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An Initial Value Technique for the Solution of Fifth Order Singularly Perturbed Boundary Value Problem

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Abstract: This paper envisages initial value technique, which is fast and accurate numerical technique for the solution of fifth order singularly perturbed boundary value problem and illustrate several examples for the reliability and implementation of the initial value technique. Moreover the technique is shown that the approximation produced by our method is better than some existing methods.
Keywords: Singular perturbation, Fifth order boundary value problem, Initial value technique.

1. Introduction

Singular perturbation boundary value problems (SPBVPs) occur frequently in many areas of applied science and engineering for example fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, optimal control and electrical network etc. The solution of singular perturbed boundary value problem have a two character First, thin transition layer where the solution varies very rapidly and second away from the layer the solution behaves regularly and varies slowly. Which is solved by analytical and numerically, to solve such problem we refer the book O'Malley[1], Doolen et al.[2], Roos et al.[3] and Miller et al.[4].

The solution of fifth order boundary value problem arises in mathematical modeling of viscoelastic flows[5,6]. The conditions for existence and uniqueness of solutions of boundary value problems are explained by the theorem presented by Agrawal[7]. Cagalar et al.[8] solved fifth order linear and non linear boundary value problems using the sixth degree B-spline. Usmani and Sakai[9] solved the linear special case third order boundary value problems using the Quartic spline. Siddiqi and Akram[10] solved fifth order linear special case boundary value problems using the sextic spline. Doha, Abd Elhameed and Youssri[11] are solving third and fifth order two point boundary value problems governed by homogeneous and non-homogeneous boundary condition using a dual Petrov-Galerkin method.

In this paper, we extend the technique proposed by Mishra and Sonali[12], for the solution of fifth order singularly perturbed boundary value problem with boundary layer at the left of the interval by initial value technique.
The approach presented in this paper has focus on the approximation to the solution of the following special case of fifth order boundary value problem.

$$\varepsilon u'''''(x) + c(x)u'(x) + d(x)u(x) = f(x), \quad x \in [a,b],$$

with boundary condition

$$u(a) = \alpha, \ u(b) = \beta, \ u'(a) = \gamma, \ u''(a) = \delta, \ u'''(a) = \eta,$$

where ε is a small as small as possible $(0 < \varepsilon < 1)$ and $\alpha, \beta, \gamma, \delta, \eta$ are known constants.

The purpose of this paper, using this technique to reduced fifth order boundary value problem into three initial value problems and for the accurate and easy numerical solution we use Runge-Kutta of fourth order.
The paper is organized as follows. Section 2 introduces the method in details. Section 3 used a few examples to investigate the applicability of the method and compares the result to exact solutions. There are some conclusions in the last section.

2. An Initial-Value Technique for a Singular Boundary Value Problem (SBVP)

Consider a linear fifth order singular perturbed boundary value problem of the following form

$$\varepsilon u'''''(x) + c(x)u'(x) + d(x)u(x) = f(x), \ x \in [a,b], \tag{2.1}$$

with boundary conditions

(2.2) $$u(a)=\alpha,\quad u(b)=\beta,\ u'(a)=\gamma,\ u''(a)=\delta,\ u'''(a)=\eta,$$

where ε is a small as small as possible $(0<\varepsilon<1)$ and $\alpha,\beta,\gamma,\delta,\eta$ are known constants.

Many numerical technique have been applied to solve such type of boundary value problem but we use initial value technique so that the boundary value problem have the following assumptions:

(a) *c(x), d(x)* and *f(x)* are sufficiently continuously differentiable functions in the interval *[a,b]*.

(b) *c(x)* ≥ *M* > *0* where *M* is positive constant throughout in *[a,b]*.

In above these assumption implies that the boundary value problem with boundary layer will be in the neighborhood at *x=a*. Suppose that the solution of equation (2.1) and (2.2) is given by

(2.3) $$u(x)=m(x)+y(x)e^{-\frac{t(x)}{\varepsilon}},$$

where

(2.4) $$t(x)=\int_a^x c(x)\,dx,\quad m(x)=\sum_{n=0}^{\infty} m_n(x)\varepsilon^n \text{ and } y(x)=\sum_{n=0}^{\infty} y_n(x)\varepsilon^n,$$

which implies

(2.5) $$u(x)=\sum_{n=0}^{\infty} m_n(x)\,\varepsilon^n+\left(\sum_{n=0}^{\infty} y_n(x)\varepsilon^n\right)e^{-\frac{t(x)}{\varepsilon}},$$

with

(2.6) $$t(x)=\int_a^x c(x)\,dx.$$

Differentiating equation (2.5) successively five times with respect to *x*, putting in equation (2.1) and comparing the coefficient of ε^k on both side we get

(A) $$m_{k-1}^{v}+y_{k-1}^{v}(x)e^{-\frac{t(x)}{\varepsilon}}-y_k^{iv}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)-4y_k^{'}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)$$

$$
\begin{aligned}
&-4y_k(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)-4y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)+6y'''_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)\\
&+12y''_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c'(x)-6y''_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c^3(x)+12y''_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c'(x)\\
&+12y'_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c'^2(x)+12y'_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c''(x)\\
&-12y'_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)c'(x)-4y''_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c^3(x)\\
&-12y_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)c^2(x)+4y'_{k+3}(x)e^{-\frac{t(x)}{\varepsilon}}c^4(x)\\
&-6y'''_k(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)-6y''_k(x)e^{-\frac{t(x)}{\varepsilon}}c''(x)+6y''_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c'(x)\\
&-4y''_k(x)e^{-\frac{t(x)}{\varepsilon}}c''(x)-4y'_k(x)e^{-\frac{t(x)}{\varepsilon}}c'''(x)+4y'_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)c''(x)\\
&-6y'_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)c'(x)-12y_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c'^2(x)c(x)\\
&-6y_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)c''(x)+6y_{k+3}(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)c^3(x)\\
&+4y'_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c''(x)+4y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)c''(x)\\
&+4y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c'''(x)-4y_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c''(x)c^2(x)\\
&+y'_{k+4}(x)e^{-\frac{t(x)}{\varepsilon}}c^4(x)+4y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c^3(x)c'(x)\\
&-y_{k+5}(x)e^{-\frac{t(x)}{\varepsilon}}c^5(x)+3y'_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c'^2(x)c^2(x)\\
&+6y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c'(x)c''(x)-3y_{k+2}(x)e^{-\frac{t(x)}{\varepsilon}}c(x)c'^2(x)\\
&-y'_k(x)e^{-\frac{t(x)}{\varepsilon}}c'''(x)-y_k(x)e^{-\frac{t(x)}{\varepsilon}}c''''(x)+y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c'''(x)c(x)\\
&+m'_k(x)c(x)+y'_k(x)e^{-\frac{t(x)}{\varepsilon}}c(x)-y_{k+1}(x)e^{-\frac{t(x)}{\varepsilon}}c^2(x)\\
&+m_k(x)d(x)+y_k(x)e^{-\frac{t(x)}{\varepsilon}}d(x)=f_k(x)
\end{aligned}
$$

Again comparing the coefficient of $e^{-\frac{t(x)}{\varepsilon}}$ in equation (A) and ignoring $e^{-\frac{t(x)}{\varepsilon}}$, we obtain

(2.7) $$m^{v}_{k-1}(x)+m'_k(x)c(x)+m_k(x)d(x)=f_k(x),$$

(2.8) $$\begin{aligned}&y^{v}_{k-1}(x)-y''''_k(x)c(x)-4y'_k(x)c(x)-4y_k(x)c'(x)\\&-4y_{k+1}(x)c^2(x)+6y'''_{k+1}(x)c^2(x)+12y''_{k+1}(x)c(x)c'(x)\end{aligned}$$

$$
\begin{aligned}
&-6y''_{k+2}(x)c^3(x)+12y''_{k+1}(x)c(x)c'(x)+12y'_{k+1}(x)c'^2(x)\\
&+12y'_{k+1}(x)c''(x)c(x)-12y'_{k+2}(x)c^2(x)c'(x)-4y''_{k+2}(x)c^3(x)\\
&-12y_{k+2}(x)c^2(x)c'(x)+4y'_{k+3}(x)c^4(x)-6y'''_k(x)c'(x)\\
&-6y''_k(x)c''(x)+6y''_{k+1}(x)c'(x)c(x)-4y''_k(x)c''(x)\\
&-4y'_k(x)c'''(x)+4y'_{k+1}(x)c''(x)c'(x)-6y'_{k+2}(x)c'(x)c^2(x)\\
&-12y_{k+2}(x)c'^2(x)c(x)-6y_{k+2}(x)c''(x)c^2(x)+6y'_{k+3}(x)c'(x)c^3(x)\\
&+4y'_{k+1}(x)c''(x)c(x)+4y_{k+1}(x)c''(x)c'(x)+4y_{k+1}(x)c'''(x)c(x)\\
&-4y_{k+2}(x)c''(x)c^2(x)+y'_{k+4}(x)c^4(x)+4y_{k+1}(x)c^3(x)c'(x)\\
&+3y'_{k+1}(x)c'^2(x)+6y_{k+1}(x)c''(x)c'(x)-3y_{k+2}(x)c'^2(x)c(x)\\
&+y'_k(x)c'''(x)-y_k(x)c''''(x)+y_{k+1}(x)c''''(x)c(x)-y_{k+5}(x)c^5(x)\\
&+y'_k(x)c(x)-y_{k+1}(x)c^2(x)+y_k(x)d(x)=0\,.
\end{aligned}
$$

Taking $k=0$ and considering only terms with zero indices (that is restricting the series in equation to their terms) in equation (2.7) and (2.8), we get the following

(2.9) $$m_0' c+m_0 d = f_k\,,$$

and

(2.10) $$-y_0'''' c-6c' y_0'''-10c'' y_0''+(-5c'''-3c)y_0'+(d-c'''-4c')y_0=0\,.$$

The representing equation (2.5) and (2.6) can be inserted into the boundary conditions. Then the boundary conditions become

$$m_0(a)+y_0(a)=\alpha,\ m_0(b)=\beta,\ m_0'(a)+y_0'(a)=\gamma,$$

$$m_0''(a)+y_0''(a)=\delta,\qquad m_0'''(a)+y_0'''(a)=\eta.$$

In the above expression, the exponential small terms $e^{-\frac{t(x)}{\varepsilon}}$ has been neglected(which is asymptotically zero) in obtaining the boundary condition at $x=b$. First, the differential equation can be solved along with the boundary condition to determine $m_0(x)$. Now, $y_0(a)$ is determined by solving equation subject to the condition

$$y_0=\alpha-m_0(a),\ y_0'=\gamma-m_0'(a),\ y_0''=\delta-m_0''(a)\ \text{ and }\ y_0'''=\eta-m_0'''(a);$$

where $m_0(x), m_0^{'}(x), m_0^{''}(x)$ and $m_0^{'''}(x)$ are already determined. Now from equation (2.6), *t(x)* take the form

$$t(x) = \int_a^x c(x)\,dx \text{ that is, } t^{'}(x) = c(x) \text{ with } t(a) = 0.$$

Therefore, the three initial value problem (IVP) corresponding to are given by

(IVP 1)

(2.11) $$c(x)m_0^{'} + d(x)m_0 = f_0 \text{ with } m_0(b) = \beta,$$

(IVP 2)

(2.12) $$-y_0^{''''}c(x) - 6c^{'}(x)y_0^{'''} - 10c^{''}(x)y_0^{''} + (-5\,c^{'''}(x)$$
$$-3c(x))\,y_0^{'} + (d - c^{''''}(x) - 4c^{'}(x))\,y_0 = 0,$$

with

$$y_0(a) = \alpha - m_0(a), \quad y_0^{'}(a) = \gamma - m_0^{'}(a),$$
$$y_0^{''}(a) = \delta - m_0^{''}(a), \text{ and } y_0^{'''}(a) = \eta - m_0^{'''}(a).$$

(IVP 3)

(2.13) $$\grave{}\,t^{'}(x) = c(x) \text{ with } t(a) = 0.$$

Note 2.1: In above we get three initial value problems which are independent of perturbation parameter ε and these are solve for accurate numerical solution by Runge-Kutta method of fourth order.

Note 2.2: In above equation (2.11) is solved at $x = b$ and in similar way we solved at $x = a$ and this value is used to solve the equation (2.12) at $x = a$. Hence the interval *[a, b]* is divided into *[a, x]* and *[x, b]*.

Note 2.3: The complete solution is obtained by putting $\varepsilon = 0$ in (2.11),(2.12) and (2.13), (2.11) which gives outer solution and (2.12),(2.13) gives inner solution and combining these solution.

Note 2.4: Runge-kutta method is used for solving these initial value problems first we find the values of $m_0(x), y_0(x)$ and $t(x)$.

The solution of equation (2.1) and (2.2) from (2.3) are as follows.

(2.14) $$u(x) = m_0(x) + y_0(x)e^{-\frac{t(x)}{\varepsilon}} .$$

3. Numerical Examples

In this section we apply the initial value technique developed in section 2 for solving fifth order singularly perturbed boundary value problem. For the sake and comparison, we consider the same example as that[13].

Example 3.1: Consider the following non-homogeneous singular perturbation problem.

$$y^{''''}(x) - y(x) = -(15+10x)e^x, \; 0 < x \le 1$$

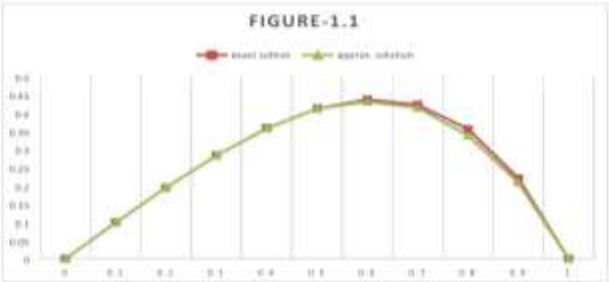
With boundary conditions

$$y(0) = y(1) = 0 \;\; \text{and} \;\; y^{'}(0) = 1, \;\; y^{'}(1) = -e, \;\; y^{''}(0) = 0.$$

The exact solution is given by

$$y(x) = (1-x)\; xe^x$$

x	Exact Solution	Approximate Solution
0	0	0
0.1	0.09946538	0.0993987
0.2	0.19542444	0.1953089
0.3	0.28347035	0.2833978
0.4	0.35803893	0.3587338
0.5	0.41218032	0.4121689
0.6	0.43730951	0.4316849
0.7	0.42288807	0.4148964
0.8	0.35608755	0.3389753
0.9	0.22136428	0.2098687
1.0	0.0	0.0



Example 3.2: Consider the following non-homogeneous singular perturbation problem.

$$y''''(x) + x\,y(x) = 19x\cos(x) + 2x^3\cos(x) + 41\sin(x) - 2x^2\sin(x), \; -1 < x \le 1$$

With boundary conditions

$$y(-1) = y(1) = \cos(1), \quad y'(-1) = -y'(1) = -4\cos(1) + \sin(1),$$

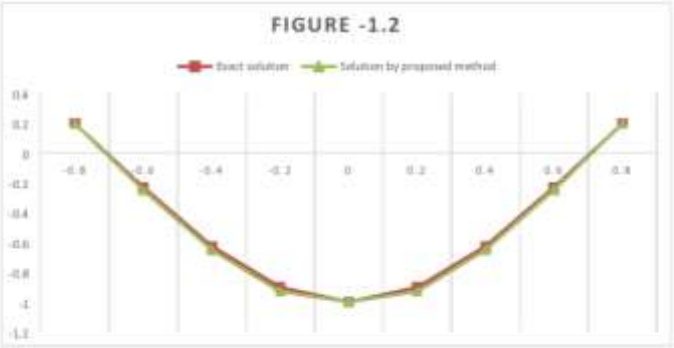
and

$$y''(-1) = 3\cos(1) - 8\sin(1),$$

The exact solution is given by

$$y(x) = (2x^2 - 1)\cos(x).$$

x	Exact Solution	Approximate Solution
-0.8	0.1950778	0.1956385
-0.6	-0.2310939	-0.2486744
-0.4	-0.6263214	-0.6498536
-0.2	-0.9016612	-0.9289641
0.0	-1.0000000	-0.9991583
0.2	-0.9016612	-0.9289641
0.4	-0.6263214	-0.6498536
0.6	-0.2310939	-0.2486744
0.8	0.1950778	0.1956385



4.Conclusion

In this paper we have used initial value technique for solving the fifth order singular perturbation boundary value problem with boundary conditions and illustrate two selected examples to show computational accuracy. Thus we conclude that initial value technique can be considered as an efficient technique for solving boundary value problem and will open new research directions.

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