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Fixed Point Theorems for Compatible and Non Compatible Self Maps

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Abstract: The purpose of this paper is to prove common fixed point theorems using the concept of weak reciprocal continuity for compatible and non-compatible maps in metric space. These results generalize the common fixed point theorems by Pant, Bisht and Arora[1] and thereby generalize several fixed point theorems.

Keywords: Fixed points, R - weakly commuting maps, compatible and non compatible maps, weak reciprocal continuity.
Mathematics Subject Classification: 54H25.

1. Introduction

Kannan[2] in 1968 proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. From then onwards, the study of common fixed points of mappings satisfying contractive conditions has been an area of vigorous research activity. Sessa[3] defined concept of weakly commuting. Then Jungck generalized this idea first to compatible mappings[4,5] and then to weakly compatible mappings[6]. In 1999, Pant[7] introduced concept of reciprocal continuous and obtained a common fixed theorem for compatible mappings in which the fixed point was a point of discontinuity for all the mappings. The present paper employs the recent notion of weak reciprocal continuity to obtain new fixed point theorems for compatible as well as non compatible mappings.

Definition: 1. Two self maps f and g of a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. Thus the mappings f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent.

Definition: 2. Two selfmappings f and g of a metric space (X, d) are called R-weakly commuting[8] at a point x in X if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R > 0$. The two self-maps f and g are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq R\, d(fx, gx)$.

Definition: 3. Two selfmappings f and g of a metric space (X, d) are called R-weakly commuting of type (A_g) if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, two selfmappings f and g of a metric space (X, d) are called R-weakly commuting of type (A_f) if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X.

Definition: 4. Two selfmappings f and g of a metric space (X, d) are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

Definition: 5. Two selfmappings f and g of a metric space (X, d) are called weakly reciprocally continuous if $\lim_n fgx_n = ft$ or $\lim_n gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

2. Main Results

Theorem: 1. *Let f and g be weakly reciprocally continuous self mappings of a complete metric space (X, d) such that*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq a\, d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy) + e[\, d(fx, gy) + d(fy, gx)]$ *with* $0 \leq a, b, c, e < 1$ *and* $0 \leq a + b + c + 2e < 1$.

If f and g are either compatible or R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *then f and g have a unique common fixed point.*

Proof: Let x_o be any point in X. Then since $fX \subseteq gX$, there exists a sequence of points $x_o, x_1, x_2, \ldots x_n, \ldots$ such that x_{n+1} is in the preimage under g of fx_n that is,

$$fx_o = gx_1, fx_1 = gx_2, \ldots, fx_n = gx_{n+1}, \ldots$$

Also define a sequence $\{y_n\}$ in X by

(1) $$y_n = fx_n = gx_{n+1} \qquad n = 0,1,2,\ldots$$

We claim that $\{y_n\}$ is a Cauchy sequence. Using (ii) we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\leq a\, d(gx_n, gx_{n+1}) + b\, d(fx_n, gx_n) + c\, d(fx_{n+1}, gx_{n+1}) \\ &\quad + e[\, d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)] \\ &= a\, d(y_{n-1}, y_n) + b\, d(y_n, y_{n-1}) + c\, d(y_{n+1}, y_n) \\ &\quad + e[\, d(y_n, y_n) + d(y_{n+1}, y_{n-1})] \end{aligned}$$

or

$$d(y_n, y_{n+1}) \leq \left(\frac{a+b+e}{1-(c+e)}\right) d(y_{n-1}, y_n) = k\, d(y_{n-1}, y_n), \text{ where } k = \left(\frac{a+b+e}{1-(c+e)}\right) < 1.$$

Also for every integer $p > 0$, we get

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+p-1}, y_{n+p}) \\ &\leq (1 + k + k^2 + \ldots + k^{p-1})\, d(y_n, y_{n+1}) \\ &\leq \left(\frac{1-k^p}{1-k}\right) k^n\, d(y_o, y_1). \end{aligned}$$

That is $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$ as $n \to \infty$. Moreover, $y_n = fx_n = gx_{n+1} \to t$.

Suppose that f and g are compatible mappings. Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Firstly let $gfx_n \to gt$.

Then compatibility of f and g gives $lim_n\, d(fgx_n, gfx_n) = 0$. As $n \to \infty$ we get $fgx_n \to gt$. Using (1) this yields $fgx_{n+1} = ffx_n \to gt$. Using (ii) we get

$$d(ft, ffx_n) \leq a\, d(gt, gfx_n) + b\, d(ft,gt) + c\, d(ffx_n, gfx_n) + e[\, d(ft, gfx_n) + d(ffx_n,gt)].$$

On letting $n \to \infty$ we get $ft = gt$, since $b+e < 1$. Since compatibility implies commutativity at coincidence point, we get $fft = fgt = gft = ggt$. Using (ii) we get

$$d(ft, fft) \leq a\, d(gt, gft) + b\, d(ft,gt) + c\, d(fft, gft) + e[\, d(ft, gft) + d(fft,gt)] = (a + 2e)\, d(ft, fft),$$

that is, $ft = fft$. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgx_n \to gu$. Compatibility of f and g implies $gfx_n \to gu$. By virtue of (1) this gives $fgx_{n+1} = ffx_n \to gu$. Using (ii) we get

$$d(fu, ffx_n) \leq a\, d(gu, gfx_n) + b\, d(fu, gu) + c\, d(ffx_n, gfx_n) + e[\, d(fu, gfx_n) + d(ffx_n, gu)].$$

As $n \to \infty$ we get $fu = gu$, since $b+e < 1$. Compatibility of f and g gives $fgu = ggu = ffu = gfu$. Finally, using (ii), we get

$$d(fu, ffu) \leq a\, d(gu, gfu) + b\, d(fu, gu) + c\, d(ffu, gfu) + e[\, d(fu, gfu) + d(ffu, gu)] = (a + 2e)\, d(fu, ffu),$$

that is, $fu = ffu$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Now suppose that f and g are R-weakly commuting of type (A_g). Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. Then R-weak commutativity of type (A_g) of f and g gives $d\,(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $ffx_n \to gt$. Also, using (ii) we get

$$d(ft, ffx_n) \leq a\, d(gt, gfx_n) + b\, d(ft,gt) + c\, d(ffx_n, gfx_n) + e[\, d(ft, gfx_n) + d(ffx_n,gt)].$$

On letting $n \to \infty$ we get $ft = gt$, since $b + e < 1$. R- weak commutativity of type (A_g) implies $d(fft, gft) \leq Rd(ft, gt)$. This gives $fft = gft$ or $fft = fgt = gft = ggt$. Using (ii) we get

$$\begin{aligned} d(ft, fft) &\leq a\, d(gt, gft) + b\, d(ft,gt) + c\, d(fft, gft) \\ &\quad + e[\, d(ft, gft) + d(fft,gt)] \\ &= (a + 2e)\, d(ft, fft), \end{aligned}$$

that is, $ft = fft$. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgx_n \to gu$. By virtue of (1) this gives $ffx_n \to gu$. R-weak commutativity of f and g of type (A_g) gives $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $gfx_n \to gu$. Now, using (ii) we have

$$\begin{aligned} d(fu, ffx_n) &\leq a\, d(gu, gfx_n) + b\, d(fu, gu) + c\, d(ffx_n, gfx_n) \\ &\quad + e[\, d(fu, gfx_n) + d(ffx_n, gu)]. \end{aligned}$$

As $n \to \infty$ we get $fu = gu$, since $b+e < 1$. Again, R-weak commutativity of type (A_g) implies

$d(ffu, gfu) \leq Rd(fu, gu)$. This gives $ffu = gfu$ and $ffu = fgu = gfu = ggu$.

Finally, using (ii) we get

$$\begin{aligned} d(fu, ffu) &\leq ad(gu, gfu) + bd(fu, gu) + c\, d(ffu, gfu) \\ &\quad + e[\, d(fu, gfu) + d(ffu, gu)] \\ &= (a + 2e)\, d(fu, ffu), \end{aligned}$$

that is, $fu = ffu$, since $a + 2e < 1$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, suppose that f and g are R-weakly commuting of type (A_f). Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. Then R-weak commutativity of type (A_f) gives $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ and by (1), we get $fgx_n \to gt$. Also, using (ii) we get

$$\begin{aligned} d(ft, fgx_n) &\leq a\, d(gt, ggx_n) + b\, d(ft, gt) + c\, d(fgx_n, ggx_n) \\ &\quad + e[\, d(ft, ggx_n) + d(fgx_n, gt)]. \end{aligned}$$

On letting $n \to \infty$ we get $ft = gt$, since $b + e < 1$. By the R- weak commutativity of type (A_f), we have $d(fgt, ggt) \leq Rd(ft, gt)$. This gives $fgt = ggt$ and $fft = fgt = gft = ggt$. Using (ii) we get

$$\begin{aligned} d(ft, fft) &\leq a\, d(gt, gft) + b\, d(ft,gt) + c\, d(fft, gft) \\ &\quad + e[\, d(ft, gft) + d(fft,gt)] \\ &= (a + 2e)\, d(ft, fft), \end{aligned}$$

that is, $(1 - a - 2e)\, d(ft, fft) \leq 0$. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgx_n \to gu$. R-weak commutativity of type (A_f) now implies $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $ggx_n \to gu$. Now, using (ii)

$$\begin{aligned} d(fu, fgx_n) &\leq a\, d(gu, ggx_n) + b\, d(fu,gu) + c\, d(fgx_n, ggx_n) \\ &\quad + e[d(fu,ggx_n) + d(fgx_n,gu)]. \end{aligned}$$

As $n \to \infty$ we get $fu = gu$, since $b+e < 1$. Again, R-weak commutativity of type (A_f) implies
$d(fgu, ggu) \leq Rd(fu, gu)$. This gives $fgu = ggu$ and $ffu = fgu = gfu = ggu$. Finally, using(ii), we get

$$\begin{aligned} d(fu, ffu) &\leq a\, d(gu, gfu) + b\, d(fu, gu) + c\, d(ffu, gfu) \\ &\quad + e[\, d(fu, gfu) + d(ffu, gu)] \\ &= (a + 2e)\, d(fu, ffu), \end{aligned}$$

that is, $(1 - a - 2e)d(fu, ffu) \leq 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Uniqueness of the common fixed point theorem follows easily in each of the three cases. We now give examples, one for compatible mappings and one for noncompatible mappings, to illustrate Theorem 1.

Example: 1. Let X = [0,10] and d be the usual metric on X. Define f, g : X → X by

$$fx = \begin{cases} 3 & \textit{if } x = 3 \textit{ and } x > 4 \\ 1 & \textit{if } 3 < x \leq 4 \textit{ and } x < 3, \end{cases} \qquad \textit{and}$$

$$gx = \begin{cases} 3 & \textit{if } x = 3 \\ 9 & \textit{if } 3 < x \leq 4 \textit{ and } x < 3 \\ 5 - \frac{x}{2} & \textit{if } x > 4 \end{cases}$$

Then f and g satisfy all the conditions of Theorem 1 and have a unique common fixed point at $x = 3$. f and g satisfy the contraction condition (ii) for $a = \frac{1}{4}$, $b = \frac{1}{3}$, $c = \frac{1}{6}$, $e = \frac{1}{12}$ .The mappings f and g are R-weakly commuting of type (A_g) since $d(ffx, gfx) \leq d(fx, gx)$ for all x in X. And f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 3$ and either $x_n = 3$ for each n or $x_n = 4 + \frac{1}{n}$. If $x_n = 3$ for each n, $fgx_n = 3 = f3$ and $gfx_n = 3 = g3$. If $x_n = 4 + \frac{1}{n}$ then $fx_n = 3$, $gx_n = 5 - \frac{x_n}{2} \to 3$, $fgx_n = f\left(3 - \frac{1}{2n}\right) = 1 \neq f3$ and $gfx_n = g3 = 3$. Thus $lim_{n\to\infty}\, gfx_n = g3$ but $lim_{n\to\infty}\, fgx_n \neq f3$. Hence f and g are weakly reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings. These computations also show that f and g are non compatible.

Example: 2. Let X = [1, 10] and d be the usual metric on X. Define f, g : X → X by

$$fx = \begin{cases} 5 & \text{if } x \leq 5 \\ 3 & \text{if } x > 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{x+5}{2} & \text{if } x \leq 5 \\ 9 & \text{if } x > 5. \end{cases}$$

Then f and g are weak reciprocally continuous and compatible mappings which satisfy all the conditions of Theorem 1 and have a unique common fixed point at $x = 5$. f and g satisfy the contraction condition (ii) for $a = \frac{1}{4}$, $b = \frac{1}{6}$, $c = \frac{1}{6}$, $e = \frac{1}{4}$. To see that f and g are weak reciprocally continuous, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 5$ and either $x_n = 5$ for each n or $x_n = 5 - \frac{1}{n}$ where $\frac{1}{n} \to 0$ as $n \to \infty$. If $x_n = 5$ for each n, $fgx_n = 5 = f5$ and $gfx_n = 5 = g5$. If $x_n = 5 - \frac{1}{n}$ then $fx_n = 5$, $gx_n =$

$\frac{x_n+5}{2} \to 5$, $fgx_n = f\left(5 - \frac{1}{2n}\right) = 5 = f5$ and $gfx_n = g5 = 5$.Hence f and g are reciprocally continuous and, therefore, weak reciprocally continuous mappings. To see that f and g are compatible, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$ and $gx_n \to t$ for some t. Then $t = 5$ and either $x_n = 5$ for each n or $x_n = 5 - \frac{1}{n}$ where $\frac{1}{n} \to 0$ as $n \to \infty$.If $x_n = 5$ for each n, $fgx_n = 5 = f5$, $gfx_n = 5 = g5$ and $d(fgx_n, gfx_n)=0$. If $x_n = 5 - \frac{1}{n}$ then $fx_n = 5$, $gx_n = \frac{x_n+5}{2} = 5 - \frac{1}{2n} \to 5$, $fgx_n = f\left(5 - \frac{1}{2n}\right) = 5 = f5$, $gfx_n = g5 = 5$ and $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$. Hence f and g are compatible.

Example 1 given above pertains to noncompatible mappings. For noncompatible mappings we can extend Theorem 1 to nonexpansive condition. We do this in the next theorem by letting a=1 in condition (ii) of Theorem 1.

Theorem: 2. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy) + e[\, d(fx, gy) + d(fy, gx)]$ with $0 \leq b, c, e < 1$, and $b+c+2e < 1$

(iii) $d(fx, f^2x) < d(gx, g^2x)$ whenever $gx \neq g^2x$.

If f and g are R-weakly commuting of type (A_g) *or R - weakly commuting of type* (A_f) *then f and g have a common fixed point.*

Proof: Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_n d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Since $fX \subseteq gX$, for each x_n there exists y_n in X such that $fx_n = gy_n$. Thus $fx_n \to t$, $gx_n \to t$, $gy_n \to t$ as $n \to \infty$. By this and using (ii) we obtain $fy_n \to t$. Therefore, we have

(2) $$fx_n \to t,\ \ gx_n \to t,\ \ gy_n \to t,\ \ fy_n \to t\,.$$

Suppose that f and g are R-weakly commuting of type (A_g). Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.

Similarly, $fgy_n \to ft$ or $gfy_n \to gt$. Let us assume that $gfy_n \to gt$. Then R-weak commutativity of type (A_g) of f and g gives $d(ffy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$ we get $ffy_n \to gt$. Also, using (ii) we get

$$d(ffy_n, ft) \leq d(gfy_n, gt) + bd(ffy_n, gfy_n) + cd(ft, gt) + e[d(ffy_n, gt) + d(ft, gfy_n)].$$

On letting $n \to \infty$ we get $d(gt, ft) \leq c\, d(ft, gt) + e\, d(ft, gt)$. This implies $ft = gt$, since $c + e < 1$. Again, by the R- weak commutativity of type (A_g), $d(fft, gft) \leq Rd(ft, gt)$. This gives $fft = gft$ and $fft = fgt = gft = ggt$. If $ft \neq fft$ then using (iii) we get $d(ft, fft) < d(gt, ggt) = d(ft, fft)$, a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgy_n = ffx_n \to ft = gu$. Thus $fgy_n \to ft = gu$ and $ffx_n \to gu$. Hence R-weak commutativity of type (A_g) of f and g yields $d(ffx_n, gfx_n) \leq R\, d(fx_n, gx_n)$. As $n \to \infty$ we get $gfx_n \to gu$ that is $ggy_n \to gu$. Now, using (ii)

$$d(fgy_n, fu) \leq d(ggy_n, gu) + b\, d(fgy_n, ggy_n) + c\, d(fu, gu) + e[d(fgy_n, gu) + d(fu, ggy_n)].$$

As $n \to \infty$ we get $d(gu, fu) \leq (c + e)\, d(gu, fu)$. This implies that $fu = gu$, since $c+e < 1$. Again, R-weak commutativity of type (A_g) of f and g implies, $d(ffu, gfu) \leq R\, d(fu, gu)$. This gives $ffu = gfu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then using (iii), we get $d(fu, ffu) < d(gu, ggu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, suppose that f and g are R-weakly commuting of type (A_f). Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Similarly, $fgy_n \to ft$ or $gfy_n \to gt$. Let $gfx_n \to gt$. Then we have $ggy_n = gfx_n \to gt$. Hence, R-weak commutativity of type (A_f) gives $d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n)$. As $n \to \infty$ we get $fgy_n \to gt$ that is, $ffx_n \to gt$. Also, using (ii) we get

$$d(ffx_n, ft) \leq d(gfx_n, gt) + b\, d(ffx_n, gfx_n) + c\, d(ft, gt) + e[d(ffx_n, gt) + d(ft, gfx_n)].$$

On letting $n \to \infty$ we get $d(gt, ft) \leq (c + e)\, d(ft, gt)$. This implies $ft = gt$, since $c + e < 1$. By the R- weak commutativity of type (A_f), $d(fgt, ggt) \leq Rd(ft, gt)$. This gives $fgt = ggt$ and $fft = fgt = gft = ggt$. If $ft \neq fft$ then by using (iii) we get $d(ft, fft) < d(gt, ggt) = d(ft, fft)$, a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. Hence R-weak commutativity of type (A_f) implies $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $ggx_n \to gu$. Now, using (ii)

$$d(fgx_n, fu) \leq d(ggx_n, gu) + b\, d(fgx_n, ggx_n) + c\, d(fu, gu) + e[d(fgx_n, gu) + d(fu, ggx_n)].$$

As $n \to \infty$ we get $fu = gu$, since $c+e < 1$. Again, R-weak commutativity of type (A_f) of f and g implies, $fgu = ggu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then using (iii), we get $d(fu, ffu) < d(gu, ggu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Example: 3. Let $X = [0, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 3 & \text{if } x \leq 3 \\ 5 & \text{if } x > 3 \end{cases} \quad \text{and} \quad gx = \begin{cases} 6 - x & \text{if } x \leq 3 \\ 10 & \text{if } x > 3 \end{cases}$$

Then f and g satisfy all the conditions of Theorem 2 and have a common fixed point at $x = 3$, f and g satisfy the contraction condition (ii), for any b, c, e such that $0 \leq b, c, e < 1$ and $b+c+2e < 1$ together with condition (iii). The mappings f and g are R-weakly commuting of type (A_g) since $d(ffx, gfx) \leq d(fx, gx)$ for all x in X. And f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 3$ and $x_n = 3$ for each n or $x_n = 3 - \frac{1}{n}$ where $\frac{1}{n} \to 0$ as $n \to \infty$. If $x_n = 3$ for each n, $fgx_n = 3 = f3$ and $gfx_n = 3 = g3$. If $x_n = 3 - \frac{1}{n}$ then $fx_n = 3$, $gx_n = 6 - x_n \to 3$ as $n \to \infty$, $fgx_n = f\left(3 + \frac{1}{n}\right) = 5 \neq f3$ and $gfx_n = g3 = 3$. Thus $\lim_{n\to\infty} gfx_n = g3$ but $\lim_{n\to\infty} fgx_n \neq f3$. Hence f and g are weak reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings. To see that f and g are noncompatible, consider a sequence $\{x_n\}$ in X such that $x_n = 3 - \frac{1}{n}$. Then $fx_n \to 3$, $gx_n \to 3$, $fgx_n \to 5$, $gfx_n \to 3$ as $n \to \infty$. Therefore $\lim_{n \to \infty} d(fgx_n, gfx_n) \neq 0$, Hence f and g are noncompatible.

The next theorem further generalizes Theorem 1 and Theorem 2 by allowing a to take any non-negative value.

Theorem: 3. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq ad(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy) + e[\, d(fx, gy) + d(fy, gx)]$ *with* $a \geq 0$, $0 \leq b, c, e < 1$ *and* $b + c + 2e < 1$

(iii) $d(fx, f^2x) < d(gx, g^2x)$ *whenever* $gx \neq g^2x$.

If f and g are R-weakly commuting of type (A_g) or R- weakly commuting of type (A_f) then f and g have a common fixed point.

Proof: Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_n d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Since $fX \subseteq gX$, for each x_n there exists y_n in X such that $fx_n = gy_n$. Thus, $fx_n \to t$, $gx_n \to t$ and $gy_n \to t$ as $n \to \infty$. By this and using (ii) we obtain $fy_n \to t$. Therefore, we have

(3) $\qquad fx_n \to t,\ gx_n \to t,\ gy_n \to t,\ fy_n \to t.$

Suppose that f and g are R-weakly commuting of type (A_g). Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Similarly, $fgy_n \to ft$ or $gfy_n \to gt$. Let us assume that $gfy_n \to gt$. Then R-weak commutativity of type (A_g) of f and g gives $d(ffy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$ we get $ffy_n \to gt$. Also, using (ii) we get

$$d(ffy_n, ft) \leq ad(gfy_n, gt) + b\, d(ffy_n, gfy_n) + c\, d(ft, gt) + e[\, d(ffy_n, gt) + d(ft, gfy_n)].$$

On letting $n \to \infty$ we get $d(gt, ft) \leq c\, d(ft, gt) + e\, d(ft, gt)$. This implies $ft = gt$, since $c + e < 1$. Again, by the R- weak commutativity of type (A_g), $d(fft,$

$gft) \leq Rd(ft, gt)$. This gives $fft = gft$ and $fft = fgt = gft = ggt$. If $ft \neq fft$ then using (iii) we get $d(ft, fft) < d(gt, ggt) = d(ft, fft)$, a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Next, suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgy_n = ffx_n \to ft$. Thus $fgy_n \to ft = gu$ and $ffx_n \to gu$. R-weak commutativity of type (A_g) of f and g yields $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $gfx_n \to gu$ that is $ggy_n \to gu$. Now, using (ii) we get

$$d(fgy_n, fu) \leq ad(ggy_n, gu) + b\, d(fgy_n, ggx_n) + c\, d(fu, gu) + e[d(fgy_n, gu) + d(fu, ggy_n)].$$

As $n \to \infty$ we get $d(gu, fu) \leq (c + e)\, d(gu, fu)$. This implies that $fu = gu$, since $c+e < 1$. Again, R-weak commutativity of type (Ag) of f and g implies $d(ffu, gfu) \leq Rd(fu, gu)$. This gives $ffu = gfu$ *and* $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then using (iii), we get $d(fu, ffu) < d(gu, ggu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally, suppose that f and g are R-weakly commuting of type (A_f). Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Similarly, $fgy_n \to ft$ or $gfy_n \to gt$. Let $gfx_n \to gt$. Then we have $ggy_n = gfx_n \to gt$. R-weak commutativity of type (A_f) gives $d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n)$. As $n \to \infty$ we get $fgy_n \to gt$ that is $ffx_n \to gt$. Also, using (ii) we get

$$d(ffx_n, ft) \leq ad(gfx_n, gt) + bd(ffx_n, gfx_n) + c\, d(ft, gt) + e[d(ffx_n, gt) + d(ft, gfx_n)].$$

On letting $n \to \infty$ we get $d(gt, ft) \leq (c + e)\, d(ft, gt)$. This implies $ft = gt$, since $c + e < 1$. R- weak commutativity of type (A_f) implies $d(fgt, ggt) \leq Rd(ft, gt)$. This gives $fgt = ggt$ and $fft = fgt = gft = ggt$. If $ft \neq fft$ then by using (iii) we get $d(ft, fft) < d(gt, ggt) = d(ft,fft)$, a contradiction. Hence ft = fft = gft and ft is a common fixed point of f and g.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. R-weak commutativity of type (A_f) now implies $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$. As $n \to \infty$ we get $ggx_n \to gu$. Now, using (ii) we have

$$d(fgx_n, fu) \leq ad(ggx_n, gu) + b\, d(fgx_n, ggx_n) + c\, d(fu, gu) + e[d(fgx_n, gu) + d(fu, ggx_n)].$$

As $n \to \infty$ we get $fu = gu$, since $c+e < 1$. Again, R-weak commutativity of type (A_f) of f and g implies, $fgu = ggu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then using (iii), we get $d(fu, ffu) < d(gu, ggu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.
The next example illustrates Theorem 3.

Example: 4. Let $X = [0, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 3 & \text{if } x \leq 3 \\ 7 & \text{if } x > 3 \end{cases} \quad \text{and} \quad gx = \begin{cases} 6 - x & \text{if } x \leq 3 \\ 7 & \text{if } x > 3. \end{cases}$$

Then f and g satisfy all the conditions of Theorem 3 and have two common fixed points at $x = 3$ and $x = 7$. f and g satisfy condition (ii) for $a = 3$, $b = 0$, $c = 0$, $e = 1/5$ together with the condition (iii). The mappings f and g are R-weakly commuting of type (A_g) since $d(ffx, gfx) \leq d(fx, gx)$ for all x in X. And f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 3$ or $t=7$ and $x_n = 3$ for each n or $x_n = 3 - \frac{1}{n}$ or $x_n > 3$ for each n. If $x_n = 3$ for each n, $fgx_n = 3 = f3$ and $gfx_n = 3 = g3$. If $x_n = 3 - \frac{1}{n}$ then $fx_n = 3$, $gx_n = 6 - x_n \to 3$, $fgx_n = f\left(3 + \frac{1}{n}\right) = 7 \neq f3$ and $gfx_n = g3 = 3$.Thus $\lim_{n\to\infty} gfx_n = g3$ but $\lim_{n\to\infty} fgx_n \neq f3$. If $x_n > 3$ for each n then $fx_n = 7$, $gx_n = 7$, $fgx_n = f7 = 7$ and $gfx_n = g7 = 7$. Thus $\lim_{n\to\infty} gfx_n = g7$ and $\lim_{n\to\infty} fgx_n = f7$. Hence f and g are weak reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings. To see that f and g are noncompatible, consider a sequence $\{x_n\}$ in X such that $x_n = 3 - \frac{1}{n}$. Then $fx_n \to 3$, $gx_n \to 3$, $fgx_n \to 7$, $gfx_n \to 3$ as $n \to \infty$. Therefore $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$, Hence f and g are noncompatible.

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