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Pseudo-Differential Operators on $W^{\Omega}(C^n)$ – Space*

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Abstract: A pseudo-differential operator on $W(C^n)$ space is defined and using the theory of Fourier transformation its various properties are studied.

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1. Introduction

The spaces $W_M(R^n), W^{\Omega}(C^n)$ were investigated by Friedman[1] and Gel'fand and Shilov[2]. It was shown that the Fourier transformation

$$F: W_M(R^n) \to W^{\Omega}(C^n), F: W^{\Omega}(C^n) \to W_M(R^n)$$

is linear and continuous, where M, Ω are convex functions and R^n, C^n are spaces of n- dimensional real and complex numbers.

The theory of pseudo-differential operators is given by Wong[3], Zaidman[4], Pathak[5] and others. They studied pseudo-differential operator by exploiting the theory of Fourier transformation on Schwartz space, Geverey type space and other spaces also.

Pseudo-differential operators on certain Gel'fand and Shilov space were studied by Cappiello, Gramchev and L. Rodino[6] by using theory of Fourier transformation.

Our main aim in this paper is to define the pseudo-differential operator on $W^{\Omega}(C^n)$-space and to study its various properties by the Fourier transformation tool because its distributional space $[W^{\Omega}(C^n)']$ is more general than Schwartz distributional space $[S(R^n)]'$.

Now, we recall the definitions of $W_M(R^n), W^{\Omega}(C^n)$-spaces and pseudo-differential operator from the papers[1,2] on $W^{\Omega}(C^n)$-space.

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Let M_j and Ω_j be the convex functions such that

(1.1) $$M_j(x_j)=\int_0^{x_j}\mu_j(\xi_j)\,d\xi_j \ (x_j\geq 0)$$

(1.2) $$\Omega_j(y_j)=\int_0^{y_j}\omega_j(\eta_j)\,d\eta_j \,(y_j\geq 0)$$

for $j = 1, 2, 3, \ldots.n$.
We set

$$\mu(\xi)=((\mu_1(\xi_1)),\ldots,(\mu_n(\xi_n)),$$
$$\omega(\eta)=((\omega_1(\eta_i)),\ldots,(\omega_\eta(\eta_n))$$

and

(1.3) $$M_j(-x_j)=M_j(x_j),M_j(x_j)+M_j(x_j^{'})\leq M_j(x_j+x_j^{'})$$

(1.4) $$\Omega_j(-y_j)=\Omega_j(y_j),\ \Omega_j(y_j)+\Omega_j(y_j^{'})\leq\Omega_j(y_j+y_j^{'}).$$

The space $W_M(R^n)$ consists of all C^∞-functions which satisfy the inequalities:

(1.5) $$\left|D_x^{(k)}\phi(x)\right|\leq C_k\exp[-M(ax)]$$

where $D_x^{(k)}=D_x^{(k_1)}D_x^{(k_2)}\ldots D_x^{(k_n)},\ k=(k_1,k_2,\ldots,k_n)$ and
$\exp[-M(ax)] = \exp[-M_1(a_1x_1) - M_2(a_2x_2) \ldots - M_n(a_nx_n)]$ and C_k, $a > 0$ are constants depending on the function .

A function $\phi\in W^{\Omega}(C^n)$ if and only if for $b > 0$ there exists a constant $C_k > 0$ such that

(1.6) $$|z^k\phi(z)|\leq C_k\exp[\Omega(by)],$$

where $z^k=z_1^{k_1}z_2^{k_2}z_3^{k_3}\ldots.z_n^{k_n}$,

$$\exp[\Omega(by)]=\exp[\Omega_1(b_1y_1)+\Omega_2(b_2y_2)+\ldots.+\Omega_n(b_ny_n)]$$

and constants $C_k > 0$, $b > 0$ depend on function ϕ.

Now, we define the duality of the functions $M(x)$ and $\Omega(y)$ in the following way:

Let $M_j(x_j)$ and $\Omega_j(y_j)$ be defined by (1.1) and (1.2) respectively and let $\mu_j(\xi_j)$ and $\omega_j(\eta_j)$ be mutually inverse, i.e. $\mu_j(\omega_j(\eta_j)) = \eta_j$ and $\omega_j(\mu_j(\xi_j)) = \xi_j$, then the corresponding functions $M_j(x_j)$ and $\Omega_j(y_j)$ are called dual in sense of Young. The Young inequality is

(1.7) $$x_jy_j\leq M_j(x_j)+\Omega_j(x_j^{'}),x_j\geq 0,\ y_j\geq 0,$$

where the equality holds if and only if $y_j=\mu_j(x_j)$ and x_j varies in the interval $x_j^0<x_j<\infty$ and y_j varies in the interval $y_j^0<y_j<\infty$. That equality will be

(1.8) $$x_j y_j = M_j^0(x_j)+\Omega_j(y_j)$$

and

(1.9) $$x_j y_j = M_j(x_j)+\Omega_j^0(y_j).$$

From the papers[1,2] the Fourier-duality relation is given by

$$F\left[W^{\Omega}(C^n)\right]=W_M(R^n),\ F\left[W_M(R^n)\right]=\left[W^{\Omega}(C^n)\right].$$

A linear partial differential operator P (z, D) for $z = x + iy \in C^n$ is given by

(1.10) $$P(z,D)=\sum_{|\xi|\le m} a_\alpha(z)\,D^{(\alpha)}$$

where $D^{(\alpha)} = D^{(\alpha_1)}\, D^{(\alpha_2)} \ldots D^{(\alpha_n)}$

If we replace $D^{(\alpha)}$ by a monomial $\xi^\alpha \in R^n$ then we get a symbol of (1.10). This symbol is

(1.11) $$P(z,\xi)=\sum_{|\alpha|\le m} a_\alpha(z)\,\xi^\alpha$$

We take $\phi\in W^{\Omega}(C^n)$ then by the property of Fourier transformation from (1.1) and (1.2)

$$\begin{aligned}
\left(P(z,D)\phi\right)(z) &= \sum_{|\alpha|\le m} a_\alpha(z)\,(D^{(\alpha)}\phi)(z)\\
&= \sum_{|\alpha|\le m} a_\alpha(z)(\xi^\alpha\,\hat{\phi})^\vee(z)\\
&= \sum_{|\alpha|\le m} a_\alpha(z)\,(2\pi)^{-n/2}\int_{R^n} \xi^\alpha e^{i<z,\xi>}\,\hat{\phi}(\xi)\,d\xi\\
&= (2\pi)^{-n/2}\int_{R^n} e^{i<z,\xi>}\left(\sum_{|\alpha|\le z} a_\alpha(z)\xi^\alpha\right)\hat{\phi}(\xi)\,d\xi\\
&= (2\pi)^{-n/2}\int_{R^n} e^{i<z,\xi>}\,p(z,\xi)\,\hat{\phi}(\xi)\,d\xi
\end{aligned}$$

Hence,

(1.12) $$(P(z,D)\phi)(z)=(2\pi)^{-n/2}\int_{R^n} e^{i<z,\xi>}\,p(z,\xi)\,\hat{\phi}(\xi)\,d\xi$$

which implies a representation of partial differential operator in terms of symbol p(z, ξ) by means of Fourier transformation. Instead of p(z, ξ), we take the general symbol θ (z, ξ) for $z\in C^n$, $\xi\in R^n$ which are no longer polynomial in ξ. The operator is so called pseudo-differential operator.

Thus, the pseudo-differential operator associated with symbol θ(z, ξ) is defined by

$$(A_\theta\phi)(z)=(2\pi)^{-n/2}\int_{R^n} e^{i<z,\xi>}\theta(z,\xi)\hat{\phi}(\xi)\,d\xi. \tag{1.13}$$

The function $\theta(z,\xi)\in C^\infty(C^n\times R^n)$ which is assumed to be an entire analytic function with respect to $z=x+iy$, $\xi\in R^n$ is said to be in the class V^m iff for any two multi-indices α and β and there exists positive constant $C_{\alpha,\beta}$, depending on α and β such that

$$\left|D_z^{(\alpha)}D_\xi^{(\beta)}\theta(z,\xi)\right|\le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, m\in R \tag{1.14}$$

and $z\in C^n$, $\xi\in R^n$.

2. Properties of the Pseudo-differential Operator

In this section, we study the various properties of the pseudo-differential operators on $W^\Omega(C^n)$-space.

Theorem 2.1. *Let σ (z, ξ) be a symbol belonging to V^m. Then pseudo-differential operator A_θ maps $W^\Omega(C^n)$-into itself.*

Proof. We have

$$(A_\theta\phi)(z)=(2\pi)^{-n/2}\int_{R^n} e^{i<z,\xi>}\ \theta(z,\xi)\hat{\phi}(\xi)\,d\xi\cdot$$

Now,

$$(iz)^k(A_\theta\phi)(z)=(2\pi)^{-n/2}\int_{R^n} D_\xi^{(k)}(e^{i<z,\xi>})\theta(z,\xi)\hat{\phi}(\xi)\,d\xi\cdot$$

Integration by parts we get

$$(iz)^k(A_\theta\phi)(z)=(2\pi)^{-n/2}(-1)^{|k|}\int_{R^n} e^{i<z,\xi>}D_\xi^{(k)}[\theta(z,\xi)\hat{\phi}(\xi)]d\xi$$

$$=(-1)^{|k|}(2\pi)^{-n/2}\sum_{|r|\le k}\binom{k}{r}\int_{R^n} e^{i<z,\xi>}\left(D_\xi^{(k-r)}\theta\right)(z,\xi)D_\xi^{(r)}\hat{\phi}(\xi)d\xi$$

Hence,

$$(iz)^k(A_\theta\phi)(z)=(-1)^{|k|}(2\pi)^{-n/2}\sum_{|r|\le k}\binom{k}{r}\int_{R^n} e^{i<z,\xi+1>}\prod_{i=1}^n(\xi_i+1)^{-|\alpha_i|}\prod_{i=1}^n(\xi_i+1)^{|\alpha_i|}$$

$$e^{-i<z,l>}\left(D_{\xi}^{(k-r)}\theta\right)(z,\xi)\ D_{\xi}^{(r)}\ \hat{\phi}(\xi)\,d\xi$$

$$=(-1)^{|k|}\,(2\pi)^{-n/2}\sum_{|r|\le k}\binom{k}{r}\int_{R^n} D_z^{(\alpha)}(e^{i<z,\xi+1>})\,e^{-i<z,l>}\prod_{i=1}^{n}(1+\xi_i)^{-|\alpha_i|}$$

$$\left(D_{\xi}^{(k-r)}\theta\right)(z,\xi)\,D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,d\xi\,.$$

Again, integration by parts we obtain

$$(iz)^k\,(A_\theta\phi)(z)=(-1)^{|k|}(2\pi)^{-n/2}\sum_{|r|\le k}\binom{k}{r}(-1)^{|\alpha|}\int_{R^n} e^{i<z,\xi+1>}$$

$$\prod_{i=1}^{n}(1+\xi_i)^{-|\alpha_i|}D_z^{(\alpha)}\left[e^{-i<z,l>}\left(D_z^{(k-r)}\theta\right)(z,\xi)\right]D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,d\xi$$

$$=(-1)^{|k|+|r|}\,(2\pi)^{-n/2}\sum_{|r|\le k}\binom{k}{r}\sum_{\delta\le\alpha}\binom{\alpha}{\delta}\int_{R^n} e^{i<\xi+1,z>}\,D_z^{(\delta)}e^{-i<z,l>}$$

$$\left(D_z^{(\alpha-\delta)}D_{\xi}^{(k-r)}\theta\right)(z,\xi)\prod_{i=1}^{n}\ (1+\xi_i)^{-|\alpha_i|}\,D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,d\xi\,.$$

Hence

$$(iz)^k\,(A_\theta\phi)(z)=(2\pi)^{-n/2}(-1)^{|k|+|\alpha|}\sum_{|r|\le k}\ \sum_{|\delta|\le\alpha}\binom{k}{r}\binom{\alpha}{\delta}\int_{R^n} e^{i<\xi+1,z>}(1)^{|\delta|}e^{-i<z,l>}$$

$$\left(D_z^{(\alpha-\delta)}D_{\xi}^{(k-r)}\theta\right)(z,\xi)\prod_{i=1}^{n}\ (1+\xi_i)^{-|\alpha_i|}\,D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,d\xi$$

$$=(2\pi)^{-n/2}\,(-1)^{|k|+|\alpha|}\sum_{|r|\le k}\sum_{|\delta|\le\alpha}\binom{k}{r}\binom{\alpha}{\delta}(-1)^{|\delta|}$$

$$\int_{R^n} e^{i<\xi,z>}\left(D_z^{(\alpha-\delta)}D_z^{(k-r)}\theta\right)(z,\xi)\prod_{i=1}^{n}(1+\xi_i)^{-|\alpha_i|}\,D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,d\xi.$$

Taking absolute of above expression we get

$$|\,z^k\,(A_\theta\phi)(z)\,|\le(2\pi)^{-n/2}\sum_{|r|\le\alpha}\sum_{|\delta|\le\alpha}\ \binom{k}{r}\binom{\alpha}{\delta}\int_{R^n}\exp[-\,|\,y\,||\,\xi\,|]$$

$$|\left(D_z^{(\alpha-\delta)}D_{\xi}^{(k-r)}\theta\right)(z,\xi)\,|(1+|\,\xi\,|)^{-|\alpha|}|D_{\xi}^{(r)}\,\hat{\phi}(\xi)\,|d\xi\,.$$

Using arguments[1,2], the above expression yields

$$|\,z^k(A_\theta\phi)(z)\,|\le(2\pi)^{-n/2}\sum_{|r|\le k}\sum_{|\delta|\le\alpha}\ \binom{k}{r}\binom{\alpha}{\delta}\int_{R^n} C_{\alpha-\delta,k-r}(1+|\,\xi\,|)^{m-|k|+|r|-|\alpha|}$$

$$D_r\,\exp[-M[(a\xi)]-|\,y\,||\,\xi\,|]\,d\xi$$

$$\le(2\pi)^{-n/2}\sum_{|r|\le k}\sum_{|\delta|\le\alpha}\binom{k}{r}\binom{\alpha}{\delta}C_{\alpha-\delta,k-r}.D_r$$

$$\int_{R^n} \exp[M(a_0\xi)]\exp[-M[(a\xi)]+|y||\xi|]d\xi \cdot$$

From (1.3), we have

$$|z^k(A_\theta\phi((z)| \le (2\pi)^{-n/2}\sum_{|r|\le k}\sum_{|\delta|\le\alpha}\binom{k}{r}\binom{\alpha}{\delta}D_{\alpha-\delta,k-r}$$

$$\int_{R^n} \exp[-M[(a-a_0)\xi]+|y||\xi|]d\xi$$

From the paper[7], we have

$$|z^k(A_\theta\phi)(z)| \le (2\pi)^{-n/2}\sum_{|r|\le k}\sum_{|\delta|\le\alpha} D''_{\alpha-\delta,k-r}\exp[\Omega(a-2a_0)^{-1}y]$$

$$\int_{R^n}\exp[-M(a_0\xi)]d\xi$$

$$\le D_n \exp[\Omega[(a-2a_0)^{-1}y].$$

This implies that

$$(A_\theta\phi)(z) \in W^\Omega(C^n).$$

Theorem 2.2. *A_θ is continuous linear mapping from [$W^\Omega(C^n)$] into [$W^\Omega(C^n)$].*

Proof. From the paper[2], if $\phi_\nu \to 0$ converges to zero uniformly in any bounded domain of the z-plane and in addition it satisfies the inequalities

$$|z^k\phi_\nu(z)| \le C_k \exp[\Omega(by)],$$

where the constants C_k and b do not depend on the index ν, then $\phi_\nu \in [W^\Omega(C^n)]$ is said to converge to zero as $\nu \to \infty$. If $\phi_\nu \in [W^\Omega(C^n)]$ then from Theorem 2.1 the pseudo-differential operator $A_\theta\,\phi_\nu$ is a linear mapping from $W^\Omega(C^n)$ into itself. Now using the above arguments $A_\theta\,\phi_\nu$ converges uniformly to zero in any bounded domain of the z-plane and in addition it satisfies the inequalities

$$|z^k(A_\theta\phi_\nu)(z)| \le C_k \exp[\Omega(by)],$$

where the constants C_k and b do not depend on the index ν. Then $A_\theta\,\phi_\nu \to 0$ uniformly as $\nu \to \infty$. This implies that $A_\theta\,\phi$ maps $W^\Omega(C^n)$ continuously into itself.

Now we define a distribution of $A_\theta f$ in $[W^\Omega(C^n)]'$ by

(2.1) $$\langle A_\theta f, \phi\rangle = \langle f, A_\theta^*\bar{\phi}\rangle, \phi \in [W^\Omega(C^n)] \text{ and } f \in [W^\Omega(C^n)]',$$

where A_θ^* is a formal adjoint of A_θ.

Theorem 2.3. A_θ *is a linear mapping from* $[W^{\Omega}(C^n)]'$ *into* $[W^{\Omega}(C^n)]'$.

Proof. Let $f \in [W^{\Omega}(C^n)]'$. Then, for any sequence of functions $\{\phi_v\} \in [W^{\Omega}(C^n)]$ converging uniformly to zero in any bounded domain of the z-plane in $[W^{\Omega}(C^n)]$ as $\nu \to \infty$. Then from (15), we have

(2.2) $$\langle A_\theta f, \phi_v \rangle = \langle f, A_\theta^* \overline{\phi_v} \rangle, \ \phi \in W^{\Omega}(C^n) \text{ and } \nu = 1, 2, 3 \ldots$$

By theorem 2.2 we have $A_\theta^* \overline{\phi_v} \to 0$ as $\nu \to \infty$. Hence using (2.2) we observe that $f \in [W^{\Omega}(C^n)]'$. Therefore, we conclude that $\langle A_\theta f, \phi_v \rangle \to 0$ as $\nu \to \infty$. This implies that $A_\theta f \in [W^{\Omega}(C^n)]'$.

Definition 2.2. A sequence of distributions $\{f_v\} \in \left[W^{\Omega}(C^n)\right]'$ is said to converge to zero in $\left[W^{\Omega}(C^n)\right]'$ if $\langle A_\theta f_v, \phi \rangle \to 0$ as $\nu \to \infty$, for all $\phi \in [W^{\Omega}(C^n)]$.

Theorem 2.4. *A_θ is a continuous linear mapping from* $\left[W^{\Omega}(C^n)\right]'$ *into* $\left[W^{\Omega}(C^n)\right]'$.

Proof. Let $\phi \in W^{\Omega}(C^n)$. Then using definition 2.2 and (2.1) we find that

$$\langle A_\theta f_v, \phi \rangle = \langle f_v, A_\theta^* \overline{\phi}_v \rangle \to 0 \text{ as } \nu \to \infty .$$

Hence, $A_\theta f_v \to 0$ in $\left[W^{\Omega}(C^n)\right]'$ as $\nu \to \infty$.

This implies that A_θ is a continuous linear mapping from $[W^{\Omega}(C^n)]'$ into itself.

Theorem 2.5. *Let σ (z, ξ) be a symbol in V^0. Then,* $A_\theta : L^p(R^n) \to L^p(R^n)$ *is bounded linear operator for $1 \le p < \infty$.*

Proof. The proof of above theorem is similar to Wong[3] (pp. 79-88).

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