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A Lifting Problem on Zero Dimensional Ideals over Laurent Polynomial Ring

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Abstract. Let R be a commutative Noetherian ring I and be a zero dimensional ideal of Laurent polynomial ring $R\left[X, X^{-1}\right]$ that contains a doubly monic polynomial. Define. $I(1) = < \{f(1) : f \in I\} >$. Suppose $\mu(I) = n$ and $I(1)/I(1)^2$ is a free module of rank $n \geq 2$ over $R/I(1)$. Then a set of n generators of $I(1)$ can be lifted to a set of n generators of I iff every unit of $R/_{(1)}$ can be lifted to a unit of $R\left[X, X^{-1}\right]/I$.

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1. Introduction

Let X be an integral, projective variety of co-dimension two, degree d and dimension r and Y be its general hyperplane section. The problem of lifting generators of minimal degree σ from the homogeneous ideal of Y to the homogeneous ideal of X has been discussed in the paper [1]. Answers in terms of relations between d and σ are known for $r = 1, 2$. Laudal [2], Gruson and Peskine [3] proved "Generalized trisecant lemma" for $r = 1$. For $r = 2$, there is an analogous result in the paper[4]. Let $X \subset P^N$ be an integral, projective variety of dimension n and degree d, defined over an algebraically closed field K of characteristic zero. Consider the hyperplane section $Y = X \cap K$ of X, where $K \cong P^{N-1}$ is a general hyperplane in P^N. The "Lifting problem" is the problem of finding conditions on d, N, n and s such that any degree s hypersurface in P^{N-1} containing Y can be lifted to a hypersurface in P^N containing X[5].

The ring of Laurent polynomials $R\left[X, X^{-1}\right]$ is a natural ground ring for the study of generators of an ideal. It is obtained by inverting the variables in polynomial ring. A Laurent polynomial ring is an extension of the polynomial ring $R[X]$. A Laurent polynomial over C may be viewed as a Laurent series in which finitely many coefficients are non-zero [6].

Let R be a commutative and Noetherian ring with identity and I be an ideal of the Laurent polynomial ring $R\left[X, X^{-1}\right]$. We consider only rings with finite Krull dimension. Let M be a finitely generated R -module. We denote $\mu(M)$ to be the least number of elements in M required to generate M as an R -module. The conormal module I/I^2 of an ideal I in a ring $R\left[X, X^{-1}\right]$ is viewed as an $R\left[X, X^{-1}\right]/I$ -module. Define $I(1) = <\{f(1) : f \in I\}>$. In general, $\mu(I(1)) \leq \mu(I)$ and $\mu\left(I/I^2\right) \leq \mu(I) \leq \mu\left(I/I^2\right) + 1$.

If I is an ideal of $R[X]$, then $I(0) = <\{f(0) : f \in I\}>$ is an ideal of R, so it is natural to ask whether a set of generators for $I(0)$ can be lifted to a set of generators for I. Since lifting of generators is a natural problem which has been studied by many mathematicians in various fashion [7,8].

Now we discuss the following problem: In [11], S. Mandal proved that if R is a commutative Noetherian ring, I is an ideal of Laurent polynomial ring $R\left[X, X^{-1}\right]$ containing a doubly monic polynomial, and I/I^2 is generated by n elements over $R\left[X, X^{-1}\right]/I$,
where $n \geq \dim\left(R\left[X, X^{-1}\right]/I\right) + 2$, then I is also generated by n elements over $R\left[X, X^{-1}\right]$. Hence $I(1)$ is also generated by n elements, where $I(1)$ is an ideal of R by putting $X = 1$. Now a question arises here that whether we can lift any set of n generators of $I(1)$ to a set of n generators of I. The objective of paper is to prove that generators of $I(1)$ can be lifted to

generators of I if and only if the units of $R/I(1)$ can be lifted to units of $R\left[X, X^{-1}\right]/I(1)$ under the natural map.

2. Preliminary Notes

In this section we define some terms used in this paper and state certain standard results without proof. We hope that this will improve the readability and understanding of the proof of the paper.

An ideal I of a ring R is called zero dimensional if every prime ideal of R containing I is a maximal ideal, that is, dim$(R/I) = 0$. Note that if I is the intersection of finitely many maximal ideals, then dim$(R/I) = 0$ and the converse is also true if I is a reduced ideal.

A polynomial f in Laurent polynomial ring $R\left[X, X^{-1}\right]$ is said to be doubly monic polynomial if the coefficient of the highest degree term and the lowest degree term are unit.

Definition [9] 2.1. Let R be a ring. For any $\in R$, let $E_{ij}(\omega)$ $(i \neq j)$ be the $n \times n$ matrix $I_n + \omega\left(e_{ij}\right)$, where e_{ij} denotes the matrix which has its $(i, j)^{th}$ entry ω and has zeros elsewhere. Let $E_n(R)$ be the subgroup of $SL_n(R)$ generated by $E_{ij}(\omega)$, $\omega \in R$.

Now we state a theorem of S. D. Kumar and S. Mandal [10]. This is a main ingredient in the proof of the Theorem 3.2.

Theorem [10] 2.2. *Let R be a commutative Noetherian ring and I be an ideal of the Laurent polynomial ring $R\left[X, X^{-1}\right]$ that contains a doubly monic polynomial. Suppose P is a projective R - module of rank $n \geq \dim\left(R\left[X, X^{-1}\right]/I\right) + 2$. Let $S : P \to I(1)$ and $\phi : R\left[X, X^{-1}\right] \to I/I^2$ be two surjective homomorphisms such that $\phi(1) \cong S$ mod $(X - 1, I(1)^2)$. Then there exists a surjective lifts ϕ.*

If $\mu(I) = I/I^2$, we say that I is efficiently generated. If R is any arbitrary commutative Noetherian ring, then equality may not hold. Take a Dedekind domain R, which is not a principal ideal. Then $\mu(I) = 2 > 1 = \mu(I)/I^2$. We now state a theorem of S. Mandal [11] about efficient generation of ideals.

Theorem*[11] *2.3. *Let I be an ideal of $R\left[X, X^{-1}\right]$ over a commutative noetherian ring R. suppose that I contains a doubly monic polynomial and $\mu\left(I/I^2\right) \geq \dim \dfrac{R\left[X, X^{-1}\right]}{I} + 2$. Then ideal I is efficiently generated.*

3. Main Theorem

Theorem 3.1. *Let I be a commutative Noetherian ring and I be a zero dimensional ideal of $R\left[X, X^{-1}\right]$ that contains doubly monic polunomial. Suppose $\mu(I) = n$ and $\mu\left(I(1)/(1)^2\right)$ is a free module of rank $n \geq 2$ over $R/I(1)$. Then a set of n generators of $I(1)$ can be lifted to a set of n generators of I iff every unit of $R/I(1)$ can be lifted to a unit of $R\left[X, X^{-1}\right]/I$.*

Proof. Let unit of $R/I(1)$ can be lifted to a unit of $R\left[X, X^{-1}\right]/I$. Let I be an ideal of $R\left[X, X^{-1}\right]$ containing doubly monic polynomial and I (1) be an ideal of R. Let $<\alpha_1, \alpha_2, ..., \alpha_n>$ generates I (1) and let $<f_1, f_2, ..., f_n>$ generates I such that $f_i = \beta_i$ for i = 1, 2, ...n, consider a generating set $<\overline{\beta_1}, \overline{\beta_2}, ..., \overline{\beta_n}>$ of $I(1)/I(1)^2$. Since $I(1)/I(1)^2$ is free module of rank $n \geq 2$ over $R/I(1)$, there exists $\delta \in GL_n\left(R/I(1)\right)$ and operate on $\{\overline{\beta_1}, \overline{\beta_2}, ..., \overline{\beta_n}\}$, then we get a new set of generators of $I(1)/I(1)^2$ is $<\overline{\alpha_1}, \overline{\alpha_2}, ...\overline{\alpha_n}>$, such that

$$\delta = \{\overline{\beta_1}, \overline{\beta_2}, ..., \overline{\beta_n}\} = \{\overline{\alpha_1}, \overline{\alpha_2}, ..., \overline{\alpha_n}\}.$$

Suppose $\tau = \det(\delta)$. Consider the following diagram

$$\begin{tikzcd}
GL_n\left(\frac{R[X,X^{-1}]}{I}\right) \arrow[r, "\varphi"] \arrow[d, "\det"'] & GL_n\left(\frac{R}{I}\right) \arrow[d, "\det"] \\
U\left(\frac{R[X,X^{-1}]}{I}\right) \arrow[r, "\vartheta"] & U\left(\frac{R}{I(1)}\right)
\end{tikzcd}$$

where φ and ϑ are the maps induced by evaluation at $X = 1$. Now $\delta \subset GL_n\left(R/I(1)\right)$ implies that $\det(\delta)$ is a unit in $R/I(1)$. Let $\tau = \det(\delta)$ and consider the matrix

$$\gamma = \begin{bmatrix} \tau^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \delta.$$

Then $\det(\gamma) = 1$ and $\gamma \in SL_n\left(R/I(1)\right)$. Since dim($R/I(1)$) = 0 and $R/I(1)$ is a Noetherian ring, it is an Artinian ring, it is isomorphic to finite direct product of local rings. For semi local rings we have[12] $SL_n\left(R/I(1)\right) = E_n\left(R/I(1)\right)$. Since elementary matrices have lifting property, there exist a matrix $M \in GL_n\left(R\left[X, X^{-1}\right]/I\right)$ such that $\varphi(M) = \gamma$ (In fact $M \in E_n\left(R\left[X, X^{-1}\right]/I\right)$. By assumption τ can be lifted so τ^{-1} can also be lifted. Let $W \in U\left(R\left[X, X^{-1}\right]/I\right)$ be such that $\vartheta(w) = \tau^{-1}$. Note that ϑ is also a induced map by $p(X-1) + I$ going to $p(1) + I(1)$. Then

$$\varphi = \begin{bmatrix} \epsilon^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} M = \delta.$$

Take

$$\tilde{\delta} = \begin{bmatrix} \epsilon^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} M \in GL_n(R[X, X^{-1}]/I).$$

Then $\varphi(\tilde{\delta}) = \delta$. Since $< \overline{f_1}, \overline{f_2}, ..., \overline{f_n} >$ also generates I/I^2 over $R[X, X^{-1}]/I$, and $\{\alpha_1, \alpha_2 \cdots, \alpha_n\} = \{\beta_1, \beta_2 \cdots, \beta_n\}$ modulo $I(1)^2$, we apply Theorem 2.2, $\{\alpha_1, \alpha_2 \cdots, \alpha_n\}$ can be lifted to a set of generators of I.
Conversely, suppose $< \alpha_1, \alpha_2, \cdots, \alpha_n >$ is a generating set of $I(1)$. Then, by assumption, there exist a generating set $< p_1, p_2, \cdots, p_n >$ of $p_i(1) = \alpha_i$, for $i \in \{1, 2, \cdots, n\}$. Take any $\gamma \in GL_r(R/I(1))$ such that

$\{\beta_1, \beta_2 \cdots, \beta_n\} = \gamma\{\alpha_1, \alpha_2 \cdots, \alpha_n\}$. These generators $<\beta_1, \beta_2 \cdots, \beta_n>$ of $I(1)$ can also be lifted to a generating set $<f_1, f_2 \cdots, f_n>$ of I such that $f_i(1) = \beta_i$ for $i \in \{1, 2, \cdots, n\}$. Note that $I/I^2 = \{\overline{f_1}, \overline{f_2}, \cdots, \overline{f_n}\}$ over $R[X, X^{-1}]/I$ and $I(1)/I(1)^2 = \{\overline{f_1(1)}, \overline{f_2(1)}, \cdots, \overline{f_n(1)}\} = \{\overline{\beta_1}, \overline{\beta_2}, \cdots, \overline{\beta_n}\}$.
Since $\{\overline{f_1(1)}, \overline{f_2(1)}, \cdots, \overline{f_n(1)}\} = \gamma\{\overline{\beta_1}, \overline{\beta_2}, \cdots, \overline{\beta_n}\}$, we have $\{\overline{f_1(1)}, \overline{f_2(1)}, \ldots, \overline{f_n(1)}\} = N\{\overline{P_1}, \overline{P_2}, \ldots, \overline{P_n}\}$, for some $N \in GL_r(\frac{R[X, X^{-1}]}{I})$. Therefore, commutativity of the diagram shows that N is a lift of γ and $det(N)$ is a lift of $det(\gamma)$.

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