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Modified Approach to Multiobjective Matrix Game with Vague Payoffs*

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Abstract: Zhou et. al. (ICFIE ASC 40, pp.543-550, 2007), in their work on multi-objective two person zero sum matrix game, used a single order function to convert a vague set to a crisp one. Their order function was based only on the membership function for favourable events. This work defines a new order function based on non-membership function to deal separately with unfavourable values. Our approach of using two order functions for vague to crisp conversion makes the crisp value more appropriate. Numerical computations on Zhou's problem have also been performed according to our approach that lead to an improvement over their results in terms of optimized pay-off value of the game.

Keywords: Vague set, Matrix games, Order function, Multiobjective game.

2000 Mathematics Subject Classification No.: 68M15

1. Introduction

Game theory usually optimizes those systems that involve more than one decision makers. In conventional game theory one studies the conflicting interests among several decision makers (players), each of them trying to maximize a scalar payoff. On the other hand in Multi-Objective Mathematical Programming (MOMP) the conflicts within a single decision maker are handled. Multi-objective matrix games are capable of dealing with both the types of conflicts. The simplest matrix game is a zero sum game involving two players only.

In fact, most of the real world problems that can be modeled as games have imprecise or vague information about its elements. In such cases fuzzy

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set theory provide an adequate tool that can be used to handle the imprecision in decision making problems. Fuzzy set theory was introduced for the first time in game theory by Butnariu[1]. Thereafter, many researchers such as Compas[2], Sakawa and Nishizaki[3], Collins and Hu[4], Bector et al.[5], Vijay et al.[6] studied matrix games, specially, two person zero sum game in fuzzy environment.

Gau and Buchrer[7] introduced the concept of vague sets. Zhou et al.[8] studied multi-objective two person zero sum matrix game having payoff values as vague sets. They defined an order function based on the membership function for favourable events and used this function to convert the vague set to the crisp one. This paper carries forward the work of Zhou et al. by defining one more order function based on non-membership function. The non-membership function assigns membership grades on the basis of unfavourable evidences. Our approach of using both the order functions for vague to crisp conversion in multi-objective two person zero sum matrix game with vague payoff values makes the crisp value more appropriate. Similar to Zhou et al.[8], two methods to transform a multiobjective game problem to a game with single objective have been discussed. We have applied our approach to the numerical example given in[8] and observed that there was change in optimal strategies of the two players leading to a higher value of the game.

2. Vague set and its order functions

Vague set: A vague set $\tilde{A}$ in the universe of discourse X is characterized by a true membership function $t_{\tilde{A}} : X \rightarrow [0,1]$ and a false membership function $f_{\tilde{A}} : X \rightarrow [0,1]$. The degree of membership for any element x in the vague set is bounded by a sub interval [$t_{\tilde{A}}(x)$, 1 - $f_{\tilde{A}}(x)$], where the degree $t_{\tilde{A}}(x)$ is called lower bound of membership degree of x derived from evidences for x and $f_{\tilde{A}}(x)$ is the lower bound of membership degree on the negation of x derived from the evidences against x ; $t_{\tilde{A}}(x) + f_{\tilde{A}}(x) \leq 1$. In the extreme case of equality $t_{\tilde{A}}(x) = 1 - f_{\tilde{A}}(x)$, the vague set reduces to the fuzzy set with interval value of the membership degree reducing to a single value $t_{\tilde{A}}(x)$. In general, however,

$$t_{\tilde{A}}(x) \leq \text{exact membership degree of } x \leq 1 - f_{\tilde{A}}(x) .$$

The following expressions can be used to represent a vague set $\tilde{A}$ for finite and infinite universe of discourse X respectively.

$$\tilde{A} = \sum_{k=1}^{n} [t_{\tilde{A}}(x_k), 1 - f_{\tilde{A}}(x_k)] / x_k \quad \text{and} \quad \tilde{A} = \int_X [t_{\tilde{A}}(x_k), 1 - f_{\tilde{A}}(x_k)] / x_k$$

A vague set is represented pictorially as

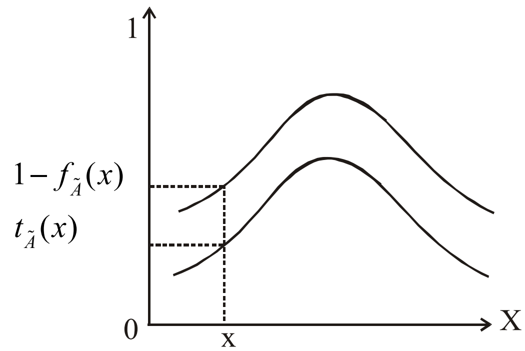

Fig-1

Order functions: The membership functions based on favourable and unfavourable evidences that is, the true and false membership functions are related to each other by the expression $t_{\tilde{A}}(x) + f_{\tilde{A}}(x) \leq 1$. We define the remaining unknown part

(2.1) $$\pi_A(x) = [1 - t_A(x) - f_A(x)]$$

as the *hesitation function.* This function may be considered as the degree of uncertainty that could not be resolved through any type of evidences. If $\pi_A(x) = 0$, for every x then the vague set reduces to a fuzzy set. For example, a vague value $[0\cdot 4, 0\cdot 7]$, means $t_{\tilde{A}}(x) = 0\cdot 4, 1 - f_{\tilde{A}}(x) = 0\cdot 7$. It can be interpreted as "the degree of element x belongs to vague set $\widehat{A}$ is $0\cdot 4$; the degree of element x does not belong to vague set $\widehat{A}$ is 0.3 and the degree of hesitation is 0.3". In a voting model, it can be interpreted as "the vote for resolution is 4 in favor, 3 against and 3 abstentions". Let us look into the voting model more deeply. It may be possible that some members in the abstained group may be inclined to vote for the proposal; some may be clearly against while the rest may be indecisive. If a criterion could be established to have such classification in abstentions then the degree in favour or against could be increased by reducing the indecisive part. Order function mainly tries to reduce the degree of the hesitation function $\pi_A(x)$ in proportion of $t_A(x) / f_A(x)$ in the favour of true/false membership function. In[8], the authors have used a single order function defined as given below:

(2.2) $$O_1(t_A(x)) = t_A(x) + t_A(x)\,\pi_A(x).$$

We define a new order function $O_2(t_A(x))$ by first reducing the hesitation function in favour of false membership function as given below:

(2.3) $$O(f_A(x)) = f_A(x) + f_A(x)\,\pi_A(x)$$

(2.4) $$O_2(t_A(x)) = 1 - O(f_A(x)).$$

In the present work we shall use a new order function $O(t_A(x))$ that combines the order functions given in (2.2) and (2.4), that is,

(2.5) $$O(t_A(x)) = \frac{1}{2}[O_1(t_A(x)) + O_2(t_A(x))].$$

3. Problem formulation and solution concepts

Let R^n denote the n-dimensional Euclidean space and R^n_+ be its non-negative part. Let $A \in R^{m\times n}$ be a $(m \times n)$ real matrix and $e^T = (1,1,...,1)$ be a vector of "ones" whose dimension is specified as per the specific context. Mathematically, a two person zero sum crisp matrix game is a triplet $G = (S^m, S^n, A)$, where $S^m = \{x \in R^m_+, e^T x = 1\}$ and $S^n = \{y \in R^n_+, e^T y = 1\}$. Here S^m / S^n is the strategy space for Player I / Player II and A is called the pay-off matrix. It is a convention to assume that Player I is a maximizing player and Player II is the minimizing one. Further for $x \in S^m$, $y \in S^n$ the scalar $x^T Ay$ is the pay-off to Player I and as the game G is zero sum, the pay-off of the Player II is $-x^T Ay$.

Definition 3.1: The triplet $(x^*, y^*, v^*) \in S^m \times S^n \times R$ is called a solution of the game G if

(i) $(x^{*T} Ay) \geq v^*$ for all $y \in S^n$,

and

(ii) $(x^T Ay^*) \leq v^*$ for all $x \in S^m$.

Here x^* and y^* are called the optimal strategies for Player I and Player II respectively and v^* is known as the value of the game G.

In the present paper, on every choice of strategies by the two Players, the payoff value for each of them is represented as a vague set. The outcome is assumed to have a zero sum structure. Although the sum of the payoffs of the two players may not appear to be zero in vague sets, yet the basis of zero sum assumption is the fact that the gains of one player are the losses of the other. Let both the players have N objectives. The payoff matrix of the k^{th} objective of Player I is denoted by

$$\tilde{A}^k = \begin{pmatrix} \tilde{a}_{11}{}^k & \tilde{a}_{12}{}^k \cdots \tilde{a}_{1n}{}^k \\ \tilde{a}_{21}{}^k & \tilde{a}_{22}{}^k \cdots \tilde{a}_{2n}{}^k \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \tilde{a}_{m1}{}^k & \tilde{a}_{m2}{}^k \cdots \tilde{a}_{mn}{}^k \end{pmatrix}$$

where $\tilde{A}^k = (\tilde{a}_{ij}^k)_{m\times n}, \tilde{a}_{ij}^k = [t_{ij}, t_{ij}*], t_{ij}* = 1 - f_{ij}, i = 1,...,m; j = 1,...n, k = 1,...,N.$
We give below the two approaches that modify the methods used by Zhou et al.[8] to solve a multiobjective matrix game whose payoff values are vague sets.

Approach I

In the first step the vague payoff matrix $\tilde{A}^k$ has to be converted to crisp payoff matrix for each objective using the order function. Unlike[8], we define two order functions given by (2.2) and (2.4) and combine these in the form as given in (2.5). The order function (2.5) is then used so that the $(i,j)^{th}$ vague payoff $\tilde{a}_{ij}^k$ of k^{th} objective payoff matrix of Player I is reduced to the $(i,j)^{th}$ crisp payoff.

The second step of the approach is similar to[8] that transform the multiobjective game to a single objective game by selecting proper objective weights. Suppose both the players use the same weight vector $w = (w_1, w_2, ..., w_N)^T$; $w_k \geq 0$ and $\sum_{k=1}^{N} w_k = 1$, where $w_k (k = 1, 2, ..., N)$ denotes the weight of objective f_k. Then the relative membership degree of all objectives can be converted into a linear weighted sum

$$b_{ij} = \sum_{k=1}^{N} w_k b_{ij}^k$$

where b_{ij}^k is crisp payoff value of each objective, $i = 1, 2, ..., m; j = 1, 2, ..., n$, obtained by using the order function (2.5). Thus the multiobjective two-person zero-sum matrix game is converted into a single objective two person zero-sum matrix game with payoff matrix $B = (b_{ij})_{m\times n}$. According to the solution of matrix game, the equilibrium strategy $x*$ and $y*$ of the two Players and the expected value of the game v can be calculated.

Approach II

In this approach the weights are also considered to be the vague sets. In the first step of this approach, the two given vague matrices are linearly aggregated with respect to vague weights to obtain a single vague matrix. In the second step the aggregated vague pay-off matrix is converted to the crisp one using order function (2.5). It can then be solved similar to first approach.

In both the approaches after acquiring the equilibrium strategies x^* and y^*, minimum and maximum possible equilibrium values can be achieved as following:

$$\underline{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{*T}\, \underline{\tilde{a}}_{ij}\, y_j^* \text{ and } \overline{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{*T}\, \overline{\tilde{a}}_{ij}\, y_j^*$$

where $\underline{\tilde{a}}_{ij}$ and $\overline{\tilde{a}}_{ij}$ are the lower and upper bounds respectively for the vague value of optimum strategy of vague pay-off matrix $\tilde{A}$.

4. Numerical example

Following numerical example[8] has been worked out according to our approaches to demonstrate the numerical improvement in the value of the game for the winning player.

Suppose there are two companies I and II aiming to enhance the sales amount and market share of a product in a targeted market. Under the circumstance that the demand amount of the product in the targeted market basically is fix, the sales amount and market share of one company increases, following the decrease of the sales amount and market share of another company, but the sales amount is not certain to be proportional to the market share. The two companies are considering about the three strategies to increase the sales amount and market share:

x_1: advertisement; x_2: reduce the price; x_3: improve the package.

This problem is a multiobjective two person zero sum matrix game. Further let Company I be Player I, adopting the strategy (x_1, x_2, x_3) and Company II be Player II, adopting strategy (y_1, y_2, y_3). Under the three strategies, the payoff matrices $\tilde{A}^1, \tilde{A}^2$ of targeted sales quantity f_1 (million) and market share f_2 (percentage) are separately indicated by vague value:

$$\tilde{A}^1 = \begin{pmatrix} [0\cdot 1, 0\cdot 2] & [0\cdot 2, 0\cdot 4] & [0\cdot 5, 0\cdot 7] \\ [0\cdot 2, 0\cdot 3] & [0\cdot 3, 0\cdot 5] & [0\cdot 4, 0\cdot 6] \\ -[0\cdot 3, 0\cdot 5] & -[0\cdot 2, 0\cdot 6] & [0\cdot 1, 0\cdot 2] \end{pmatrix},$$

$$\tilde{A}^2 = \begin{pmatrix} [0\cdot 2, 0\cdot 3] & [0\cdot 5, 0\cdot 6] & [0\cdot 8, 0\cdot 9] \\ [0\cdot 3, 0\cdot 4] & [0\cdot 2, 0\cdot 3] & [0\cdot 4, 0\cdot 7] \\ -[0\cdot 3, 0\cdot 4] & -[0\cdot 2, 0\cdot 3] & [0, 0\cdot 1] \end{pmatrix}.$$

Here, the negative vague value of the type "$-[v_1, v_2]$" in a payoff matrix is interpreted to indicate an unfavorable payoff between v_1 and v_2 for Company I and favorable in the same range for Company II.

Approach I

Suppose that the two companies agree that the weight of objective f_1 and f_2 separately are $0\cdot 4$ and $0\cdot 6$. Using the order functions (2.2) and (2.4) the defuzzification values of $\tilde{A}^1$ are

$$B_{11} = \begin{pmatrix} 0\cdot 11 & 0\cdot 24 & 0\cdot 60 \\ 0\cdot 22 & 0\cdot 36 & 0\cdot 48 \\ -0\cdot 36 & -0\cdot 28 & 0\cdot 11 \end{pmatrix}$$

and

$$B_{12} = \begin{pmatrix} 0\cdot 12 & 0\cdot 28 & 0\cdot 64 \\ 0\cdot 23 & 0\cdot 40 & 0\cdot 52 \\ -0\cdot 40 & -0\cdot 44 & 0\cdot 12 \end{pmatrix}.$$

Now according the (2.5) we get following matrix

$$B_1 = \frac{1}{2}[B_{11} + B_{12}]$$

$$= \begin{pmatrix} 0\cdot 115 & 0\cdot 260 & 0\cdot 620 \\ 0\cdot 225 & 0\cdot 380 & 0\cdot 500 \\ -0\cdot 380 & -0\cdot 360 & 0\cdot 115 \end{pmatrix}.$$

Similarly for the second objective $\tilde{A}^2$ we get

$$B_2 = \frac{1}{2}\left[B_{21} + B_{22}\right]$$

$$= \begin{pmatrix} 0\cdot 225 & 0\cdot 555 & 0\cdot 885 \\ 0\cdot 335 & 0\cdot 225 & 0\cdot 565 \\ -0\cdot 335 & -0\cdot 225 & 0\cdot 005 \end{pmatrix}$$

Considering the weight of each objective, the two objectives can be linearly aggregated as

$$B = 0\cdot 4B_1 + 0\cdot 6B_2$$
$$= \begin{pmatrix} 0\cdot 181 & 0\cdot 437 & 0\cdot 779 \\ 0\cdot 291 & 0\cdot 287 & 0\cdot 539 \\ -0\cdot 353 & -0\cdot 279 & 0\cdot 049 \end{pmatrix}.$$

This now becomes a traditional two players zero sum game with crisp payoff values given by *B*. According to the solution of classical matrix game, the equilibrium strategy x^* is (0.015, 0.985, 0) and y^* is (0, 577, 0.423, 0) and the expected payoff value v^* is $0\cdot 289$, which is better from Zhou et al.'s expected payoff value $0\cdot 282$.

Approach II

Choose the vague weights for objectives f_1, f_2 to be $w_1 = [0\cdot 3, 0\cdot 5]$ and $w_2 = [0\cdot 5, 0\cdot 7]$ respectively.

As the first step, the pay-off matrix $\tilde{A}^1, \tilde{A}^2$ can be aggregated as

$$\tilde{A} = \begin{pmatrix} [0\cdot 17, 0\cdot 289] & [0\cdot 295, 0\cdot 536] & [0\cdot 490, 0\cdot 760] \\ [0\cdot 201, 0\cdot 388] & [0\cdot 181, 0\cdot 408] & [0\cdot 296, 0\cdot 643] \\ -[0\cdot 227, 0\cdot 460] & -[0\cdot 154, 0\cdot 447] & [0\cdot 030, 0\cdot 163] \end{pmatrix}.$$

Using order function (2.5) as described earlier in Approach I, required crisp pay-off matrix is obtained as given below

$$B = \begin{pmatrix} 0\cdot 161 & 0\cdot 395 & 0\cdot 659 \\ 0\cdot 257 & 0\cdot 248 & 0\cdot 459 \\ -0\cdot 307 & -0\cdot 242 & 0\cdot 069 \end{pmatrix}.$$

According to the solution of classical matrix game, the equilibrium strategy x^* is $(0\cdot 037, 0\cdot 963, 0)$ and y^* is $(0\cdot 605, 0\cdot 395, 0)$ and expected pay-off value v^* is $0\cdot 253$, which is better from Zhou et al.'s expected pay-off value $0\cdot 232$.

According to second approach, the minimum possible equilibrium value and maximum possible equilibrium value can be achieved as

$$\underline{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i{}^{*T}\, \underline{\tilde{a}}_{ij} y_j{}^* = 0\cdot 193 \ \textit{and}\ \overline{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i{}^{*T}\, \overline{\tilde{a}}_{ij} y_j{}^* = 0\cdot 395.$$

5. Conclusion

In this paper, we have dealt a model of multiobjective two person zero sum matrix game whose pay-off values are vague sets. Our approach of using an order function different from[8] which is based on true as well as false membership functions, improves the payoff value for the winning player significantly.

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