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Index of W_2-symmetric Semi-Riemannian Manifolds

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Abstract: The index of $\tilde{\nabla}-\tilde{W}_2$-symmetric semi-Riemannian manifolds are studied, where $\tilde{\nabla}$ is Ricci symmetric metric connection.

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1. Introduction

Pokhariyal and Mishra[1] defined the W_2-curvature tensor and studied its physical and geometrical properties. In[2,3,4], authors proved that W_2-curvature flat space-time is an Einstein manifold. In[5], the authors compute the W_2-curvature tensor on warped product manifolds, generalized Robertson-Walker and standard static space-times. W_2-curvature tensor was also studied by many authors[6-12] .

In 1923, L.P. Eisenhart[13] derived the condition for the existence of a second order parallel symmetric tensor in a Riemannian manifold and proved that if a positive definite Riemannian manifold admits a second order parallel symmetric tensor other than a constant multiple of the Riemannian metric, then it is reducible. In 1925, H. Levy[14] proved that a second order parallel symmetric non-singular tensor in a real space form is always proportional to the Riemannian metric. As an improvement of the result of Levy, R. Sharma[15] proved that any second order parallel tensor (not necessarily symmetric) in a real space form of dimension greater than 2 is proportional to the Riemannian metric. In 1939, T.Y. Thomas[16] defined the index of a Riemannian manifold. The problem of existence of a second order parallel symmetric tensor is closely related with the index of Riemannian manifolds. In 1968, J. Levine and G.H. Katzin[17] studied the index of conformally flat Riemannian manifolds. They proved that the index of an n-dimensional conformally flat manifold is $n(n+1)/2$ or 1 according as it is a flat manifold or a manifold of non-zero constant curvature. In 1981, P. Stavre[18] proved that if the index of an $n-$dimensional conformally symmetric Riemannian manifold (except the four cases of being conformally flat, of constant curvature, an Einstein manifold or with covariant constant Einstein tensor) is greater than one, then it must be between 2 and $n+1$. In 1982, P. Starve and D. Smaranda[19] found the index of a conformally symmetric Riemannian manifolds with respect to a semi-symmetric metric connection of K. Yano[20]. M. M. Tripathi et al.[21] studied the index of quasi- conformally symmetric, conformally symmetric and concircularly symmetric semi-Riemannian manifolds with respect to any metric connection and discussed some of applications in theory of relativity. In[22], the present author studied the index of pseudo-projectively symmetric, projectively symmetric semi-Riemannian manifolds with respect to Ricci symmetric metric connection.

The W_2-curvature tensor is important tensor from the differential geometric point of view. In this paper, we study the index of W_2-symmetric semi-Riemannian manifolds with respect to the Ricci symmetric metric connection $\tilde{\nabla}$.

2. Preliminaries

Let M be an $n-$dimensional differentiable manifold and $\tilde{\nabla}$ be a linear connection in M. The torsion tensor $\tilde{T}$ and curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z,$$

for all $X, Y, Z \in X(M)$, where $X(M)$ is the Lie algebra of vector fields in M. By a semi-Riemannian metric[23] on M, we understand a non-degenerate symmetric (0, 2) tensor field g.

Let (M, g) be an n-dimensional semi-Riemannian manifold. A linear connection $\tilde{\nabla}$ in M is called a metric connection with respect to the semi-Riemannian metric g if $\tilde{\nabla}g = 0$. If the torsion tensor of the metric connection $\tilde{\nabla}$ is zero, then it becomes Levi-Civita connection ∇, which is unique by the fundamental theorem of Riemannian geometry. If the torsion tensor of the metric connection $\tilde{\nabla}$ is not zero, then it is called a Hayden connection[24,25]. Semi-symmetric metric connections[20] and quarter symmetric metric connections[26] are some well-known examples of Hayden conections. Let (M, g) be an $n-$dimensional semi-Riemannian manifold. For a metric connection $\tilde{\nabla}$ in M, the curvature tensor $\tilde{R}$ with respect to the $\tilde{\nabla}$ satisfies the following condition

(2.1) $$\tilde{R}(X,Y,Z,V) + \tilde{R}(Y,X,Z,V) = 0,$$

(2.2) $$\tilde{R}(X,Y,Z,V) + \tilde{R}(X,Y,V,Z) = 0,$$

for all $X,Y,Z,V \in \mathfrak{X}(M)$, where

(2.3) $$\tilde{R}(X,Y,Z,V) = g(\tilde{R}(X,Y)Z,V).$$

The Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of the semi-Riemannian manifold with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\tilde{S}(X,Y) = \sum_{i=1}^{n} \varepsilon_i \tilde{R}(e_i, X, Y, e_i),$$

$$\tilde{r} = \sum_{i=1}^{n} \varepsilon_i \tilde{S}(e_i, e_i),$$

where $\{e_1,\ldots,e_n\}$ is any orthonormal basis of vector fields in the manifold M and $\varepsilon_i = g(e_i,e_i)$. The Ricci operator $\tilde{Q}$ with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y), \qquad X,Y \in \mathfrak{X}(M).$$

Let

(2.4) $$\tilde{e}X = \tilde{Q}X - \frac{\tilde{r}}{n}X,\ \ X \in \mathfrak{X}(M),$$

(2.5) $$\tilde{E}\left(X,Y\right) = g\left(\tilde{e}X,Y\right),\ X,Y \in \mathfrak{X}(M).$$

Then

(2.6) $$\tilde{E} = \tilde{S} - \frac{\tilde{r}}{n}g.$$

The $(0,2)$ tensor $\tilde{E}$ is called tensor of Einstein[27] with respect to the metric connection $\tilde{\nabla}$. If $\tilde{S}$ is symmetric then $\tilde{E}$ is also symmetric.

Definition 2.1: *A metric connection $\tilde{\nabla}$ with symmetric Ricci tensor $\tilde{S}$ is called a Ricci-symmetric metric connection*[21].

Definition[16] **2.2:** *A set of metric tensors (a metric tensor on a differentiable manifold is a symmetric non-degenerate parallel $(0,2)$ tensor field on the differentiable manifold) $\{H_1,\dots,H_\ell\}$ is said to be linearly independent} if*

$$c_1H_1 + \cdots + c_\ell H_\ell = 0,\quad c_1,\dots,c_\ell \in R,$$

implies that $c_1 = \cdots = c_\ell = 0.$

The set $\{H_1,\dots,H_\ell\}$ is said to be a complete set if any metric tensor H can be written as

(2.7) $$H = c_1H_1 + \cdots + c_\ell H_\ell,\ \ c_1,\dots,c_\ell \in R\ .$$

More precisely, the number of linearly independent metric tensors in a complete set of metric tensors of a Riemannian manifold is called the index of the Riemannian manifold.

Let (M,g) be an $n-$dimensional semi-Riemannian manifold equipped with a metric connection $\tilde{\nabla}$. A symmetric $(0,2)$ tensor field H, which is covariantly constant with respect to $\tilde{\nabla}$, is called a special quadratic first integral (for brevity SQFI)[28] with respect to $\tilde{\nabla}$. The semi-Riemannian metric g is always an SQFI. A set of SQFI tensors $\{H_1,\ldots,H_\ell\}$ with respect to $\tilde{\nabla}$ is said to be linearly independent if

$$c_1H_1+\cdots+c_\ell H_\ell=0, \qquad c_1,\ldots,c_\ell\in R,$$

implies that $c_1=\cdots=c_\ell=0$.

The set $\{H_1,\ldots,H_\ell\}$ is said to be a complete set if any SQFI tensor H with respect to $\tilde{\nabla}$ can be written as

$$H=c_1H_1+\cdots+c_\ell H_\ell, \qquad c_1,\ldots,c_\ell\in R,$$

The index[16] of the manifold M with respect to $\tilde{\nabla}$, denoted by $i_{\tilde{\nabla}}$, is defined to be the number ℓ of members in a complete set $\{H_1,\ldots,H_\ell\}$.

Let (M,g) be an $n-$dimensional semi-Riemannian manifold equipped with the metric connection $\tilde{\nabla}$. Let $\tilde{R}$ be the curvature tensor of M with respect to the metric connection $\tilde{\nabla}$. If H is a parallel symmetric $(0,2)$ tensor with respect to the metric connection $\tilde{\nabla}$, then we easily obtain

$$H((\tilde{\nabla}_U\tilde{R})(X,Y)Z,V)+H(Z,(\tilde{\nabla}_U\tilde{R})(X,Y)V)=0,\ X,Y,Z,V,U\in \mathfrak{X}(M). \tag{2.8}$$

The solutions H of (1.9) is closely related to the index of W_2-symmetric semi-Riemannian manifolds with respect to the $\tilde{\nabla}$.

3. Index of W_2-Symmetric Manifold

The W_2-curvature tensor[29] of the manifold with respect to the metric connection $\tilde{\nabla}$ is given by

$$\tilde{W}_2(X,Y,Z,V)=\tilde{R}(X,Y,Z,V)+\frac{1}{n-1}\left(\tilde{S}(Y,V)g(X,Z)-\tilde{S}(X,V)g(Y,Z)\right). \tag{3.1}$$

Definition 3.1: *A semi-Riemannian manifold is said to be* $\tilde{\nabla}-W_2-$*symmetric manifold if* $\tilde{\nabla}\tilde{W}_2=0$, *where* $\tilde{\nabla}$ *is Ricci symmetric metric connection.*

Lemma 3.2: *If* (M, g) *be an* $n-$*dimensional semi-Riemannian* $\tilde{\nabla}-W_2-$*symmetric manifold, then*

(3.2) $$\operatorname{trace}(\tilde{\nabla}_U\tilde{E})=0,$$

where U *is an arbitrary vector field.*

Proof: Using (2.6) in (3.1), we get

(3.3) $$\tilde{W}_2(X,Y,Z,V)=\tilde{R}(X,Y,Z,V)+\frac{1}{n-1}(\tilde{E}(Y,V)g(X,Z)-\tilde{E}(X,V)g(Y,Z))+\frac{\tilde{r}}{n(n-1)}(g(Y,V)g(X,Z)-g(X,V)g(Y,Z)).$$

Taking the covariant derivative of (3.3) and using $\tilde{\nabla}_U\tilde{W}_2=0$, we get

(3.4) $$(\tilde{\nabla}_U\tilde{R})(X,Y,Z,V)=\frac{1}{n-1}\Big((\tilde{\nabla}_U\tilde{E})(X,V)g(Y,Z)-(\tilde{\nabla}_U\tilde{E})(Y,V)g(X,Z)\Big)+\frac{(\tilde{\nabla}_U\tilde{r})}{n(n-1)}\Big(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)\Big).$$

Contracting Y and Z in (3.4) and using the condition (2.1) and (2.2), we have

(3.5) $$(\tilde{\nabla}_U\tilde{S})(X,V)=(\tilde{\nabla}_U\tilde{E})(X,V)+\frac{(\tilde{\nabla}_U\tilde{r})}{n}g(X,V).$$

Taking $X=V=e_i$ in (3.5), we obtain

(3.6) $$\operatorname{trace}(\tilde{\nabla}_U\tilde{E})=0.$$

Theorem 3.3: *If* (M, g) *is an* $n-$*dimensional semi-Riemannian* $\tilde{\nabla}-W_2-$*symmetric manifold with* $n>2$, *then the equation (2.7) has maximum number of solution and consequently* $i_{\tilde{\nabla}} = \frac{1}{2}n(n-1)$.

Proof: Using (2.8) and (3.4), we find

(3.7)
$$\begin{aligned}
&\frac{1}{n-1}(H((\tilde{\nabla}_U\tilde{e})X,V)g(Y,Z)-H((\tilde{\nabla}_U\tilde{e})Y,V)g(X,Z)\\
&+H((\tilde{\nabla}_U\tilde{e})X,Z)g(Y,V)-H((\tilde{\nabla}_U\tilde{e})Y,Z)g(X,V))\\
&+\frac{(\tilde{\nabla}_U\tilde{r})}{n(n-1)}(g(Z,Y)H(X,V)-g(Z,X)H(Y,V)\\
&+g(V,Y)H(X,Z)-g(V,X)H(Y,Z))=0.
\end{aligned}$$

Taking $X = Z = e_i$ in (3.7) and using (3.6), we get

(3.8)
$$\begin{aligned}
&\frac{1}{n-1}(-nH((\tilde{\nabla}_U\tilde{e})Y,V)+g(Y,V)\,H((\tilde{\nabla}_U\tilde{e})e_i,e_i)\\
&=\frac{(\tilde{\nabla}_U\tilde{r})}{n(n-1)}(nH(Y,V)-g(Y,V)\mathrm{trace}(H)).
\end{aligned}$$

Interchanging V with Y in (3.8) and then subtracting the resulting equation from (3.8), we obtain

(3.9)
$$H((\tilde{\nabla}_U\tilde{e})Y,V)=H((\tilde{\nabla}_U\tilde{e})V,Y).$$

Now, interchanging X with Z, and Y with V in (3.7) and taking the sum of the resulting equation and (3.7) and using (3.8), we see that the equation (2.8) is satisfied identically. Thus the equation has the maximum number of solutions for a $\tilde{\nabla}-W_2-$symmetric semi-Riemannian manifold.

Consequently, M admits the maximum number of linearly independent SQFI. So, the index of a $\tilde{\nabla}-W_2-$symmetric semi-Riemannian manifold is

$$i_{\tilde{\nabla}} = \frac{1}{2}n(n-1).$$

Remark 3.1. A Riemannian manifold is decomposable if and only if its index is greater than one[16]. A complete Riemannian manifold is reducible if and only if its index is greater than one[16]. Therefore by using Theorem 3.3, we can say that a $\tilde{\nabla}-W_2-$symmetric Riemannian manifold (where $\tilde{\nabla}$ is any Ricci symmetric metric connection, not necessarily Levi-Civita connection) is decomposable and it is reducible if the manifold is complete.

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