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# A Study of Julia Sets Using Switching Processes*

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**Abstract:** The intent of this paper is to study the Julia sets of certain complex-valued polynomials by using the switching processes. Further, symmetry for such processes is also discussed in the paper. Such processes are of importance in the study of dynamical systems in which several free-standing processes may be involved. Our work extends the work of Lakhtakia[1], Shirriff[2] & Negi et al[3] and opens up an area having interesting possibilities for investigations.

**Keywords:** Fractals, Julia sets, Iteration and Fixed points.
**2000 Mathematics Subject Classification No.**: 28A80, 37Fxx, 39B12, 26A18, 37C25.

## 1. Introduction

Generation of Julia sets offer some of the most remarkable illustrations of how apparently simple processes can lead to highly intricate sets. In 1918, French Mathematician Julia[4] investigated the iteration processes of a complex function intensively, and obtained the Julia sets, a very important and useful concept. At present, the Julia sets have been applied widely in computer graphics, Biology, engineering and various branches of Mathematical sciences. Lakhtakia[1,5] studied the rotational symmetry of Julia sets of switched processes for two Polynomials i.e. $f(z_n)=(z_n)^p+c_f$ & $g(z_n)=(z_n)^q+c_g$, where $z_n$, $c_f$, $c_g$ are complex Numbers, while p>1, q>1 are integers. The study of Shirriff[2] describes a method of generating fractals by composing two simple polynomial functions. In 2008, Negi et al[3] studied the effect of switched processes on Superior Julia sets. The concept of Superior iterations was introduced by Rani and Kumar[6,7,8]. Lakhtakia[1] and Negi et al[3] have studied the Julia sets of

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switched processes for polynomials $f(z_n)=(z_n)^p+c_f$ and $g(z_n)=(z_n)^q+c_g$. In this paper we study the Julia sets of switching processes by considering the polynomials $f(z_n)=\left(z_n\right)^p+c_f$ and $g(z_n)=c_g.z_n(1-z_n)$, where $z_n$, $c_f$, $c_g$ are complex Numbers and p>1. We also compare the Julia sets and Superior Julia sets by using switching processes considered by us for certain values.

Let X be a non-empty set and $f:X\rightarrow X$. For a point $x_0$ in X, the Picard orbit or the Picard iteration (generally called orbit of *f* or trajectory of *f*) is the set of all iterates of a point $x_0$[6], that is:

$$O(f, x_0):=\{ x_n : x_n =f(x_{n-1}), n=1, 2, \ldots\}.$$

Let C be the set of complex numbers, consider the complex-valued function

$$z_{n+1}=z^p{}_n+c_f,\ c_f\in C, \tag{1.1}$$

where n is the iteration number. The set of points K whose orbits are bounded under the function iteration (Picard iteration) of Q(z) is called the filled Julia set. Julia set of Q is the boundary of the filled Julia set K. The boundary of a set is the collection of points for which every neighborhood contains an element of the set as well as an element, which is not in the set[6]. Let $z_0$ be an arbitrary element of C. Construct a sequence $\{z_n\}$ of points in C in the following manner:

$$z_n=sf(z_{n-1})+(1-s)z_{n-1}, n=1,2,... \tag{1.2}$$

where *f* is a function on a subset of C and the parameter *s* (Mann constant[9]) lies in the closed interval [0, 1]. The sequence $\{z_n\}$ constructed above, denoted by SO (*f*, $z_0$, s) is called the superior orbit for the complex-valued function *f* with an initial choice $z_0$ and parameter s[6,7,8]. The process (1.2) is also called the Superior iteration of $z_0$ under *f*[6]. The fractals for polynomials generated by using the superior iteration instead of the Picard iteration are called superior Julia sets[7,8]. This procedure is essentially due to W. R. Mann[9]. We remark that the Julia set is the special case of the superior Julia set, i.e., the superior Julia set with s=1 is the Julia set.

Now we define the Julia set for a function using Mann iteration (Superior iteration) and call it Superior Julia set. The set of points SK whose orbits are bounded under superior iteration of a function Q(z) is called filled superior Julia set. Superior Julia set of Q is the boundary of filled superior Julia set SK.

## 2. Preliminaries

The logistic function

$$z_{n+1} = c_g . z_n .(1 - z_n), \text{where } c_g \in C, \tag{2.1}$$

is a very interesting model for the population dynamics. This simple looking one-dimensional quadratic function embodies much of the complexity of chaotic systems and experiments with the same have generated interest and excitement in complex dynamics as well. For the beauty, usefulness, various applications and of course, complexity of the logistic functions, we refer to[10]. In 1987 Lakhtakia examined the self-replicating properties of the Julia sets for the iterative process $z_{n+1} = z_n^p + c$, where integer p>1. He has shown that the corresponding Mandelbrot sets contain (p-1)-fold symmetries, which results in Julia sets having p-fold symmetries.

Escape criterions play a crucial role in generation and analysis of the Julia set for polynomials. The general escape criterion for the polynomial $z_{n+1} = z^p{}_n + c$, using superior iterates is $\max\left\{|c|, \left(\frac{2}{s}\right)^{\frac{1}{(p-1)}}\right\}, p \geq 2$ (see Rani[6]).

## 3. Mathematical Analysis

In 1991, Lakhtakia investigated the behavior of filled Julia sets for switched processes and described the rotational symmetry of various types for these sets. Among the several processes and switched processes [3, Eqs. (1) - (7)], Lakhtakia emphasizes on the special switched processes given by

$$\left.\begin{aligned} z_{n+1} &= f(z_n) = (z_n)^p + c_f, & |z_n| \leq r_0 \\ z_{n+1} &= g(z_n) = (z_n)^q + c_g, & |z_n| > r_0 \end{aligned}\right\}. \tag{3.1}$$

In this paper, we consider the special switched processes given by:

$$\left.\begin{aligned} z_{n+1} &= f(z_n) = (z_n)^p + c_f, & |z_n| \leq r_0 \\ z_{n+1} &= g(z_n) = c_g . z_n (1 - z_n), & |z_n| > r_0 \end{aligned}\right\}, \tag{3.2}$$

where $r_0$ is defined as a buffer value between the two free standing processes.

Some of the interesting new Filled Julia sets and Filled Superior Julia sets obtained by us are given below.

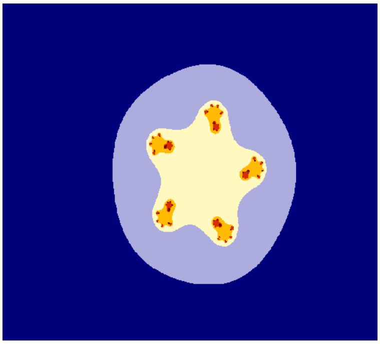

**Fig.1.** Julia Set for the iterative process $f(z_n)=z^p + c_f$, p=5, and $c_f$= -1.2-0.7i.

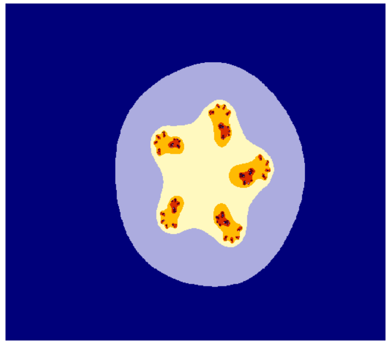

**Fig.2.** Julia Set for the iterative process $f(z_n)=z^p + c_f$, p=5, $c_f$= -1.2-0.7i and s=0.9.

Lakhtakia[1] remarks that such switched processes have applications in ecology, meteorology and control systems. Using (3.1), Lakhtakia[1] has presented some interesting filled Julia sets for $c_f = c_g$ =-(1.2, 0.7), where p=2 or 3 and q=2 whereas using (3.1), Negi et al[3] have obtained some interesting filled superior Julia sets for $c_f = c_g$ =-(1.2, 0.7), where p=2 or 3, q=2 and by taking the various values of the Mann constant s. In this paper we study both Filled Julia sets and filled superior Julia sets and also compare the behaviors of filled Julia sets and the filled superior Julia sets.

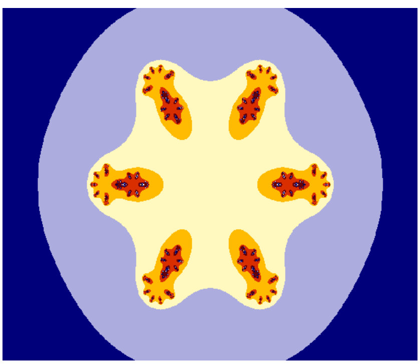

**Fig.3.** Julia Set for the iterative process $f(z_n)=z^p + c_f$, p=6, and $c_f$= -1.2+0i.

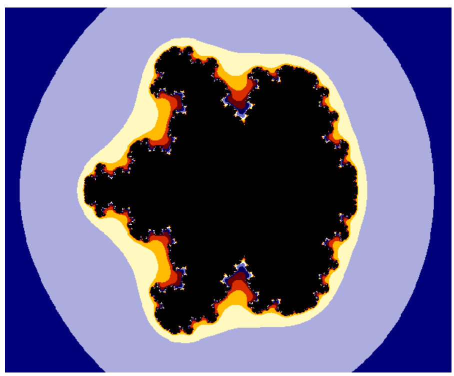

**Fig.4.** Julia Set for the iterative process $f(z_n)=z^p + c_f$, p=6, $c_f$= -1.2+0i and s=0.5.

Fig.1 shows the Julia sets for the iterative process $f(z_n) = z^p + c_f$, with p=5, $c_f$= -(1.2,0.7) (using picard iterative process) while Fig.2 shows the Julia sets of iterative process $f(z_n) = z^p + c_f$, with p=5, $c_f$ = -(1.2,0.7) and s=0.9 (using Mann iterative process). Analogous to the symmetries observed by Lakhtakia[1], In Fig.1 and Fig.2 we observe that using Picard iterative process five-fold symmetry is apparent while using the Mann iterative process we find that five-fold symmetry is not immediately apparent for values of s ranging between 0.1 and 0.8 but it appears for s=0.9 (see Fig.2).

Similar results are obtained for p=6, $c_f$= -1.2+0.i and s=0.5 [see Fig.3 and Fig.4]. Fig.3 gives the filled Julia sets by using the Picard iterative process while Fig.4 shows the filled Superior Julia sets by using the Mann iterative process. By Fig.3 & Fig.4 it is clear that filled Superior Julia sets are effectively different from filled Julia sets, comparison of Fig.3 and Fig.4 should be made with Figs.11 and 12, which are the Julia sets of process (3.2). Fig.5 shows the Julia sets for the iterative process $g\left(z_n\right)=c_g.z.(1-z)$ with $c_g$=-1.2-0.7i while Fig.6 shows the Superior Julia sets for the iterative process $g\left(z_n\right)=c_g.z.(1-z)$ with $c_g$=-1.2-0.7i and s=0.5. In these figures two-fold symmetry is observed due to the logistic function which is quadratic.

Some representative examples of switched process (3.2) are presented in Figs.7-12. Shown

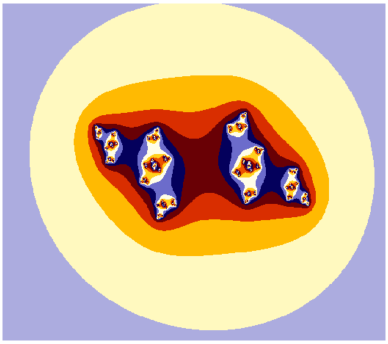

**Fig.5.** Julia Set for the iterative process $g(z_n)= c_g\ z\ (1-z)$, $c_g$= -1.2-0.7i.

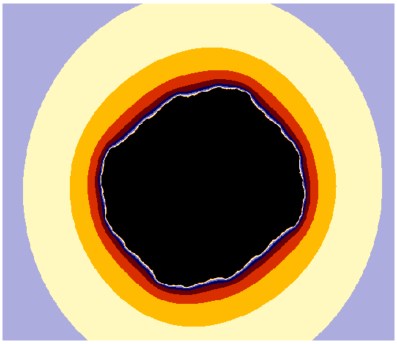

**Fig.6.** Julia Set for the iterative process $g(z_n)= c_g\ z\ (1-z)$, $c_g$= -1.2-0.7i and s=0.5.

Fig.7: is the Julia sets of switched process (3.2) with p=4, $c_f$= $c_g$=(-1.2+ 0i) and $r_0$=2, while Fig.8 also shows the Superior Julia sets of process (3.2) with p=4, $c_f$= $c_g$=(-1.2+ 0i), $r_0$=2 and s=0.6. The difference in Fig.7 & Fig.8 arises from the difference in the iterative process.

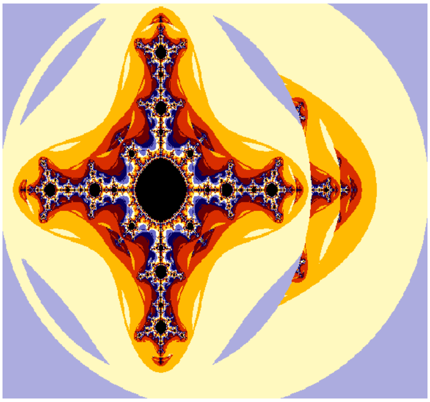

**Fig.7.** Julia Set for the iterative process (3.2); $p=4$, $c_f = c_g = -1.2+0i$, and $r_0=2$

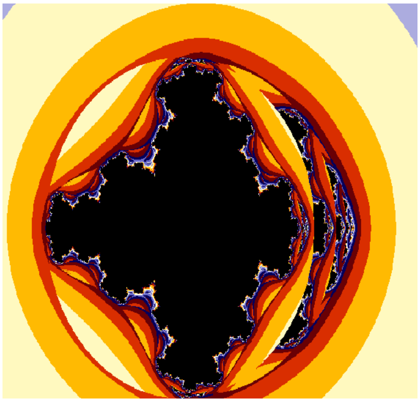

**Fig.8.** Julia Set for the iterative process (3.2) $p=4$, $c_f = c_g = -1.2+0i$, $r_0=2$ and $s=0.6$.

We have observed that the filled Julia set in Fig.7 for the switched process (3.2) is of period 5 with the fixed points (-1.2000, 0.8736, -0.6176, -1.0545, 0.0366) while in Fig.8 filled superior Julia sets for the switched process (3.2) is of period 1 with a fixed point (-0.7968) (after 42 iterations).

Fig.9 shows the filled Julia set for the switched process (3.2) for $p=6$, $c_f = c_g = -1.2-0.7i$ and $r_0=0.1$ whereas Fig.10 shows the filled superior Julia set for the switched process (3.2) for $p=6$, $c_f = c_g = -1.2-0.7i$, $r_0=0.1$ and $s=0.5$. In Fig.9 we observe that for the smaller values of $r_0$, Julia set for switched processes (3.2) shows similarity with the usual filled Julia set for $c_g$ (see Fig.5) and $g(z_n)$ becomes dominant and yields two-fold symmetry due to the logistic function while In Fig.10, we observe that the superior Julia sets for switched processes (3.2) shows similarity with filled superior Julia set for $c_g$ (see Fig.6) and $g(z_n)$ becomes dominant and yields two-fold symmetry.

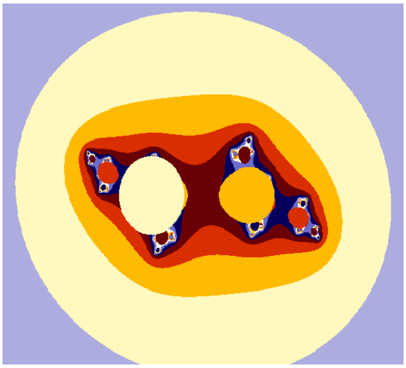

**Fig.9.** Julia Set for the iterative process(3.2) for $p=6$, $c_f = c_g = -1.2-0.7i$, & $r_0=0.1$,

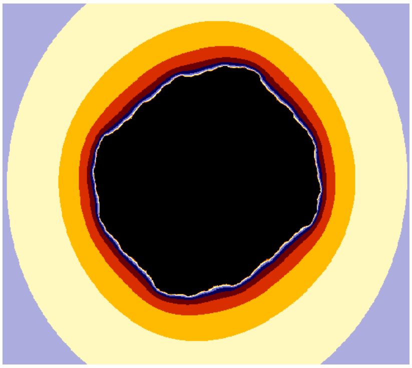

**Fig.10.** Julia Set for the iterative process (3.2) for $p=6$, $c_f = c_g = -1.2-0.7i$, $r_0=0.1$ & $s=0.5$.

Some noticeable characteristic of the filled Julia sets as well as Filled superior Julia sets for switched process (3.2) were observed for the larger values of $r_0$. Taking $r_0$=12, $c_f$= $c_g$= (-1.2+ 0i) with p=6, we observe that the Filled Julia sets for switched process (3.2) is similar to the Filled Julia sets for the iterative function $f(z_n)=z^p+c_f$, with p=6, $c_f$= $c_g$= (-1.2+ 0i) [see Fig.3]. If $r_0$=12 with s=0.1 (Mann constant) and $c_f$= $c_g$= (-1.2+ 0i) then resultant filled Superior Julia set for the switched process is a pile of layered filled Julia sets (see Fig.12), the upper most filled Superior Julia set resembles with the filled Superior Julia set for $f(z_n)=z^p+c_f$, p=6, $c_f$=(-1.2+ 0i) and s=0.5 (see Fig.4).

We observe that further increasing the value of s, all layers of the filled superior Julia set merge together to form a single Superior Julia set. This peculiar characteristic of merging of several slices of filled Superior Julia sets into one slice of Filled Superior Julia sets is primarily due to the dominance of $f(z_n)$ over $g(z_n)$.

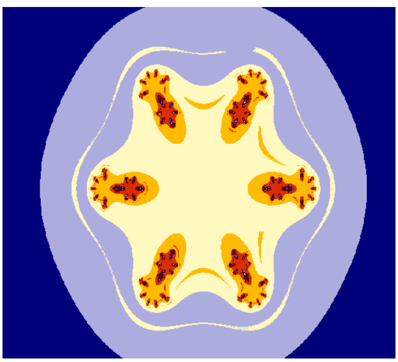

**Fig.11:** Julia Set for the iterative process(3.2) for p=6, $c_f$ = $c_g$ = -1.2+0i, and $r_0$=12.

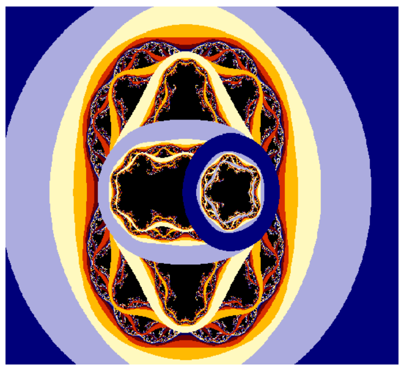

**Fig.12:**Julia Set for the iterative process (3.2) For p=6 $c_f$ = $c_g$ = -1.2+0i, $r_0$=12 and s=0.1.

## 4. Concluding Remarks

Many interesting fractals can be generated from the switching of $f(z_n)=z^p+c_f$, and $g\left(z_n\right)=c_g.z.(1-z)$. In many cases, the symmetry of the resulting fractal can be easily proved. By taking the different complex-valued polynomials, we find some interesting features. On the switching of $f(z_n)=z^p+c_f$ and $g\left(z_n\right)=c_g.z.(1-z)$ using Mann iteration, we get effectively different and converging results as compared to those obtained using Picard iteration (function iterations). It can be concluded on the basis of the present study that the orbit of the Superior Julia set for the switched process is found to converge to fixed points. Our results have shown that the superior iterates converge quickly while the Picard iterates converge slowly after a large number of iterations or overflow.

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