hence the theorem.

**Theorem2.5.** *Let there exist a projection operator m and a tensor field F on $V_{n,}$ such that (2.6),(2.3b) and (2.4b) are satisfied. Then F satisfied the equation (1.1).*

**Proof.** In consequence of (2.3b) and (2.4b), the equation (2.6) reduces to

$$m - F^2 = -I_n,.$$

Multiplying this equation by F and using (2.4b), we get

$$moF - F^3 = -I_n F,$$

$$F^3 = F.$$

Hence, we have (1.1).

**Theorem2.6.** *On Hyperbolic F –structure manifold, put*

(2.7a) $p \overset{def}{=} m+F,$

(2.7b) $q \overset{def}{=} m+F$

*Then*

(2.8a) $pq = qp = I_n,$

(2.8b) $p^2 - p + q - I_n = 0,$

(2.8c) $q^2 - p + q - I_n = 0,$

(2.8d) $p^2 = q^2.$

**Proof**. From (2.7a),(2.7b), (2.3b) and (2.4b) ,We have

$$pq = -m + F^2 = I_n.$$

Similarly, we can prove $qp = I_n$. In consequence of (2.7a),(2.7b), (2.3b),(2.4b) and (2.1b), we obtain (2.8b),(2.8c),(2.8d) proved similarly.

**Theorem2.7.** *On a Hyperbolic F-structure manifold, if p and q are defined by (2.7), then*

(2.9a) $pl = F,$

(2.9b) $pm = -m ,$

(2.9c) $p^2 l = l,$

(2.9d) $p^2 m = m.$

**Proof.** We know that

$$p \overset{def}{=} m + F.$$

Using equation (2.4a), we get

$$pl = (m+F)l,$$

$$pl = m \circ l + F \circ l,$$

$$pl = F,$$

which is the equation (2.9a).
Using equation(2.4b) and (2.3b),we have

$$pm = m^2 + F \circ m,$$

$$pm = m^2,$$

$$pm = -m,$$

which is the equation (2.9b).
Using equation (2.3a), (2.3c),(2.3b),we get

$$p^2 = m^2 + F^2 + m \circ F,$$

$$p^2 l = m^2 l + F^2 \circ l + 2\, mlF,$$

$$p^2 l = -\, ml + F^2 \circ l + 2mlF,$$

$$p^2 l = l.$$

Which is the the equation (2.9c).
The proof of (2.9d) is follows similarly.

**Theorem 2.8.** *g satisfies the relation*

(2.10) $$g(\overline{X}, \overline{Y}) = {}'m(X, Y) - g(X, Y),$$

*where*

(2.11) $${}'m(X,Y) \overset{def}{=} g(mX, Y) = g(X, mY).$$

**Proof.** Barring $X$ or $Y$ in (1.2b) separately and subtracting the second resulting equation
from the first, we have

(2.12) $$2g(\overline{X}, \overline{Y}) = g(\overline{\overline{X}}, Y) + g(X, \overline{\overline{Y}}).$$

Now,using (2.1b) and (2.11) in (2.12), we obtain (2.10).

## 3. Integrability Conditions

**Theorem 3.1.** *The necessary and sufficient condition that $V_n$ be a Hyperbolic F- structure manifold of rank $2m<n$,is that it contains a tangent bundle $\pi_m$ of dimension m, a tangent bundle $\tilde{\pi}_m$,complex conjugate to $\pi_m$ and a tangent bundle $\pi_{n-2m}$ of real dimension n-2m such that $\pi_m \cap \tilde{\pi}_m = \phi$, $\pi_m \cap \pi_{n-2m} = \phi$, $\tilde{\pi}_m \cap \pi_{n-2m} = \phi$ and $\pi_m \cup \tilde{\pi}_m \cup \pi_{n-2m} = V_n$, projection on $\pi_{m}$, $\tilde{\pi}_m$, $\pi_{n-2m}$ being given by*

$$2L \overset{def}{=} -F^2 - F, \tag{3.1a}$$

$$2M \overset{def}{=} -F^2 + F, \tag{3.1b}$$

$$m \overset{def}{=} F^2 - I_n. \tag{3.1c}$$

**Proof.** Let $V_n$ be HyperbolicF – structure manifold .Corresponding to the eigen valus1and -1, let there be m linearly independent eigen vectors $\underset{x}{P}$, $x$ =1,2,…,m and m linearly independent complex conjugate eigen vectors $\underset{x}{Q}$,and n-2m linearly independent vectors $\underset{\alpha}{U}$, $\alpha$ = 1,2,……..,n-2m .Then[2]

$$\overset{x}{a}\underset{x}{P} = 0 \Rightarrow \overset{x}{a} = 0 \quad \forall \quad x,$$

$$\overset{x}{b}\underset{x}{Q} = 0 \Rightarrow \overset{x}{b} = 0 \quad \forall \quad x,$$

$$\overset{\alpha}{e}\underset{\alpha}{U} = 0 \Rightarrow \overset{\alpha}{e} = 0 \quad \forall \quad \alpha,$$

Now

(i) $$\overset{x}{c}\underset{x}{P} + \overset{x}{d}\underset{x}{Q} + \overset{\alpha}{h}\underset{\alpha}{U} = 0,$$

(ii) $$\overset{x}{c}\underset{x}{P} + \overset{x}{d}\underset{x}{Q} = 0,$$

and

$$\overset{x}{c}\underset{x}{\overline{P}} + \overset{x}{d}\underset{x}{\overline{Q}} = 0,$$

$$\Rightarrow \quad \overset{x}{c}(\underset{x}{P}) + \overset{x}{d}(-\underset{x}{Q}) = 0$$

(iii) $\Rightarrow \overset{x}{c}\underset{x}{P} - \overset{x}{d}\underset{x}{Q} = 0.$

From the equation (i), (ii) and (iii), we obtain

$$\overset{x}{c}\underset{x}{P} = 0 \ , \qquad \overset{x}{d}\underset{x}{Q} = 0, \qquad \overset{\alpha}{h}\underset{\alpha}{U} = 0 \ ,$$

or

$$\overset{x}{c} = \overset{x}{d} = \overset{\alpha}{h} = 0 \ , \ \forall \ x \text{ and } \alpha.$$

Therefore { $\underset{x}{P}, \underset{x}{Q}, \underset{\alpha}{U}$ } is linearly independent set. From equation (2.1a),(2.1b) and (2.1c), it can be easily seen that

(3.2a) $L\underset{x}{P} = -\underset{x}{P},$ (3.2b) $L\underset{x}{Q} = 0,$ (3.2c) $L\underset{\alpha}{U} = 0,$

(3.3a) $M\underset{x}{P} = 0$ , (3.4b) $\mathrm{M}\underset{x}{Q} = -\underset{x}{Q},$ (3.3c) $\mathrm{M}\underset{\alpha}{U} = 0$ .

(3.4a) $m\underset{x}{P} = 0$ , (3.4b) $m\underset{x}{Q} = 0,$ (3.4c) $m\underset{\alpha}{U} = -\underset{\alpha}{U}$ .

Thus, we proved that on Hyperbolic F- structure manifold $V_n$ ,there is a tangent bundle $\pi_m$ of dimension m, a tangent bundle $\tilde{\pi}_m$ complex conjugate to $\pi_m$ and a real line $\pi_{n-2m}$ of real dimension n-2m , such that $\pi_m \cap \tilde{\pi}_m = \phi$, $\pi_m \cap \pi_{n-2m} = \phi$, $\tilde{\pi}_m \cap \pi_{n-2m} = \phi$, and $\pi_m \cup \tilde{\pi}_m \cup \pi_{n-2m} = V_n$, projection on $\pi_m$, $\tilde{\pi}_m$, $\tilde{\pi}_{n-2m}$ being $L$ ,$M$,$m$ respectively.

Conversely, suppose that there is a tangent bundle $\pi_m$ of dimension m , a tangent bundle $\tilde{\pi}_m$ conjugate to $\pi_m$ and real line $\pi_{n-2m}$ such that $\pi_m, \tilde{\pi}_m$ and $\pi_{n-2m}$ have no common direction and span a linear manifold of dimension n . Let $\underset{x}{P}$ be m- linearly independent vector in $\pi_m$ and $\underset{x}{Q}$ be complex conjugate to $\underset{x}{P}$ be m- linearly independent vector $\tilde{\pi}_m$ and $\underset{\alpha}{U}$ be vector in $\pi_{n-2m}$.

Let { $\underset{x}{P}, \underset{x}{Q}, \underset{\alpha}{U}$ } span tangent bundle of dimension n. Then $\{\underset{x}{P}, \underset{x}{Q}, \underset{\alpha}{U}\}$ is linearly independent set.

Let us define the inverse set $\{\overset{x}{p}, \overset{x}{q}, \overset{\alpha}{u}\}$, such that

(3.5a) $I_n = \overset{x}{p} \otimes \underset{x}{P} + \overset{x}{q} \otimes \underset{x}{Q} + \overset{\alpha}{u} \otimes \underset{\alpha}{U}$ , (3.5b) $X = \overset{x}{p}(X)\underset{x}{P} + \overset{x}{q}(X)\underset{x}{Q} + \overset{\alpha}{u}(X)\underset{\alpha}{U}$ .

Let us put

(3.6) $F X \overset{def}{=} \{ \overset{x}{p}\left(X\right)\underset{x}{P} - \overset{x}{q}\left(X\right)\underset{x}{Q} - \overset{\alpha}{u}\left(X\right)\underset{\alpha}{U} \}$.

Then

(3.7) $F F X = \{ \overset{x}{p}\left(FX\right)\underset{x}{P} - \overset{x}{q}\left(FX\right)\underset{x}{Q} - \overset{\alpha}{u}\left(FX\right)\underset{\alpha}{U} \}$.

From the equation (3.3b) and (3.6), we get

(3.8a) $\overset{x}{p}(FX) = \overset{x}{p}(X)$ ,

(3.8b) $\overset{x}{q}(FX) = -\overset{x}{q}(X)$ ,

(3.8c) $\overset{\alpha}{u}(FX) = -\overset{\alpha}{u}(X)$ .

Using the equation (3.8), we have

(3.9) $F F X = \{ \overset{x}{p}\left(X\right)\underset{x}{P} + \overset{x}{q}\left(X\right)\underset{x}{Q} + \overset{\alpha}{u}\left(X\right)\underset{\alpha}{U} \}$,

(3.10) $F F X = I_n$.

Operating F on the equation (3.10), we get

(3.11) $F^3 = F \;\Rightarrow\; F^3 - F = 0$.

Thus the manifold admits a Hyperbolic F- structure manifold.
Hence the condition is sufficient.

**Corollary3.1.** *We have*

*(3.12a)* $L = -\overset{x}{p} \otimes \underset{x}{P}$,

*(3.12b)* $M = -\overset{x}{q} \otimes \underset{x}{Q}$ ,

*(3.12c)* $n = -\overset{\alpha}{u} \otimes \underset{\alpha}{U}$ .

**Proof.** From equation (3.1a) and (3.1b) and (3.1c), we have

(3.13) $L + M + n = -I_n$

Comparing the equations (3.5a) and (3.13), we get equations (3.12).

## 4. Curvature Tensor of Different kinds

In the Hyperbolic F- structure manifold $V_n$, Pseudo Projective curvature tensor $W^3$, Pseudo Conformal curvature tensor $C^3$, Pseudo Conharmonic curvature tensor L, Pseudo Concircular curvature tensor V, Pseudo H-conharmonic curvature tensors S, Pseudo H-Projective curvature tensor P, and Pseudo Bochner curvature tensor B are defined by[4]

(4.1) $$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y],$$

(4.2) $$C(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X) + \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],$$

(4.3) $$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)r(X) - g(X,Z)r(Y)],$$

(4.4) $$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

(4.5) $$\begin{aligned} S(X,Y,Z) = K(X,Y,Z) + \frac{1}{(n+4)}[&Ric(X,Z)Y - Ric(Y,Z)X \\ &+ Ric(\overline{X},Z)\overline{Y} - Ric(\overline{Y},Z)\overline{X} + 2Ric(\overline{X},Y)\overline{Z} + g(X,Z)r(Y) \\ &- g(Y,Z)r(X) + g(\overline{X},Z)r(\overline{Y}) - g(\overline{Y},Z)r(\overline{X}) \\ &+ 2g(\overline{X},Y)r(\overline{Z})], \end{aligned}$$

(4.6) $$\begin{aligned} P(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n+2)}[&Ric(Y,Z)X - Ric(X,Z)Y - Ric(\overline{Y},Z)\overline{X} \\ &+ Ric(\overline{X},Z) - 2Ric(X,\overline{Y})\overline{Z}], \end{aligned}$$

(4.7) $$\begin{aligned} B(X,Y,Z) = K(X,Y,Z) + \frac{1}{(n+4)}[&Ric(X,Z)Y - Ric(Y,Z)X \\ &+ g(X,Z)r(Y) - g(Y,Z)r(X) + Ric(\overline{X},Z)\overline{Y} \\ &- Ric(\overline{Y},Z)\overline{X} + g(\overline{X},Z)r(\overline{Y}) - g(\overline{Y},Z)r(\overline{X}) \\ &+ 2Ric(\overline{X},Y)\overline{Z} + 2g(\overline{X},Y)r(\overline{Z})] - \frac{R}{(n+2)(n+4)} \\ &[g(X,Z)Y - g(Y,Z)X + g(\overline{X},Z)\overline{Y} \\ &- g(\overline{Y},Z)\overline{X} + 2g(\overline{X},Y)\overline{Z}]. \end{aligned}$$

The associated tensors of *W, C, L, V, S, P, B* are defined by

(4.8a) $'W\ (X,Y,Z,T) = g\ (W(X,Y,Z,),T),$

(4.8b) $'C\,(X,Y,Z,T) = g\ (C\ (X,Y,Z)\ ,T),$

(4.8c) $'L\,(X,Y,Z,T) = g(L(X,Y,Z)\ ,T),$

(4.8d) $'V\ (X,Y,Z,T) = g(V(X,Y,Z),T),$

(4.8e) $'S\ (X,Y,Z,T) = g(S(X,Y,Z),T),$

*(4.8f)* $'P\ (X,Y,Z,T) = g(P(X,Y,Z),T)$

(4.8g) $'B$(X,Y,Z,T) =g(B(X,Y,Z),T).

Consequently-

(4.9) $$`W(X,Y,Z,T) = `K(X,Y,Z,T) - \frac{1}{(n-1)}[Ric(Y,Z)g(X,T) - Ric(X,Z)g(Y,T)],$$

(4.10) $$`C(X,Y,Z,T) = `K(X,Y,Z,T) - \frac{1}{(n-2)}[Ric(Y,Z)g(X,T) - Ric(X,Z)g(Y,T) - g(X,Z)Ric(Y,T) + g(Y,Z)Ric(X,T)] + \frac{R}{(n-1)(n-2)} [g(Y,Z)g(X,T) - g(X,Z)g(Y,T)],$$

(4.11) $$`L(X,Y,Z,T) = `K(X,Y,Z,T) - \frac{1}{(n-2)}[Ric(Y,Z)g(X,T) - Ric(X,Z)g(Y,T) + g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)],$$

(4.12) $$`V(X,Y,Z,T) = `K(X,Y,Z,T) - \frac{R}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)],$$

(4.13) $$`S(X,Y,Z,T) = `K(X,Y,Z,T) + \frac{1}{(n+4)}[Ric(X,Z)g(Y,T) - Ric(Y,Z)g(X,T) + Ric(\overline{X},Z)g(\overline{Y},T) - Ric(\overline{Y},Z)g(\overline{X},T) + 2Ric(\overline{X},Y)g(\overline{Z},T) + g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)$$

$$+g(\overline{X},Z)Ric(\overline{Y},T)-g(\overline{Y},Z)Ric(\overline{X},T)$$
$$+2g(\overline{X},Y)Ric(\overline{Z},T)],$$

(4.14) $$`P(X,Y,Z,T)=`K(X,Y,Z,T)-\frac{1}{(n+2)}[Ric(Y,Z)g(X,T)$$
$$-Ric(X,Z)g(Y,T)-Ric(\overline{Y},Z)g(\overline{X},T)$$
$$+Ric(\overline{X},Z)g(\overline{Y},T)-2Ric(X,\overline{Y})g(\overline{Z},T)],$$

(4.15) $$`B(X,Y,Z,T)=`K(X,Y,Z,T)+\frac{1}{(n+4)}[Ric(X,Z)g(Y,T)$$
$$-Ric(Y,Z)g(X,T)+g(X,Z)Ric(Y,T)$$
$$-g(Y,Z)Ric(X,T)+Ric(\overline{X},Z)g(\overline{Y},T)$$
$$-Ric(\overline{Y},Z)g(\overline{X},T)+g(\overline{X},Z)Ric(\overline{Y},T)$$
$$-g(\overline{Y},Z)Ric(\overline{X},T)+2Ric(\overline{X},Y)g(\overline{Z},T)$$
$$+2g(\overline{X},Y)Ric(\overline{Z},T)]-\frac{R}{(n+2)(n+4)}$$
$$[g(X,Z)g(Y,T)-g(Y,Z)g(X,T)$$
$$+g(\overline{X},Z)g(\overline{Y},T)-g(\overline{Y},Z)g(\overline{X},T)$$
$$+2g(\overline{X},Y)(\overline{Z},T)].$$

## 5. Recurrence and Recurrence Symmetry of Different Kinds

Let $'Q$ be any one of the curvature tensors $'W$, $`C$, $'L$, $'V$, $'S$, $'P$ and $'B$ .Then we define recurrence of different kinds in $'Q$ as follows[5]:

**Definition 5.1.** A Hyperbolic F-Structure manifold is said to be (1)-recurrent in $'Q$ if

(5.1) $$(\nabla`Q)(X,Y,Z,T,U)+'Q((\nabla F)(\overline{X},U)Y,Z,T)=A_1(U)'Q(X,Y,Z,T),$$

where $A_1(U)$is a $C^\infty$ function, called recurrence parameter.

**Definition 5.2.** The Hyperbolic F-structure manifold is said to be (2)-recurrent in $'Q$ , if

(5.2) $$(\nabla`Q)(X,Y,Z,T,U)+'Q(X,((\nabla F)(\overline{Y},U),Z,T)=A_1(U)\,'Q(X,Y,Z,T).$$

**Definition 5.3.** The Hyperbolic F-structure manifold is said to be (3)-recurrent in $'Q$ ,if

(5.3) $(\nabla`Q)(X,Y,Z,T,U) + `Q(X,Y((\nabla F)(\overline{Z},U),T) = A_1(U)`Q(X,Y,Z,T)$.

**Definition 5.4.** The Hyperbolic F-structure manifold is said to be (4)-recurrent in $'Q$, if

(5.4) $(\nabla`Q)(X,Y,Z,T,U) + `Q(X,Y,Z,(\nabla F)(\overline{T},U),T) = A_1(U)`Q(X,Y,Z,T)$.

**Definition 5.5.** The Hyperbolic F-structure manifold is said to be (12)-recurrent in $'Q$, if

$$(5.5a)\quad (\nabla`Q)(\overline{X},Y,Z,T,U) + `Q(\overline{X},(\nabla F)(\overline{Y},U),Z,T) + `Q((\nabla F)(X,U),Y,Z,T) = A_1(U)`Q(\overline{X},Y,Z,T),$$

or equivalently,

$$(5.5b)\quad (\nabla`Q)(X,\overline{Y},Z,T,U) + `Q((\nabla F)(\overline{X},U),\overline{Y},Z,T) + `Q(X,(\nabla F)(Y,U),Z,T) = A_1(U)`Q(X,\overline{Y},Z,T).$$

**Definition 5.6.** The Hyperbolic F-structure manifold is said to be (13)-recurrent in $'Q$, if

$$(5.6a)\quad (\nabla`Q)(X,Y,\overline{Z},T,U) + `Q((\nabla F)(\overline{X},U),Y,\overline{Z},T) + `Q(X,Y,(\nabla F)(Z,U),T) = A_1(U)`Q(X,Y,\overline{Z},T),$$

or equivalently,

$$(5.6b)\quad (\nabla`Q)(\overline{X},Y,Z,T,U) + `Q(\overline{X},Y(\nabla F)(\overline{Z},U),T) + `Q((\nabla F)(X,U),Y,Z,T) = A_1(U)`Q(\overline{X},Y,Z,T).$$

**Definition 5.7.** The Hyperbolic F-structure manifold is said to be (14)-recurrent in $'Q$, if

$$(5.7a)\quad (\nabla`Q)(X,Y,Z,\overline{T},U) + `Q((\nabla F)(\overline{X},U),Y,Z,\overline{T}) + `Q(X,Y,Z,(\nabla F)(T,U)) = A_1(U)`Q(X,Y,Z,\overline{T}),$$

or equivalently,

$$(5.7b)\quad (\nabla`Q)(\overline{X},Y,Z,T,U) + `Q(\overline{X},Y,Z,(\nabla F)(\overline{T},U)) + `Q((\nabla F)(X,U),Y,Z,T) = A_1(U)`Q(\overline{X},Y,Z,T).$$

**Definition 5.8.** The Hyperbolic F-structure manifold is said to be (23)-recurrent in $'Q$, if

$$(5.8a)\quad (\nabla`Q)(X,Y,\overline{Z},T,U) + `Q(X,(\nabla F)(\overline{Y},U),\overline{Z},T) + `Q(X,Y,(\nabla F)(Z,U),T) = A_1(U)`Q(X,Y,\overline{Z},T),$$

or equivalently,

$$(5.8b)\qquad (\nabla'Q)(X,\overline{Y},Z,T,U)+'Q(X,\overline{Y},(\nabla F)(\overline{Z},U),T)$$
$$+'Q(X,(\nabla F)(Y,U),Z,T)=A_1(U)'Q(X,\overline{Y},Z,T),$$

**Definition 5.9.** The Hyperbolic F-structure manifold is said to be (24) recurrent in $'Q$, if

$$(5.9a)\qquad (\nabla'Q)(X,Y,Z,\overline{T},U)+'Q(X,(\nabla F)(\overline{Y},U),Z,\overline{T})$$
$$+'Q(X,Y,Z,(\nabla F)(T,U))=A_1(U)'Q(X,Y,Z,\overline{T}),$$

or equivalently,

$$(5.9b)\qquad (\nabla'Q)((X,\overline{Y},Z,T,U)+'Q(X,\overline{Y},Z,(\nabla F)(\overline{T},U))$$
$$+'Q(X,(\nabla F)(Y,U),Z,T)=A_1(U)'Q(X,\overline{Y},Z,T).$$

**Definition 5.10.** The Hyperbolic F-structure manifold is said to be (34)-recurrent in $'Q$, if

$$(5.10a)\qquad (\nabla'Q)(X,Y,Z,\overline{T},U)+'Q(X,Y,(\nabla F)(\overline{Z},U),\overline{T})$$
$$+'Q(X,Y,Z,(\nabla F)(T,U)=A_1(U)'Q(X,Y,Z,\overline{T}),$$

or equivalently,

$$(5.10b)\qquad (\nabla'Q)(X,Y,\overline{Z},T,U)+'Q(X,Y,\overline{Z},(\nabla F)(\overline{T},U)$$
$$+'Q(X,Y,(\nabla F)(Z,U),T)=A_1(U)'Q(X,Y,\overline{Z},T).$$

**Definition 5.11.** The Hyperbolic F-structure manifold is said to be (123)-recurrent in $'Q$, if

$$(5.11a)\qquad (\nabla'Q)(X,\overline{Y},\overline{Z},T,U)+'Q((\nabla F)(\overline{X},U),\overline{Y},\overline{Z},T)$$
$$+'Q(X,(\nabla F)(Y,U),\overline{Z},T)+'Q(X,\overline{Y},(\nabla F)(Z,U),T)$$
$$=A_1(U)'Q(X,\overline{Y},\overline{Z},T),$$

or equivalently,

$$(5.11b)\qquad (\nabla'Q)(\overline{X},Y,\overline{Z},T,U)+'Q(\overline{X},(\nabla F)(\overline{Y},U)\overline{Z},T)$$
$$+'Q((\nabla F)(X,U),Y,\overline{Z},T)+'Q(\overline{X},Y,(\nabla F)(Z,U),T)$$
$$=A_1(U)'Q(\overline{X},Y,\overline{Z},T),$$

or equivalently,

$$(5.11c)\quad (\nabla 'Q)(\overline{X},\overline{Y},Z,T,U)+'Q(\overline{X},\overline{Y},(\nabla F)(\overline{Z},U),T)$$
$$+'Q((\nabla F)(X,U),\overline{Y},Z,T)+Q(\overline{X},(\nabla F)(Y,U),Z,T)$$
$$=A_1(U)'Q(\overline{X},\overline{Y},Z,T).$$

**Definition 5.12.** The Hyperbolic F-structure manifold is said to be (124)-recurrent in $'Q$ ,if

$$(5.12a)\ (\nabla 'Q)(X,\overline{Y},Z,\overline{T},U)+'Q((\nabla F)(\overline{X},U),\overline{Y},Z,\overline{T})+'Q(X,(\nabla F)(Y,U),Z,\overline{T})$$
$$+'Q(X,\overline{Y},Z,(\nabla F)(T,U))=A_1(U)'Q(X,\overline{Y},Z,\overline{T}),$$

or equivalently,

$$(5.12b)\ (\nabla 'Q)(\overline{X},Y,Z,\overline{T},U)+'Q(\overline{X},(\nabla F)(\overline{Y},U)Z,\overline{T})+'Q((\nabla F)(X,U),Y,Z,\overline{T})$$
$$+'Q(\overline{X},Y,Z,(\nabla F)(T,U))=A_1(U)'Q(\overline{X},Y,Z,\overline{T}),$$

or equivalently,

$$(5.12c)\ (\nabla 'Q)(\overline{X},\overline{Y},Z,T,U)+'Q(\overline{X},\overline{Y},Z,(\nabla F)(\overline{T},U))+'Q((\nabla F)(X,U),\overline{Y},Z,T)$$
$$+'Q(\overline{X},(\nabla F)(Y,U),Z,T)=A_1(U)'Q(\overline{X},Y,Z,\overline{T}),$$

**Definition 5.13.** The Hyperbolic F-structure manifold is said to be (134)-recurrent in $'Q$ , if

$$(5.13a)\,(\nabla 'Q)(X,Y,\overline{Z},\overline{T},U)+'Q((\nabla F)(\overline{X},U),Y,\overline{Z},\overline{T})+'Q(X,Y,(\nabla F)(Z,U),\overline{T})$$
$$+'Q(X,Y,\overline{Z},(\nabla F)(T,U))=A_1(U)'Q(X,Y,\overline{Z},\overline{T}),$$

or equivalently,

$$(5.13b)\,(\nabla 'Q)(\overline{X},Y,Z,\overline{T},U)+'Q(\overline{X},Y,(\nabla F)(\overline{Z},U),\overline{T})+'Q((\nabla F)(X,U),Y,Z,\overline{T})$$
$$+'Q(\overline{X},Y,Z,(\nabla F)(T,U))=A_1(U)'Q(\overline{X},\overline{Y},Z,T),$$

or equivalently,

$$(5.13c)\,(\nabla 'Q)(\overline{X},Y,\overline{Z},T,U)+'Q(\overline{X},Y,\overline{Z},(\nabla F)(\overline{T},U))+'Q((\nabla F)(X,U),Y,\overline{Z},T)$$
$$+'Q(\overline{X},Y,(\nabla F)(Z,U),T)=A_1(U)'Q(\overline{X},Y,\overline{Z},T).$$

**Definition 5.14.** The Hyperbolic F-structure manifold is said to be (234)-recurrent in $'Q$ , if

$$(5.14a)\,(\nabla 'Q)(X,\overline{Y},Z,\overline{T},U)+'Q(X,\overline{Y},(\nabla F)(\overline{Z},U),\overline{T})+'Q(X,(\nabla F)(Y,U),Z,\overline{T})$$
$$+'Q(X,\overline{Y},Z,(\nabla F)(T,U))=A_1(U)'Q(X,\overline{Y},Z,\overline{T}),$$

or equivalently,

$$(5.14b)\ (\nabla `Q)(X,Y,\overline{Z},\overline{T},U)+`Q(X,(\nabla F)(\overline{Y},U)\overline{Z},\overline{T})+`Q(X,Y,(\nabla F)(Z,U),\overline{T})$$
$$+`Q(X,Y,\overline{Z},(\nabla F)(T,U))=A_1(U)`Q(X,Y,\overline{Z},\overline{T}),$$

or equivalently,

$$(5.14c)\ (\nabla `Q)(X,\overline{Y},\overline{Z},T,U)+`Q(X,\overline{Y},\overline{Z},(\nabla F)(\overline{T},U))+`Q(X,(\nabla F)(Y,U),\overline{Z},T)$$
$$+`Q(X,\overline{Y},(\nabla F)(Z,U),T)=A_1(U)`Q(X,\overline{Y},Z,T).$$

**Definition 5.15.** The Hyperbolic F-structure manifold is said to be (1234)-recurrent in $'Q$, if

$$(5.15a)\qquad (\nabla `Q)(X,\overline{Y},\overline{Z},\overline{T},U)+'Q(X\nabla F(\overline{X},U)\overline{Y},\overline{Z},\overline{T})$$
$$+`Q(X(\nabla F)(Y,U),\overline{Z},\overline{T})+`Q(X,\overline{Y},(\nabla F)(Z,U),\overline{T})$$
$$+`Q(X,\overline{Y},\overline{Z},(\nabla F)(T,U))=A_1(U)`Q(X,\overline{Y},\overline{Z},\overline{T}),$$

or equivalently,

$$(5.15b)\qquad (\nabla `Q)(\overline{X},Y,\overline{Z},\overline{T},U)+'Q(\overline{X},(\nabla F)(\overline{Y},U)\overline{Z},\overline{T})$$
$$+`Q(\overline{X},Y,\overline{Z},(\nabla F)(T,U)+`Q((\nabla F)(X,U),Y,\overline{Z},\overline{T})$$
$$+`Q(\overline{X},Y,(\nabla F)(Z,U),\overline{T})=A_1(U)`Q(\overline{X},Y,\overline{Z},\overline{T}),$$

or equivalently,

$$(5.15c)\qquad (\nabla `Q)(\overline{X},\overline{Y},\overline{Z},T,U)+'Q(\overline{X},\overline{Y},\overline{Z},(\nabla F)(\overline{T},U))$$
$$+`Q((\nabla F)(X,U),\overline{Y},\overline{Z},T)+`Q(\overline{X},(\nabla F)(Y,U)\overline{Z},T)$$
$$+'Q(\overline{X},\overline{Y},(\nabla F)(Z,U),T)=\mathrm{A}_1(\mathrm{U})`Q(\overline{X},\overline{Y},\overline{Z},T).$$

or equivalently,

$$(5.15d)\qquad (\nabla `Q)(\overline{X},\overline{Y},Z,\overline{T},U)+`Q(\overline{X},\overline{Y},(\nabla F)(\overline{Z},U),\overline{T})$$
$$+`Q((\nabla F)(X,U),\overline{Y},Z,\overline{T})+`Q(\overline{X},(\nabla F)(Y,U),Z,\overline{T})$$
$$+`Q(\overline{X},\overline{Y},Z,(\nabla F)(T,U))=\mathrm{A}_1(\mathrm{U})`Q(\overline{X},\overline{Y},Z,\overline{T}).$$

**Definition5.16.** The (1), (2), (3), (4), (12), (13), (14), (23), (24), (34), (123), (124), (134), (234) and (1234) - recurrent Hyperbolic F- structure manifold are said to be $'Q$ -symmetric ,if

$$(5.16)\qquad A_1(U)=0.$$

**Theorem 5.1.** *In the (1)-recurrent Hyperbolic F-structure manifold if any two of the conditions hold for the same recurrence parameter then third also holds:*

***(i)*** *It is Pseudo conharmonic – (1)-recurrent,*

***(ii)****It is Pseudo conformal – (1)-recurrent,*

***(iii)****It is Pseudo concircular – (1)-recurrent.*

**Proof.** From the equation (4.10) (4.11) and (4.12), we have

(5.17) $$`L(X,Y,Z,T) = `C(X,Y,Z,T) - \frac{n}{(n-2)}\{`K(X,Y,Z,T) - `V(X,Y,Z,T)\}.$$

Barring X in equation (5.17) we get

(5.18) $$`L(\overline{X},Y,Z,T) = `C(\overline{X},Y,Z,T) - \frac{n}{(n-2)}\{`K(\overline{X},Y,Z,T) - `V(\overline{X},Y,Z,T)\}.$$

Now from equation (5.17) we have

(5.19) $$A_1(U)`L(X,Y,Z,T) = A_1(U)`C(X,Y,Z,T) - \frac{nA_1(U)}{(n-2)} \{`K(X,Y,Z,T) - `V(X,Y,Z,T)\}.$$

Differentiating (5.18) with respect to U, using equation (5.18) and then barring X in the resulting equation, we get

(5.20) $$(\nabla`L)(X,Y,Z,T,U) + `L((\nabla F)(\overline{X},U),Y,Z,T) = (\nabla`C)(X,Y,Z,T,U) + `C((\nabla F)(\overline{X},U),Y,Z,T) - \frac{n}{(n-2)}\{(\nabla`K)(X,Y,Z,T,U) + `K((\nabla F)(\overline{X},U),Y,Z,T) - (\nabla`V)(X,Y,Z,T) - `V((\nabla F)(\overline{X},U),Y,Z,T)\}.$$

Subtracting equation (5.20) from (5.19), we get

(5.21) $$\begin{aligned}&(\nabla`L)(X,Y,Z,T,U) + `L((\nabla F)(\overline{X},U),Y,Z,T) - A_1(U)`L(X,Y,Z,T)\\ &= (\nabla`C)(X,Y,Z,T,U) + `C((\nabla F)(\overline{X},U),Y,Z,T) - A_1(U)`C(X,Y,Z,T)\\ &- \frac{n}{(n-2)}\{(\nabla`K)(X,Y,Z,T) + `K((\nabla F)(\overline{X},U),Y,Z,T)\\ &- A_1(U)`K(X,Y,Z,T) - (\nabla`V)(X,Y,Z,T,U) - `V((\nabla F)(\overline{X},U),Y,Z,T)\\ &+ A_1(U)`V(X,Y,Z,T)\}.\end{aligned}$$

If a (1)-recurrent Hyperbolic F-structure manifold is Pseudo conharmonic (1)-recurrent and Pseudo conformal (1)-recurrent for the same recurrence parameter then from equation (5.21), we get

$$(\nabla`L)(X,Y,Z,T,U) + `L((\nabla F)(\overline{X},U),Y,Z,T) = A_1(U)`L(X,Y,Z,T)$$

which shows that the manifold is Pseudo conharmonic(1)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either Pseudo conharmonic (1)-recurrent and Pseudo concircular (1)-recurrent or Pseudo conformal (1)-recurrent and Pseudo concircular(1)-recurrent then it is either

Pseudo conharmonic (1)-recurrent or Pseudo conformal (1)-recurrent for the same recurrence paramter.

**Theorem 5.2.** *In the (1)-recurrent Hyperbolic F-structure manifold if any two of the*
*conditi on hold for the same recurrence parameter then third also holds:*
***(i)** It is Pseudo conharmonic –(1)-symmetric,*
***(ii)**It is Pseudo conformal –(1)-symmetric,*
***(iii)**It is Pseudo concircular -(1)-symmetric.*

**Proof.** If a (1)-symmetric Hyperbolic F-structure manifold is Pseudo conharmonic (1) symmetric and Pseudo conformal (1)-symmetric, then from equation (5.21), we get

$$(\nabla`L)(X,Y,Z,T,U)+`L((\nabla F)(\overline{X},U),Y,Z,T)=0,$$

which shows that the manifold is Pseudo conharmonic (1)-symmetric.

Similarly, it can be shown that if the (1)-symmetric Hyperbolic F-stucture manifold is either Pseudo conharmonic (1)-symmetric and Pseudo concircular(1)-symmetric or Pseudo conformal (1)-symmetric and Pseudo concircular (1)-symmetric then it is either Pseudo conformal (1)-symmetric or Pseudo conharmonic (1)- symmetric for the same recurrence parameter.

**Theorem 5.3.** *In the (12)-recurrent Hyperbolic F-structure manifold, if any two of the conditions hold for the same recurrence parameter then third also holds:*
***(i)** It is Pseudo conharmonic –(12)-recurrent,*
***(ii)**It is Pseudo conformal –(12)-recurrent,*
***(iii)**It is Pseudo concircular -(12)-recurrent.*

**Proof.** Barring X and Y in equation (5.17), we get

$$(5.22)\quad `L(\overline{X},\overline{Y},Z,T)=`C(\overline{X},\overline{Y},Z,T)-\frac{n}{(n-2)}\{`K(\overline{X},\overline{Y},Z,T)-`V(\overline{X},\overline{Y},Z,T)\}.$$

From equation (5.22) we obtain

$$(5.23)\quad \begin{aligned} A_1(U)`L(X,\overline{Y},Z,T)&=A_1(U)`C(X,\overline{Y},Z,T)\\ &-\frac{A_1(U)n}{(n-2)}\{`K(X,\overline{Y},Z,T)-`V(X,\overline{Y},Z,T)\}.\end{aligned}$$

Differentiating equation (5.22) with respect to U and using equation (5.22) and then barring X in the resulting equation, we get

$$(5.24)\quad (\nabla`L)(X,\overline{Y},Z,T,U)+`L((\nabla F)(\overline{X},U),\overline{Y},Z,T)$$
$$+`L(X,(\nabla F)(Y,U),Z,T)=(\nabla`C)(X,\overline{Y},Z,T,U)+`C((\nabla F)(\overline{X},U),\overline{Y},Z,T)$$
$$+`C(X,(\nabla F)(Y,U),Z,T)-\frac{n}{(n-2)}\{(\nabla`K)(X,\overline{Y},Z,T,U)$$
$$+`K((\nabla F)(\overline{X},U),\overline{Y},Z,T)+`K(X,(\nabla F)(Y,U),Z,T)-(\nabla`V)(X,\overline{Y},Z,T,U)$$
$$-`V((\nabla F)(\overline{X},U),\overline{Y},Z,T)-`V(X,(\nabla F)(Y,U),Z,T)\}.$$

Subtracting equation (5.24) from (5.23), we get

$$(5.25)\quad (\nabla`L)(X,\overline{Y},Z,T,U)+`L((\nabla F)(\overline{X},U),\overline{Y},Z,T)+`L(X,(\nabla F)(Y,U),Z,T)$$
$$-A_1(U)`L(X,\overline{Y},Z,T)=(\nabla`C)(X,\overline{Y},Z,T,U)+`C((\nabla F)(\overline{X},U),\overline{Y},Z,T)$$
$$+`C(X,(\nabla F)(Y,U),Z,T)-A_1(U)`C(X,\overline{Y},Z,T)$$
$$-\frac{n}{(n-2)}\{(\nabla`K)(X,\overline{Y},Z,T,U)$$
$$+`K((\nabla F)(\overline{X},U),\overline{Y},Z,T)+`K(X,(\nabla F)(Y,U),Z,T)$$
$$-A_1(U)`K(X,\overline{Y},Z,T)-(\nabla`V)(X,\overline{Y},Z,T,U)-`V((\nabla F)(\overline{X},U),\overline{Y},Z,T)$$
$$-`V(X,(\nabla F)(Y,U),Z,T)+A_1(U)`V(X,\overline{Y},Z,T)\}.$$

If a (12)-recurrent Hyperbolic F-structure manifold is Pseudo conharmonic (12)-recurrent and Pseudo conformal (12)-recurrent for the same recurrence parameter then from equation (5.25), we get

$$(\nabla`L)(X,\overline{Y},Z,T,U)+`L((\nabla F)(\overline{X},U),\overline{Y},Z,T)+`L(X,(\nabla F)(Y,U),Z,T)$$
$$=A_1(U)`L(X,\overline{Y},Z,T).$$

Which shows that the manifold is Pseudo conharmonic (12) recurrent.

Similarly, it can be shown that if the (12)-recurrent Hyperbolic F-stucture manifold is either Pseudo conharmonic (12)-recurrent and Pseudo concircular(12)-recurrent or Pseudo conformal (12)recurrent and Pseudo concircular (12)-recurrent then it is either Pseudo conformal (12)-recurrent or Pseudo conharmonic (12)-recurrent for the same recurrence parameter.

**Theorem 5.2.** *In the (12)-recurrent Hyperbolic F-structure manifold if any two of the*
*conditions hold for the same recurrence parameter then third also holds:*

***(i)*** *It is Pseudo conharmonic – (12)-symmetric,*

***(ii)****It is Pseudo conformal – (12)-symmetric,*

***(iii)****It is Pseudo concircular - (12)-symmetric.*

**Proof.** If a (12)-symmetric Hyperbolic F-structure manifold is Pseudo conharmonic (12) symmetric and Pseudo conformal (12)-symmetric then from equation (5.25)

$$(\nabla`L)(X,\overline{Y},Z,T,U)+`L((\nabla F)(\overline{X},U),\overline{Y},Z,T)+`L(X,(\nabla F)(Y,U),Z,T)=0,$$

which shows that the manifold is Pseudo conharmonic (12)- Symmetric.

Similarly, it can be shown that if the (12)-symmetric Hyperbolic F-stucture manifold is either Pseudo conharmonic (12)-symmetric and Pseudo concircular (12)-symmetric or Pseudo conformal (12)-symmetric and Pseudo concircular (12)-symmetric then it is either Pseudo conformal (12)-symmetric or Pseudo conharmonic (12)- symmetric for the same recurrence parameter.

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