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A Delay Mathematical Model for Spread of Carrier Dependent Infectious Diseases**

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Abstract: In this paper, a non-linear delay mathematical model for spread of carrier dependent infectious diseases has been proposed and analyzed. In the modeling process it is assumed that disease spreads due to the direct contact between susceptible and infective as well as through carriers. It is also assumed that the infective individuals transmit disease to susceptible individuals after some time lag 'τ'. It is further assumed that carriers follow logistic growth, whose growth rate depends on the human related activities. The equilibriums of the model have been obtained and their stability discussed. The critical value of time delay τ for Hopf-bifurcation has been obtained analytically.
Key words: mathematical model, epidemic, carriers, delay, Hopf-bifurcation.

1. Introduction

The study of spread of infectious diseases has been initiated by well known Kermack and McKendrick model. Since then various generalizations of this model has occurred in literature[1-6]. Most of the countries are being affected by various infectious diseases. Keeping the mode of transmission of infectious diseases in mind, several researchers of different countries have proposed and analyzed epidemic mathematical models. The main assumption in most of the available epidemic models is that the disease spreads in human population by direct contact between susceptibles and infectives. With this assumption various studies have been conducted. The conditions under which the disease may be eradicated from the population have been derived.

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In real situation it is observed that some of the diseases like dysentery, gastroenteritis, diarrhoea, etc. are being transmitted to the human population by direct contact as well as through vectors [7-14]. These diseases are known as vector borne diseases. It is observed that the third world countries are most affected by such diseases due to lack of cleanliness, wide occurrence of carriers such as flies, ticks, mites, etc. which are generally present in the environment. The carriers present in the environment transfer the bacteria of various diseases to the susceptible population. In 1976, Hethcote[15] studied the spread of infectious diseases by incorporating the density of carrier population in the model. In this model it is assumed that the density of carrier population is constant. Since then some mathematical models have been analyzed by considering the density of bacteria and carrier population[8, 9, 10, 11, 12, 13, 14,16]. In particular Ghosh, et. al.[10-12] studied the spread of carrier dependent infectious diseases by assuming carrier population as a dynamic variable. They have also considered the effect of role of environment on the carrier population in the modeling process. In these models the effect of human related activities on the density of carrier population has not been considered, Singh, et.al.[13-14] studied spread of carrier dependent epidemic models by considering that the growth rate of carrier population increases due to the human related activities. Das, et. al. presented an epidemic model for cholera disease by assuming that the intrinsic growth rate of carrier population is a linear function of total human population. There are some other mathematical models for the controls of epidemic using vaccination are also available in literature[17-19].

In all the above studied, it is assumed that the infective individuals transmit the disease to the susceptible population instantaneously. But in the real situation, individual who got infection some time before may transmit the disease to the susceptible. Thus time delay plays a very important role in the studies of infectious diseases. Recently, some mathematical models have been studied involving time delay in the spread of infectious diseases[20-23]. In these delay mathematical models the role of carriers in the spread of epidemics has not been considered.

In the present study, we develop and analyze a delay nonlinear mathematical model for diseases that are transmitted in both ways, i.e. the disease spreads by direct contact between susceptibles and infectives as well as through carriers. In the modeling process, we assume that the carriers follow logistic growth in its environment. It is also assumed that with the increase in human population density, the effects of human population related activities, like discharge of household wastes, open sewage drainage, industrial effluents in residential areas, open water storage tanks and ponds

etc. makes the environment conducive for the growth of carrier population density. The intrinsic growth rate of carrier population is assumed to be a linear function of human population. It is further assumed that the susceptibles are infected by carriers in direct proportion to the density of carrier population and infective population (bilinear interaction). In the modeling process we also assume that the infected individuals will be able to transmit the disease to the susceptible individuals after some time lag. The model is analyzed qualitatively to determine the stability of its associated equilibriums.

2. Mathematical Model

Let N(t) be the total human population at any time t. The total human population is being divided into two classes, (i) susceptible class X(t) and (ii) infective class Y(t). Let C(t) be the density of carrier population at time t. In the modeling process it is assumed that individuals of susceptible class become infected when they are directly coming to the contact of infective class (direct contact). Here it is also assumed that the infected individuals who got infection before time τ will be able to transmit the disease to the susceptible individuals. In other words at time t, the individuals who got infection at time $t-\tau$ are capable of transmitting the disease among susceptible individuals. The other way of transmission of the disease is through carrier (i.e, flies, ticks, mosquito etc.) population (indirect contact). Here it is assumed that when the carriers are coming in contact with the susceptible class, the susceptible individuals will get infected and move to the infective class. In the modelling process it is assumed that there is immigration of susceptibles with a constant rate '*A*'.

Keeping in view the above considerations, the system is governed by the following differential equations.

$$\begin{aligned}
\frac{dX(t)}{dt} &= A - \beta X(t)Y(t-\tau) - \lambda X(t)C(t) - dX(t) + \nu Y(t), \\
\frac{dY(t)}{dt} &= \beta X(t)Y(t-\tau) + \lambda X(t)C(t) - (\nu + \alpha + d)Y(t), \qquad (2.1) \\
\frac{dC(t)}{dt} &= (r_0 + r_1 N(t))C(t) - r_{10}C^2(t) - s_1 C(t),
\end{aligned}$$

$X(0) = X_0 > 0, Y(\theta) = Y_0 \geq 0$ for $\theta \in [-\tau,\ 0]$, $N(0) = X(0) + Y(0) = N_0 > 0$ and $C(0) = C_0 \geq 0$,

where β and λ are transmission coefficients due to infectives and carrier population respectively. The parameters ν and d represent the therapeutic

treatment coverage (of infected individuals) and natural death respectively, α is the disease induced death constant. The constant s_1 is the death rate coefficient of carriers due to natural factors as well as by control measures. Here, $(r_0 + r_1 N(t))$ denotes per capita the growth rate of the carrier population density such that $(r_0 + r_1 N(t) - s_1)$ is its intrinsic growth rate and $r_{10} C^2(t)$ represents the death rate of carrier population due to crowding.

By using the fact that $X(t) + Y(t) = N(t)$, the above model system (2.1) may be reduced in the following form:

$$\frac{dY(t)}{dt} = \beta(N(t) - Y(t))Y(t-\tau) + \lambda(N(t) - Y(t))C(t) - (\nu + \alpha + d)Y(t),$$

$$\frac{dN(t)}{dt} = A - dN(t) - \alpha Y(t),$$

$$\frac{dC(t)}{dt} = (r_0 + r_1 N(t))C(t) - r_{10} C^2(t) - s_1 C(t), \tag{2.2}$$

$N(0) = N_0 > 0, Y(\theta) = Y_0 \geq 0$ for $\theta \in [-\tau,\ 0]$, $C(0) = C_0 \geq 0$.

Now we analyze the model system (2.2) in detail rather than model system (2.1). In the following we show that solutions of model (2) are bounded. Following[9,13], the region of attraction for all solutions initiating in the positive octant is given by

$$\text{uh}\,\Omega := \{(Y, N, C) : 0 \leq Y \leq N \leq \frac{A}{d}, 0 \leq C \leq \frac{1}{r_{10}}(r_0 + r_1 \frac{A}{d})\}$$

and attracts all solutions initiating in the positive octant.

3. Equilibrium Analysis

The model (2.1) exhibits three non-negative equilibrium, which are as follows:

(1) Disease free equilibrium $E_1(0,\ A/d,\ 0)$ always exists.

(2) Carrier free equilibrium

$$E_2\left(\frac{\beta A - d(\nu + \alpha + d)}{\beta(\alpha + d)}, \frac{\beta A + \alpha(\nu + \alpha + d)}{\beta(\alpha + d)}, 0\right) \text{exists,}$$

provided, $R_0 = \dfrac{\beta A}{d(\nu + \alpha + d)} > 1$ where R_0 is known as basic reproduction number.

(3) Endemic equilibrium $E_3(Y^*, N^*, C^*)$ exists, provided condition $r_0 + r_1 N^* - s_1 > 0$ is satisfied.

Proof: The existence of disease free equilibrium $E_1(0, A/d, 0)$ and carrier free equilibrium $E_2\left(\frac{\beta A - d(\nu+\alpha+d)}{\beta(\alpha+d)}, \frac{\beta A + \alpha(\nu+\alpha+d)}{\beta(\alpha+d)}, 0\right)$ is trivial. In the endemic equilibrium $E_3(Y^*, N^*, C^*)$, the values of Y^*, N^* and C^* may be obtained by solving the following algebraic equations:

(3.1) $$\beta(N-Y)Y + \lambda(N-Y)C - (\nu+\alpha+d)Y = 0,$$

(3.2) $$A - dN - \alpha Y = 0,$$

(3.3) $$(r_0 + r_1 N) - r_{10} C - s_1 = 0.$$

Denoting $r_0 - s_1$ by r, eliminating Y and C in equation (3.1) by using equations (3.2) and (3.3), we have the following quadratic equation in N:

$$F(N) = \beta\left(\frac{(\alpha+d)N - A}{\alpha}\right)\left(\frac{A-dN}{\alpha}\right) + \lambda\left(\frac{(\alpha+d)N - A}{\alpha}\right)\left(\frac{r + r_1 N}{r_{10}}\right)$$
$$-(\nu+\alpha+d)\left(\frac{A-dN}{\alpha}\right) = 0$$

From the above equation, we can easily note that, (i) $F\left(\frac{A}{\alpha+d}\right) < 0$,

(ii) $F\left(\frac{A}{d}\right) > 0$. Thus their exists a positive root of above equation in the open interval $\left(\frac{A}{\alpha+d}, \frac{A}{d}\right)$. Since $F(N) = 0$ is a quadratic equation N, thus this positive root will be unique. By using this value of N in equations (3.2) and (3.3), we get positive values of Y and C respectively if the following condition is satisfied:

(H_1). $$r_0 + r_1 N^* - s_1 > 0.$$

Thus there exists a unique endemic equilibrium $E_3(Y^*, N^*, C^*)$, provided condition (H_1) holds.

4. Stability Analysis

Linearizing the model system (2.2) about $E_3(Y^*, N^*, C^*)$ by using the following transformations $Y = Y^* + y$, $N = N^* + n$, $C = C^* + c$, where y, n and c are small perturbations.

The linearized system is given by:

$$\frac{du}{dt} = Au(t) + Bu(t-\tau), \tag{4.1}$$

where $u(t) = [y, n, c]^T$, $A = (a_{ij})_{3\times 3}$ and $B = (b_{ij})_{3\times 3}$. The values of a_{ij} and b_{ij} are as follows:

$a_{11} = -(\beta Y^* + \lambda C^* + \nu + \alpha + d)$, $a_{12} = \beta Y^* + \lambda C^*$, $a_{13} = \lambda(N^* - Y^*)$,

$a_{21} = -\alpha$, $a_{22} = -d$, $a_{23} = 0$, $a_{31} = 0$, $a_{32} = -r_1 C^*$, $a_{33} = -r_{10} C^*$,

$b_{11} = \beta(N^* - Y^*)$ and all other $b_{ij} = 0$.

The characteristic equation for the linearized system (4.1) is given by the following equation

$$P(\psi) - Q(\psi)e^{-\psi\tau} = 0, \tag{4.2}$$

where $P(\psi) = \psi^3 + p_1\psi^2 + p_2\psi + p_3$ and $Q(\psi) = q_1\psi^2 + q_2\psi + q_3$.

In the above expressions of $P(\psi)$ and $Q(\psi)$, p_i's and q_i's are given as follows:

$$p_1 = \beta Y^* + \lambda C^* + \nu + \alpha + d + d + r_{10} C^*,$$

$$p_2 = (\beta Y^* + \lambda C^* + \nu + \alpha + d)(d + r_{10} C^*) + d r_{10} C^* + \alpha(\beta Y^* + \lambda C^*),$$

$$p_3 = (\beta Y^* + \lambda C^* + \nu + \alpha + d) d r_{10} C^* + \alpha r_1 C^* \lambda(N^* - Y^*) + \alpha r_{10} C^* \lambda(\beta Y^* + \lambda C^*)$$

$$q_1 = \beta(N^* - Y^*), q_2 = \beta(N^* - Y^*)(d + r_{10} C^*), q_3 = \beta(N^* - Y^*) d r_{10} C^*$$

Let $g = (\beta Y^* + \lambda C^*) + \dfrac{\lambda(N^* - Y^*)C^*}{Y^*} > 0$, then for $\tau = 0$, the above characteristic equation reduces to the following form:

$$\psi^3 + A_1\psi^2 + A_2\psi + A_3 = 0, \tag{4.3}$$

where, $A_1 = p_1 - q_1 = g + d + r_{10} C^*$,

$A_2 = p_2 - q_2 = g(d + r_{10} C^*) + d r_{10} C^* + \alpha r_{10} C^* (\beta Y^* + \lambda C^*)$

$A_3 = p_3 - q_3 = g d r_{10} C^* + \alpha r_1 C^* \lambda(N^* - Y^*) + \lambda \alpha r_{10} C^* (\beta Y^* + \lambda C^*)$

Here we note that $A_1 > 0$, $A_2 > 0$. Thus using Routh-Hurwitz criterion, we can say that the roots of above polynomial will be either negative or

having negative real part iff the condition $A_1A_2 - A_3 > 0$ is satisfied. Thus we have the following theorem.

Theorem 4.1. *The equilibrium* $E_2(Y^*, N^*, C^*)$ *is locally asymptotically stable for* $\tau = 0$ *iff the following condition is satisfied:*

(H_2) $\quad A_1A_2 - A_3 > 0$

*where A*1*, A*2 *and A*3 *are defined as above.*

Remark 1: Here it may be noted that if $r_1 = 0$ the local stability condition (H_2) is automatically satisfied. This implies that the process of interaction of human population with carriers is destabilizing in nature. This result is similar as obtained by Singh et.al.[13].

Now substituting $\psi = i\omega(\omega > 0)$ into equation (4.2) and separating real and imaginary parts, we get the following transcendental equations:

(4.4) $\quad p_3 - p_1\omega^2 = \left(q_3 - q_1\omega^2\right)Cos\omega\tau + q_2\omega Sin\omega\tau.$

(4.5) $\quad p_2\omega - \omega^3 = q_2\omega Cos\omega\tau - \left(q_3 - q_1\omega^2\right)Sin\omega\tau.$

Now squaring and adding equations (4.4) and (4.5) and using $\omega^2 = \eta$, we get the following cubic equation:

(4.6) $\quad h(\eta) = \eta^3 + D_1\eta^2 + D_2\eta + D_3 = 0,$

where $D_2 = p_1^2 - q_1^2 - 2p_2$, $D_1 = p_2^2 - q_2^2 - 2p_1p_3 + 2q_1q_3$ and $D_3 = p_3^2 - q_3^2$.

In terms of coefficients of $h(\eta)$ define $\Delta = D_1^2 - 3D_2$, here it may be noted that $p_3 - q_3 > 0$, thus D_3 is always positive. It is easy to note from the characters of the cubic algebraic equation that $h(\eta)$ is a strictly monotonically increasing function of η if $\Delta \le 0$. If $\Delta > 0$ and $\eta^* = \frac{\sqrt{\Delta} - D_1}{3} > 0$, but $h(\eta^*) > 0$, then $h(\eta)$ has always no positive root.

Therefore under these conditions, equation (4.6) has no purely imaginary roots for any $\tau > 0$ and this also implies that the endemic equilibrium $E_2(Y^*, N^*, C^*)$ of system (2.2) is absolutely stable if condition (H_2) holds. Thus we can obtain easily the following result on the stability of the positive equilibrium $E_2(Y^*, N^*, C^*)$ of system (2.2).

Theorem 4.2: *Assume that* (H_1) *and* (H_2) *holds and* $\Delta \le 0$ *or* $\Delta > 0$ *and* $\eta^* = \frac{\sqrt{\Delta} - D_1}{3} < 0$ *or* $\Delta > 0$, $\eta^* > 0$ *and* $h(\eta^*) > 0$. *Then the positive*

equilibrium $E_2(Y^*, N^*, C^*)$ *of system (2.2) is absolutely stable, namely* $E_2(Y^*, N^*, C^*)$ *is asymptotically stable for any delay* $\tau \geq 0$.

In what follows, we assume that the coefficients in $h(\eta)$ *satisfy the condition* (H_3) $\Delta = D_1^2 - 3D_2$, $\eta^* = \frac{\sqrt{\Delta} - D_1}{3} > 0$, $h(\eta^*) < 0$

Then according to Lemma[24] *2.2 we know that (4.6) has at least a positive root* ω_0, *that is the characteristic equation (4.2) has a pair of purely imaginary roots* $\pm i\omega_0$.

From equation (4.4) and (4.5), we can get corresponding $\tau_k > 0$ *such that equation (4.2) has a pair of purely imaginary roots* $\pm i\omega_0$ *are given by*

(4.7) $$\tau_k = \frac{1}{\omega_0} arc\cos\left[\frac{(q_3 - q_1\omega_0^2)(p_3 - p_1\omega_0^2) + q_2\omega_0^2(p_2 - \omega_0^2)}{(q_3 - q_1\omega_0^2)^2 + (q_2\omega_0)^2}\right] + \frac{2k\pi}{\omega_0},$$

where $k = 0, 1, 2, ...$

Let $\psi(\tau) = \nu(\tau) + i\omega(\tau)$ *be the roots of the equation (4.2) such that when* $\tau = \tau_k$ *satisfying* $\nu(\tau_k) = 0$ *and* $\omega(\tau_k) = \omega_0$. *We claim that*

(4.8) $$\operatorname{sgn}\left[\frac{d(\operatorname{Re}\psi)}{d\tau}\right]_{\tau=\tau_k} = \operatorname{sgn}\{h'(\omega_0^2)\}$$

Proof: Differentiating equation (4.1), with respect to τ, after simple manipulation, we get

(4.9) $$\left(\frac{d\psi}{d\tau}\right)^{-1} = \frac{(2q_1\psi + q_2) - (3\psi^2 + 2p_1\psi + p_2)e^{\psi\tau}}{\psi(q_1\psi^2 + q_2\psi + q_3)} - \frac{\tau}{\lambda}$$

$$\operatorname{sgn}\left[\frac{d(\operatorname{Re}\psi)}{d\tau}\right]_{\tau=\tau_k} = \operatorname{sgn}\left[\operatorname{Re}\left(\frac{d\psi}{d\tau}\right)^{-1}\right]_{\psi=i\omega_0}$$

$$= \operatorname{sgn}\left[\operatorname{Re}\left(\frac{(2q_1\psi + q_2) - (3\psi^2 + 2p_1\psi + p_2)e^{\psi\tau}}{\psi(q_1\psi^2 + q_2\psi + q_3)}\right)\right]_{\psi=i\omega_0}$$

$$= -\operatorname{sgn}\left[\frac{q_2^2\omega_0 - 2q_1\omega_0\left((q_2 - q_1\omega_0^2)\right)}{\omega_0\left[(q_3 - q_1\omega_0^2)^2 + \left(q_2\omega_0\right)^2\right]}\right.$$

$$\left. - \frac{(p_2 - 3\omega_0^2)\left[q_2\omega_0\cos\omega_0\tau - (q_3 - q_1\omega_0^2)\sin\omega_0\tau\right]}{\omega_0\left[(q_3 - q_1\omega_0^2)^2 + \left(q_2\omega_0\right)^2\right]}\right],$$

Using equation (4.4) and (4.5), the above equation reduces to the following form

$$\operatorname{sgn}\left[\frac{d\left(\operatorname{Re}\psi\right)}{d\tau}\right]_{\tau=\tau_k} = \frac{1}{\Lambda}\operatorname{sgn}\left[\omega_0^4 + \left(2p_1^2 - q_1^2 - 2p_2\right)\omega_0^2\right.$$

$$\left. + \left(p_2^2 - q_2^2 - 2p_1p_3 + 2q_1q_3\right)\right],$$

$$\operatorname{sgn}\left[\frac{d\left(\operatorname{Re}\psi\right)}{d\tau}\right]_{\tau=\tau_k} = \frac{1}{p_1^2}\operatorname{sgn}\left\{h'\left(\omega_0^2\right)\right\},$$

where, $\Lambda = \left(q_3 - q_1\omega_0^2\right)^2 + \left(q_2\omega_0\right)^2$

It follows from the hypothesis $\left(H_3\right)$ that $h'\left(\omega_0^2\right) \neq 0$ and therefore the transversality condition holds. According to the Hopf-bifurcation theorem for functional differential equations, we have the following result.

Theorem 4.3. *Suppose that the conditions* $\left(H_1\right), \left(H_2\right)$ *and* $\left(H_3\right)$ *are satisfied:*

(i) If $\tau \in [0, \tau_0)$, *then the positive equilibrium* $E_2(Y^*, N^*, C^*)$ *of system (2.2) is asymptotically stable and unstable when* $\tau > \tau_0$.

(ii) System (2.2) can undergo a Hopf -bifurcation at the positive equilibrium $E_2(Y^*, N^*, C^*)$, *when* $\tau_0 = \tau_k \ \ (k = 0, 1, 2, \cdots)$, *where* τ_k *is defined by (4.7).*

5. Conclusion

In this paper, we have proposed and analyzed a nonlinear delay mathematical model for the spread of carrier dependent infectious diseases. It is assumed that the infectious individuals transmit the disease to the susceptible with a time lag τ. It is assumed that the carrier population follows logistic growth and its intrinsic growth rate also depends on the human related activities. The existence of endemic equilibrium has been shown and its local stability condition has been derived. The local stability

condition shows that due to the human related activities disease becomes more endemic. Further, it is shown that the endemic equilibrium, if exists, remains stable for all $\tau \geq 0$ under certain conditions, which are given in theorem 4.2. The critical value of delay parameter has been obtained. It has been shown that the endemic equilibrium, which is locally asymptotically stable without delay, remains locally asymptotically stable under certain conditions when the time delay parameter is less than the critical value, otherwise this stable equilibrium may become unstable. The condition for the Hopf-bifurcation has been derived analytically.

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