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Conformal Randers Change of Douglas Space of Second Kind with Special (α, β)-Metrics

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Abstract: A change of Finsler metric $L(\alpha, \beta) \rightarrow \bar{L}(\bar{\alpha}, \bar{\beta}) = e^{\sigma}\left[L(\alpha, \beta)+\beta\right]$ is called conformal Randers change where σ is a function of position of x^i only and known as a conformal factor, α is a Riemannian metric and β is differential 1-form. The present article is devoted to study the necessary and sufficient condition for a conformal randers change of douglas space of second kind of a Finsler space to be a douglas space of second kind. Further we are discussed the conformal randers change of Finsler space with special $(\alpha, \beta)-$metrics of douglas space of second kind.

Keywords: Finsler space, $(\alpha, \beta)-$metrics, Conformal change, Conformal Randers change, Douglas space, Douglas space of second kind.

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1. Introduction

The notation of Douglas space was introduced by S. Bacso and M. Matsumoto[1] as a generalization of Berwlad space from the view point of geodesic equations. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically. The condition for Finsler space with $(\alpha, \beta)-$metric to be of Douglas type studied by M. Matsumoto[2]. I. Y. Lee[3], defined a Douglas space of second kind and he proved that the Finsler space with the Matsumoto metric to be a Douglas space of second kind. S. K. Narasimhamurthy and G. N. Latha kumari[4], worked on Douglas space of second kind of Finsler space with $(\alpha, \beta)-$metric.

Conformal transformation of Douglas space with special $(\alpha, \beta)-$metric have been studied by S. K. Narasimhamurthy[5]. And also S. K. Narasimhamurthy, Ajith and C. S. Bagewadi[6], studied on the necessary and sufficient condition for Douglas space with $(\alpha, \beta)-$metric under conformal $\beta-$change. S. K. Narasimhamurthy, Ajith and C. S. Bagewadi[7] worked on conformal change of Douglas space of second kind with $(\alpha, \beta)-$metrics. H. S. Sukhla, O. P. Pandey and H. D. Joshi[8], finds the condition that conformal Randers change of Finsler space with $(\alpha, \beta)-$metric of Douglas type yields a Finsler space of Douglas type.

The present article is organized as follows: 'In the first part, we are devoted to study the necessary and sufficient condition for conformal randers change of Douglas space of second kind to be Douglas space of second kind, i.e., K^{im}_{m} are homogeneous polynomial in (y^m) of degree two. In the next part, we are discussing about conformal randers change of Finsler space with special $(\alpha, \beta)-$metrics of Douglas space of second kind.

2. Preliminaries

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space, where M^n be a $n-$dimensional differential manifold and $L(x, y)$ is the fundamental function defined on the manifold TM_0 of non zero tangent vectors. We assume that $L(x, y)$ is positive and the metric tensors $g_{ij}(x) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$ is positive definite, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

A Finsler space $F^n = (M^n, L)$ is said to have an $(\alpha, \beta)-$metric, if $L(\alpha, \beta)$ is positively homogeneous function of degree one in two variables

$\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$, where α is Riemannian metric and β is differential 1-form. From the differential 1-form $\beta(x, y) = b_i(x)y^i$, we define:

$$2r_{ij} = b_{i|j} + b_{j|i},\ r^i_j = a^{ih} r_{hj},\ r_j = b_i r^i_j,$$

$$2s_{ij} = b_{i|j} - b_{j|i},\ \ s^i_j = a^{ih} s_{hj},\ s_j = b_i s^i_j,$$

$$b^i = a^{ih} b_h,\ b^2 = b^i b_i.$$

The geodesic of an $n-$dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations[9]:

$$\frac{d^2x^i}{dt^2} y^j - \frac{d^2x^j}{dt^2} y^i + 2\left(G^i y^j - G^j y^i\right) = 0, \quad y^i = \frac{dx^i}{dt},$$

in a parameter t. The function $G^i(x, y)$ is given by

$$2G^i(x, y) = g^{ij}\left(y^r \dot{\partial}_j \partial_r F - \partial_j F\right) = \gamma^i_{jk} y^j y^k,$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$, $F = \dfrac{L^2}{2}$ and $g^{ij}(x, y)$ be the inverse of the Finsler metric tensor $g_{ij}(x, y)$.

According to[1], a Finsler space F^n is of Douglas type if and only if the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}\left(G_{ijk} y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j\right),$$

vanishes identically, where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the $hv-$curvature tensor of the Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$, $G_{ij} = G^r_{ijr}$ and $G_{ijk} = \dot{\partial}_k G_{ij}$[10].

A Finsler space F^n is said to be a Douglas space[1], if

(2.1) $$D_{ij} = G^i(x, y)y^j - G^j(x, y)y^i,$$

is homogeneous polynomial in (y^i) of degree three. Thus, a Finsler space with an $(\alpha, \beta)-$metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are homogeneous polynomial in (y^i) of degree three.

Further differentiating (2.1) by y^m and contracting with m and j in the obtained equation, we have

(2.2) $$D^{im}_m = (n+1)G^i - G^m_m y^i.$$

A Finsler space F^n is said to be Douglas space of second kind if it satisfies the condition that $D^{im}_m = (n+1)G^i - G^m_m y^i$ be a homogeneous polynomials in (y^i) of degree two. And also a Finsler space with an $(\alpha, \beta)-$metric is said to be a Douglas space of the second kind if and only if

(2.3) $$B^{im}_m = (n+1)B^i - B^m_m y^i,$$

are homogeneous polynomials in y^i of degree two, where B^m_m is given by[11]. Again differentiating the above with respect to y^h, y^j and y^k, we get

$$B^{im}_{hjkm} = B^i_{hjk} = 0.$$

3. Douglas Space of Second Kind with $(\alpha, \beta)-$Metric

In this section, we study the condition for a Finsler space F^n with an $(\alpha, \beta)-$metric to be a Douglas space of second kind. Now let us consider the function $G^i(x, y)$ of F^n with an $(\alpha, \beta)-$metric. According to[12], $G^i(x, y)$ can be written as,

(3.1)
$$2G^i = \gamma^i_{00} + 2\bar{A}^i,$$
$$B^i = \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 + C^* \left[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right)\right],$$

where,

(3.2) $$C^* = \frac{\alpha\beta\left(r_{00}L_\alpha - 2\alpha s_0 L_\beta\right)}{2\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)}, \quad \gamma^2 = b^2\alpha^2 - \beta^2 .$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ are homogeneous polynomial in $\left(y^i\right)$ of degree two, equation (3.1) yields

(3.3) $$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}\left(s^i_0 y^j - s^j_0 y^i\right) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\left(b^i y^j - b^j y^i\right).$$

By means of (2.1) and (3.3), we use the following lemma proved by M. Matsumoto[2],

Lemma 3.1: *A Finsler space F^n with an $(\alpha, \beta)-$metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are* $\mathrm{hp}(3)$.

Further differentiating (3.3) by y^m and contracting with m and j in the obtained equation, we have

(3.4) $$B^{im}_m = \frac{(n+1)\alpha L_\beta}{L_\alpha} s^i_0 + \frac{a\left\{(n+1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\right\}}{2\Omega^2} r_{00}$$
$$- \frac{a^2\left\{(n+1)\alpha^2 \Omega L_\beta L_{\alpha\alpha} b^i + B y^i\right\}}{L_\alpha \Omega^2} s_0 - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\omega} r_0 ,$$

where,

$$\Omega = \left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right),$$

(3.5) $$A = \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha\left(L_{\alpha\alpha}\right)^2,$$

$$B = \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\left\{\left(3\gamma^2 - \beta^2\right)L_\alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\right\}$$
$$\times L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}$$

Further, we use the following theorem proved by I. Y. Lee[3]:

Theorem 3.1: *The necessary and sufficient condition for a Finsler space* F^n *with an* $(\alpha, \beta)-$*metric to be a Douglas space of the second kind is that,* B^{im}_m *are homogeneous polynomial in* (y^m) *of degree two, where* B^{im}_m *is given by (3.4) and (3.5), provided that* $\Omega \neq 0$.

4. Conformal Randers Change of Douglas Space of Second Kind

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on the same underlying manifold M^n such that $\bar{L}(\bar{\alpha}, \bar{\beta}) = e^{\sigma}\left[L(\alpha, \beta) + \beta\right]$, then F^n is called Conformaly Randers to $\bar{F}^n$, and the change $L \to \bar{L}$ of metric is called conformal Randers change of $(\alpha, \beta)-$metric. A conformal Randers change of $(\alpha, \beta)-$metric is expressed as $(\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$, where $\bar{\alpha} = e^{\sigma}\alpha$, $\bar{\beta} = e^{\sigma}\beta$. Therefore, we have

$$\text{(4.1)} \qquad \begin{cases} \bar{a}_{ij} = e^{2\sigma} a_{ij}, \quad \bar{a}^{ij} = e^{-2\sigma} a^{ij}, \\ \bar{b}_i = e^{\sigma} b_i, \qquad \bar{b}^i = e^{-\sigma} b^i, \\ \bar{b}^2 = b^2, \; \bar{y}^i = y^i, \; \bar{y}_i = e^{2\sigma} y_i. \end{cases}$$

Proposition 4.1: *A Finsler space with* $(\alpha, \beta)-$*metric the length* b *of* b_i *with respect to the Riemannian metric* α *is invariant under any conformal change of* $(\alpha, \beta)-$*metric.*

According to[13], Conformal Randers Change of Christoffel symbols γ^i_{jk} are same as conformal change. From (4.1), it follows that conformal Randers change of Christoffel symbols is given by[9];

$$\text{(4.2)} \qquad \bar{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk},$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij}\sigma_j$. From (4.1) and (4.2), we have the following under conformal Randers change:

$$\bar{b}_{i|j} = e^{\sigma}\left(b_{i|j} + \rho a_{ij} - \sigma_i b_j\right),$$

$$\bar{r}_{ij} = e^{\sigma}\left[r_{ij} + \rho a_{ij} - \frac{1}{2}\left(b_i\sigma_j + b_j\sigma_i\right)\right],$$

(4.3)
$$\bar{s}_{ij} = e^{\sigma}\left[s_{ij} + \frac{1}{2}\left(b_i\sigma_j - b_j\sigma_i\right)\right],$$

$$\bar{s}^i_j = e^{-\sigma}\left[s^i_j + \frac{1}{2}\left(b^i\sigma_j - b_j\sigma^i\right)\right],$$

$$\bar{s}_j = s_j + \frac{1}{2}\left(b^2\sigma_j - \rho b_j\right),$$

where $\rho = \sigma_r b^r$.

From (4.2) and (4.3), we can easily obtain the following:

$$\text{(4.4)}\qquad \begin{cases} \bar{\gamma}^i_{00} = \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma^i, \\ \bar{r}_{00} = e^{\sigma}\left[r_{00} + \rho\alpha^2 - \sigma_0\beta\right], \\ \bar{s}^i_0 = e^{-\sigma}\left[s^i_0 + \frac{1}{2}\left(b^i\sigma_0 - \beta\sigma^i\right)\right], \\ \bar{s}_0 = s_0 + \frac{1}{2}\left(b^i\sigma_0 - \rho\beta\right). \end{cases}$$

Next, to find the conformal Randers change of B^{ij} given in (3.3), we first find the conformal Randers change of C^* given in (3.2). Since $\bar{L}\left(\bar{\alpha}, \bar{\beta}\right) = e^{\sigma}\left[L\left(\alpha, \beta\right) + \beta\right]$, we have

$$\text{(4.5)}\qquad \begin{cases} \bar{L}_{\bar{\alpha}} = L_{\alpha}, \bar{L}_{\bar{\alpha}\bar{\alpha}} = e^{-\sigma}L_{\alpha\alpha}, \\ \bar{L}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}} = e^{-2\sigma}L_{\alpha\alpha\alpha}, \\ \bar{L}_{\bar{\beta}} = L_{\beta} + 1, \bar{\gamma}^2 = e^{2\sigma}\gamma^2. \end{cases}$$

By using (3.2), (4.4) and (4.5), we obtain,

$$\bar{C}^* = e^{\sigma}\left(C^* + D^*\right),$$

where $$D^* = \frac{\alpha\beta\left[\left(\rho\alpha^2 - \sigma_0\beta\right)L_\alpha - 2\alpha s_0 - \alpha\left(b^2\sigma_0 - \rho\beta\right)\left(L_\beta + 1\right)\right]}{\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)}.$$

Hence under the conformal Randers change of B^{ij} can be written in the form:

$$\bar{B}^{ij} = B^{ij} + C^{ij},$$

where

(4.6) $$C^{ij} = \frac{\alpha}{L_\alpha}\left(s_0^i y^j - s_0^j y^i\right) - \frac{\alpha\beta\left(L_\beta + 1\right)}{2L_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right)$$

$$+\left[\frac{\alpha\left(L_\beta + 1\right)\sigma_0}{2L_\alpha} + \frac{\alpha^3 L_{\alpha\alpha}\left(\rho\alpha^2 - \sigma_0\beta\right)}{\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)} - \frac{2\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)}\right.$$

$$\left.- \frac{\alpha^4 L_{\alpha\alpha}\left(L_\beta + 1\right)\left(b^2\sigma_0 - \rho\beta\right)}{L_\alpha\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)}\right]\left(b^i y^j - b^j y^i\right).$$

From the equation (3.5), it is clear that

(4.7) $$\bar{\Omega} = e^{2\Omega},\ \bar{A} = e^{-\sigma} A,\ \bar{B} = e^{2\sigma} B.$$

Now apply conformal transformation to B_m^{im}, we obtain

(4.8) $$\bar{B}_m^{im} = B_m^{im} + K_m^{im},$$

where,

(4.9) $$K_m^{im} = \frac{(n+1)\alpha}{L_\alpha} s_0^i + \frac{(n+1)\alpha\left(L_\beta + 1\right)}{2L_\alpha}\left(\sigma_0 b^i - \beta\sigma^i\right)$$

$$+\frac{\alpha\left\{(n+1)\alpha^2\Omega L_{\alpha\alpha}b^i+\beta\gamma^2Ay^i\right\}}{2\Omega^2}\left(\rho\alpha^2-\sigma_0\beta\right)$$

$$-\frac{\alpha^2\left[(n+1)\alpha^2\Omega L_{\alpha\alpha}b^i+\begin{pmatrix}\alpha\beta\gamma^2L_\alpha L_{\alpha\alpha\alpha}+\beta\begin{Bmatrix}\left(3\gamma^2-\beta^2\right)L_\alpha\\-4\alpha\gamma^2L_{\alpha\alpha}\end{Bmatrix}\\L_{\alpha\alpha}+\Omega\beta L_{\alpha\alpha}\end{pmatrix}y^i\right]}{L_\alpha\Omega^2}S_0$$

$$+\frac{\alpha^2\left[(n+1)\alpha^2\Omega\left(L_\beta+1\right)L_{\alpha\alpha}b^i+\begin{pmatrix}B+\alpha\beta\gamma^2L_\alpha L_{\alpha\alpha\alpha}\\+\beta\begin{Bmatrix}\left(3\gamma^2-\beta^2\right)L_\alpha\\-4\alpha\gamma^2L_{\alpha\alpha}\end{Bmatrix}L_{\alpha\alpha}\end{pmatrix}y^i\right]}{2L_\alpha\Omega^2}$$

$$-\frac{\Omega\alpha^2\beta L_{\alpha\alpha}y^i}{2L_\alpha\Omega^2}\left(b^2\sigma_0-\rho\beta\right)+\frac{\alpha^3L_{\alpha\alpha}y^i}{2\Omega}\left(b^2\sigma_0-\rho\beta\right).$$

Thus we state the following:

Theorem 4.1: *The necessary and sufficient condition for a conformal Randers change of Douglas space of the second kind of Finsler space to be Douglas space of the second kind, is that* $K_m^{im}(x)$ *are homogeneous polynomial in* $\left(y^m\right)$ *of degree two.*

5. Conformal Randers Change of Douglas Space of Second Kind with special $(\alpha,\beta)-$metrics

In this section, we extend the study on conformal Randers change of Douglas space of second kind and obtain the conditions for Douglas space of second kind with the $(\alpha,\beta)-$metrics as mentioned below to be a Douglas space of second kind under conformal Randers change.

Applying Conformal Randers change for the following $(\alpha,\beta)-$metrics:

(5.1) $$L=\alpha\,,\ L=\frac{\alpha^2}{\beta}+\beta\ .$$

5.1 Conformal Randers Change of Douglas Space of Second Kind with Riemannian metric: First we consider the metric, $L=\alpha$ called as Riemannian metric. Then we have

(5.2) $$L_\alpha=1,\ L_{\alpha\alpha}=0,\ L_{\alpha\alpha\alpha}=0,\ L_\beta=0.$$

Since from (3.5), we have

(5.3) $$\Omega=\beta^2,\ A=0,\ B=0.$$

Under the conformal Randers change for Riemannain metric, B_m^{im} can be written in the form:

(5.4) $$\bar{B}_m^{im}=B_m^{im}+K_m^{im},$$

where K_m^{im} from (4.9) is reduced to,

(5.5) $$K_m^{im}=\frac{(n+1)\alpha\left(\sigma_0 b^i-\beta\sigma^i\right)}{2}.$$

Now to show the Riemannian metric is Douglas space of second kind under conformal Randers change, just we have to show that K_m^{im} is a $\mathrm{hp}(2)$.

Since α is irrational function in $\left(y^i\right)$ from (5.5), it follows that K_m^{im} is a $\mathrm{hp}(2)$ if and only if

(5.6) $$\left(\sigma_0 b^i-\beta\sigma^i\right)=0.$$

Now the terms of (5.5) reduced to $K_m^{im}=0$.

So $K_m^{im}=0$, if and only if $\sigma_k b^i-b_k\sigma^i=0$. i.e. $b_i\sigma_j-b_j\sigma_i=0$ which gives $\sigma_i=\frac{\rho}{b^2}b_i$. Conversely, if $\sigma_i=\frac{\rho}{b^2}b_i$, then $\sigma_0=\frac{\rho}{b^2}\beta$ and (5.5) implies that $K_m^{im}=0$.

Thus we state that,

Theorem 5.1: *Let* $\bar{F}^n(n>2)$ *be a Finsler space which is transformed by a conformal Randers change of a Finsler space* F^n *with Riemannian metric is of Douglas space of second kind to be also Douglas space of second kind if and only if* $\sigma_i = \frac{\rho}{b^2} b_i$.

5.2 Conformal Randers Change of Douglas Space of Second Kind with the metric $L = \frac{\alpha^2}{\beta} + \beta$:

For an $(\alpha, \beta)-$metric, $L = \frac{\alpha^2}{\beta} + \beta$. Then we have

$$L_\alpha = \frac{2\alpha}{\beta}, \; L_{\alpha\alpha} = \frac{2}{\beta}, \; L_{\alpha\alpha\alpha} = 0, \; L_\beta = \frac{\beta^2 - \alpha^2}{\beta^2}. \tag{5.7}$$

Since from (3.5), we have

$$\Omega = \frac{2b^2\alpha^3}{\beta}, \quad A = 0, \quad B = \frac{8b^2\alpha^5}{\beta^3}. \tag{5.8}$$

Under the conformal Randers change for a metric $L = \frac{\alpha^2}{\beta} + \beta$, B_m^{im} can be written in the form:

$$\bar{B}_m^{im} = B_m^{im} + K_m^{im}, \tag{5.9}$$

where K_m^{im} from (4.9) is reduced to,

$$\begin{aligned} K_m^{im} = {} & \frac{(n+1)\beta}{2} s_0^i + \frac{(n+1)(2\beta^2 - \alpha)}{4\beta}\left(\sigma_0 b^i - \beta\sigma^i\right) \\ & + \frac{(n+1)b^i}{2b^4}\left(\rho\alpha^2 - \sigma_0\beta\right) - \frac{(n+1)\beta}{2b^2} s_0 b^i + \frac{1}{2b^2}\left(b^2\sigma_0 - \rho\beta\right) \\ & - \left[\frac{(n+1)(2\beta^2 - \alpha)}{4b^2\beta} b^i + \frac{1}{2b^2} y^i\right]\left(b^2\sigma_0 - \rho\beta\right). \end{aligned} \tag{5.10}$$

From the equation (5.10), the terms $\frac{(n+1)\beta}{2}s_0^i+\frac{(n+1)\beta}{2}\sigma_0 b^i+\frac{(n+1)b^i}{2b^4}$
$\times\left(\rho\alpha^2-\sigma_0\beta\right)-\frac{(n+1)\beta}{2b^2}s_0 b^i-\left[\frac{(n+1)\beta}{2b^2}b^i+\frac{1}{2b^2}\right]\left(b^2\sigma_0-\rho\beta\right)+\frac{1}{2b^2}\left(b^2\sigma_0-\rho\beta\right)$
are homogeneous polynomial in $\left(y^i\right)$ of degree 2. So these terms may be neglected in our discussion and we consider only the terms

$$K_m^{im}=-\frac{(n+1)\alpha^2}{4\beta}\sigma_0 b^i+\frac{(n+1)\alpha^2}{4\beta}\sigma_0 b^i=0. \tag{5.11}$$

Since $K_m^{im}=0$, from (4.8), we have $\bar{B}_m^{im}=B_m^{im}$.
Thus we state that,

Theorem 5.2: *Let $\bar{F}^n(n>2)$ be a Finsler space which is transformed by a conformal Randers change of a Finsler space F^n with the metric $L=\frac{\alpha^2}{\beta}+\beta$ is of Douglas space of second kind to be also Douglas space of second kind if and only if $K_m^{im}=0$.*

6. Conclusion

Let $\left(M^n, L\right)$ be a Finsler space where M is $n-$dimensional differentiable manifold equipped with a fundamental function L. The conformal transformation between L and $\bar{L}$ is defined by $\bar{L}=e^{\sigma}L$ where $\sigma=\sigma(x)$ is a scalar function on M . we call such two metrics L and $\bar{L}$ are conformally related. In 1941, Randers has introduced the Finsler change $L(x, y)=L(x, y)+\beta$ is called as Randers change, where L is the fundamental Finsleriean function.

Let $F^n=\left(M^n, L\right)$ and $\bar{F}^n=\left(\bar{M}^n, \bar{L}\right)$ be two Finsler spaces on the same underlying manifold M^n such that $\bar{L}\left(\bar{\alpha}, \bar{\beta}\right)=e^{\sigma}\left[L(\alpha, \beta)+\beta\right]$, then F^n is called conformaly Randers to $\bar{F}^n$, and the change $L\to\bar{L}$ of metric is called conformal Randers change of $(\alpha, \beta)-$metric.

The conformal Randers change can be considered as generalization of conformal as well as Randers change. Because if $\beta=0$, it reduces to conformal change and if $\sigma(x)=0$, it reduces to Randers change. It is composition of Randers change and conformal change. By using the above results, we are trying to generalize conformal Randers change which is of the form $\bar{L}(\bar{\alpha}, \bar{\beta})=e^{\sigma}\left[L(\alpha, \beta)+\beta\right]$ is of Douglas space of second kind.

The present investigation deals with the necessary and sufficient condition for a Finsler space with an $(\alpha, \beta)-$metric to be a Douglas space of second kind under conformal Randers change. Further we are discussed the special $(\alpha, \beta)-$metrics for conformal Randers change of a Douglas space of second kind. In this regard we obtain the following results as follows:

1. The necessary and sufficient condition for a conformal Randers change of Douglas space of the second kind of Finsler space to be Douglas space of the second kind, is that $K_m^{im}(x)$ are homogeneous polynomial in (y^m) of degree two.
2. Let $\bar{F}^n(n>2)$ be a Finsler space which is transformed by a conformal Randers change of a Finsler space F^n with Riemannian metric is of Douglas space of second kind to be also Douglas space of second kind if and only if $\sigma_i=\frac{\rho}{b^2}b_i$.
3. Let $\bar{F}^n(n>2)$ be a Finsler space which is transformed by a conformal Randers change of a Finsler space F^n with the metric $L=\frac{\alpha^2}{\beta}+\beta$ is of Douglas space of second kind to be also Douglas space of second kind if and only if $K_m^{im}=0$.

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