ISSN 0974 - 9373

Vol. 14 No.4 (2010) **Journal of International Academy of Physical Sciences** pp. 413 - 423

On The Degree of Approximation of Signals (Functions) Belonging to Generalized Weighted $W\left(L_p, \xi(t)\right), (p \geq 1)$ - Class by Product Summability Method*

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(Received February 20, 2010)

Abstract: In the present paper we have studied the degree of approximation of a Signal (function) associated with Fourier series and belonging to the generalized weighted $W\left(L_p, \xi(t)\right), (p \geq 1)$ -class by product summability (C, 1)(E, q) method. Recently Lal and Kushwaha[1] obtained the degree of approximation of certain function belonging to $Lip\,\alpha$ class by (C, 1)(E, q) means of its Fourier series. We have extended this result to the functions belonging to $W\left(L_p, \xi(t)\right), (p \geq 1)$ by using $(C,1)(E,q)$ means of its Fourier series. The class $W\left(L_p, \xi(t)\right), (p \geq 1)$, we have used in the theorem includes $Lip\left(\xi(t), p\right)$ and $Lip\,\alpha$ classes.

Key words: Fourier series, Product summability (C, 1) (E, q) method, Generalized weighted $W\left(L_p, \xi(t)\right)$ - Class, Degree of Approximation.

AMS 2000 Mathematics Subject Classification: 40C99, 40G99.

1. Introduction and Notations

Let $f(x)$ be a 2π -periodic signal (function) and let $f \in L_1[0, 2\pi] = L_1$. Then the Fourier series of function (signal) f at any point x is given by

$$(1.1) \qquad f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx\right) \equiv \sum_{k=0}^{\infty} u_k(f; x),$$

with partial sums $s_n(f;x)$ -a trigonometric polynomial of degree (or order) n, of the first $(n+1)$ terms. A function (signal) $f \in Lip\,\alpha$, for $0 < \alpha \leq 1$, if $\left| f(x+t) - f(x) \right| = O(t^{\alpha})$.

*Presented at CONIAPS XI, University of Allahabad, Feb. 20-22, 2010.

A function[2] (signal) $f \in Lip(\alpha, p)$ for $p \geq 1$, $0 < \alpha \leq 1$, if

$$\left\{ \int_0^{2\pi} | f(x+t) - f(x) |^p \, dx \right\}^{1/p} = O(t^\alpha),$$

Given[3] a positive increasing function $\xi(t)$ and an integer $p \geq 1$, $f \in Lip(\xi(t), p)$, if

$$\left\{ \int_0^{2\pi} | f(x+t) - f(x) |^p \, dx \right\}^{1/p} = O(\xi(t)).$$

In case $\xi(t) = t^\alpha, 0 < \alpha \leq 1$, then $Lip(\xi(t), p)$ coincides with the class $Lip(\alpha, p)$. If $p \to \infty$ in $Lip(\alpha, p)$ class then this class reduces to $Lip\,\alpha$. For a given positive increasing function[4] $\xi(t)$, an integer $p \geq 1$, $f \in W(L_p, \xi(t))$, if

$$\left\{ \int_0^{2\pi} \left| \{ f(x+t) - f(x) \} \sin^\beta x \right|^p dx \right\}^{1/p} = O(\xi(t)), \ (\beta \geq 0). \tag{1.2}$$

We note that, if $\beta = 0$ then the weighted class $W(L_p, \xi(t))$ coincides with the class $Lip(\xi(t), p)$ and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), p)$ class coincides with the class $Lip(\alpha, p)$.

Also we observe that

$$Lip\,\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L_p, \xi(t)) \ \text{ for } 0 < \alpha \leq 1, \ p \geq 1.$$

The L_p-norm is defined by

$$\| f \|_p = \left(\int_0^{2\pi} | f(x) |^p dx \right)^{1/p}, \ p \geq 1$$

The L_∞ – norm of a function $f : R \to R$ is defined by

$$\| f \|_\infty = \sup\{ | f(x) | : x \in R \},$$

and the degree of approximation $E_n(f, x)$ is given by Zygmund[5]

$$E_n(f, x) = \underset{n}{Min} \| f(x) - \tau_n(f; x) \|_p, \tag{1.3}$$

in terms of n, where $\tau_n(f; x) = \sum_{k=0}^{n} a_{n,k} s_k(f; x)$ is a trigonometric polynomial of degree n. This method of approximation is called trigonometric Fourier Approximation (tfa).

(1.4) $$\| \tau_n(f,x) - f(x) \|_\infty = \sup_{x \in R} \left\{ | \tau_n(f,x) - f(x) | \right\}.$$

Let $\sum_{k=0}^{\infty} u_k$ be a given infinite series with the sequence of n$^{\text{th}}$ partial sums $\{s_n\}$. If

(1.5) $$(E,q) = E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s, \ as\ n \to \infty,$$

then an infinite series $\sum_{k=0}^{\infty} u_k$ with the partial sums s_n is said to be (E, q) summable to the definite number s (Hardy[6]).

An infinite series $\sum_{k=0}^{\infty} u_k$ is said to be (C, 1) summable to s if

$$(C,1) = \frac{1}{(n+1)} \sum_{k=0}^{n} s_k \to s \ \ as\ n \to \infty.$$

The (C, 1) transform of the (E, q) transform E_n^q defines the (C, 1) (E, q) transform of the partial sums s_n of the series $\sum_{k=0}^{\infty} u_k$, i.e. the product summability (C, 1) (E, q) is obtained by superimposing (C, 1) summability on (E, q) summability.
Thus, if

(1.6) $$(C\,E)_n^q = \frac{1}{(n+1)} \sum_{k=0}^{n} E_k^q = \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} s_r \to s, as\, n \to \infty,$$

where E_n^q denote the (E, q) transform of s_n, then an infinite series $\sum_{k=0}^{\infty} u_k$ with the partial sums s_n is said to be summable (C, 1) (E, q) means or simply summable (C, 1) (E, q) to the definite number s and we can write

$$(C\ E)_n^q \to s\left[(C,1)(E,q)\right], \ as\ n \to \infty.$$

We shall use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2\ f(x),$$

(1.7) $$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r+1/2)t}{\sin(t/2)} \right].$$

Furthermore C will denote an absolute positive constant, not necessarily the same at each occurrence.

2. Known Results

Various investigators such as Qureshi[7, 8], Khan[9], Qureshi and Neha[10, 11] discussed the degree of approximation of signals (functions) belonging to $Lip\,\alpha$, $Lip\,(\alpha, p)$, $Lip\left(\xi(t),\ p\right)$ and $W\left(L_p, \xi(t)\right)$-classes by using Nörlund means of an infinite series. Lal and Kushwaha[1] have determined the degree of approximation of functions of Lipschitz class by product summability mean of the form $(C,1)(E,q)$ of its Fourier series. They have proved:

Theorem[1] 2.1. *If $f: R \to R$ is $2\pi-$periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class, then the degree of approximation of f by the (C, 1) (E, q) product means of its Fourier series, satisfies for $n = 0, 1, 2 \ldots$*

$$\left\|(C\ E)_n^q(x) - f(x)\right\|_\infty = O\left((n+1)^{-\alpha}\right), \text{ for } 0<\alpha<1.$$

3. Main Result

In the present paper, we extend theorem 2.1 for the functions (signals) f of weighted $W(L_p, \xi(t))$, $(p \geq 1)$-class by using product summabilities (C, 1) (E, q) means of its Fourier series. We prove:

Theorem 3.1. *If $f: R \to R$ is a 2π-periodic, Lebesgue integrable and belonging to weighted $W(L_p, \xi(t))\,(p \geq 1)$-class, then the degree of approximation of $f(x)$ by (C, 1) (E, q) means of its Fourier series is given by*

$$\left\|(C\,E)_n^q(x) - f(x)\right\|_p = O\left((n+1)^{\beta+1/p}\,\xi(1/n+1)\right) \quad \forall n > 0, \tag{3.1}$$

provided $\xi(t)$ is positive increasing function of t satisfying the following conditions

$$\left\{\int_0^{\pi/n+1}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t\,dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{3.2}$$

$$\left\{\int_{\pi/n+1}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n+1)^{\delta} \tag{3.3}$$

and

(3.4) $\dfrac{\xi(t)}{t}$ *is non-increasing in t,*

where δ *is an arbitrary number such that* $q(1-\delta+\beta)-1>0$, *q the conjugate index of p,* $p^{-1}+q^{-1}=1$, *conditions* (3.2), (3.3) *hold uniformly in x, and* $(C\,E)_n^q$ *are* (C, 1) (E, q) *means of Fourier series (1.1).*

Note (i): Condition (3.4) implies

$$\xi\left(\pi/(n+1)\right)\leq \pi\ \xi\left(1/(n+1)\right),\ \text{for}\ \left(\pi/(n+1)\right)\geq\left(1/(n+1)\right).$$

Note (ii): The product transform (C, 1) (E, q) plays an important role in signal theory as a double digital filter[12].

In order to prove our Theorem 3.1, we require the following lemmas.

Lemma 3.1. *For* $0<t<\pi/(n+1)$, *we have* $M_n(t)=O(n+1)$.

Lemma 3.2. *For* $\pi/(n+1)<t<\pi$, *we have* $M_n(t)=O(1/t)$.

Proof of Lemma 3.1. Using $\sin nt\leq n\sin t$ for $0<t<\pi/(n+1)$, then

$$\begin{aligned}
M_n(t)&=\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\frac{(2r+1)\sin(t/2)}{\sin(t/2)}\right]\\
&\leq\frac{1}{(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}(2k+1)\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\right]\\
&=\frac{1}{(n+1)}\sum_{k=0}^{n}(2k+1) \qquad \left(\because\sum_{r=0}^{k}\binom{k}{r}q^{k-r}=(1+q)^k\right),\\
&=O(n+1).
\end{aligned}$$

This completes the proof of Lemma 3.1.

Proof of Lemma 3.2. Using $\sin(t/2)\geq\left(t/\pi\right)$ and $\sin kt\leq 1$ for $\pi/(n+1)<t<\pi$, we obtain

$$\begin{aligned}
M_n(t)&=\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\frac{1}{(t/\pi)}\right]\\
&=O\left(1/t\right). \qquad \left(\because\sum_{r=0}^{k}\binom{k}{r}q^{k-r}=(1+q)^k\right).
\end{aligned}$$

This completes the proof of Lemma 3.2.

Proof of Theorem 3.1. It is well known from Titchmarsh[13] that the n[th] partial sum s_n of Fourier series (1.1) at $t=x$ may be written as

$$s_n(f,x)=f(x)+\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+1/2)t}{\sin(t/2)}\,dt,$$

so that (E, q) means (transform) of $s_n(f,x)$ are given by

$$\begin{aligned}E_n^q(x)&=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k\\&=f(x)+\frac{1}{2\pi(q+1)^n}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+1/2)t\right\}dt.\end{aligned}$$

Now, the (C, 1) (E, q) transform of $s_n(f,x)$ is given by

$$\begin{aligned}(CE)_n^q&=\frac{1}{(n+1)}\sum_{k=0}^{n}E_k^q\\&=f(x)+\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\sin(r+1/2)t\right\}dt\right]\\&=f(x)+\int_0^{\pi}\phi(t)M_n(t)\,dt,\end{aligned}$$

where

$$M_n(t)=\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{r=0}^{k}\binom{k}{r}q^{k-r}\frac{\sin(r+1/2)t}{\sin(t/2)}\right].$$

So

$$\begin{aligned}(3.5)\qquad (CE)_n^q(x)-f(x)&=\int_0^{\pi}\phi(t)M_n(t)\,dt\\&=\left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\phi(t)M_n(t)\,dt\\&=I_1+I_2.\end{aligned}$$

Using Hölder's inequality, condition (3.2), note1, Lemma 3.1, the fact that $(\sin t)^{-1}\le\frac{\pi}{2t}$, *for* $0<t\le\pi/2$, $p^{-1}+q^{-1}=1$ and the second mean value theorem for integrals, we find

$$(3.6)\quad |I_1| \le \left[\int_0^{\pi/(n+1)} \left(\frac{t\,|\phi(t)|}{\xi(t)}\sin^\beta t\right)^p dt\right]^{1/p} \left[\int_0^{\pi/(n+1)} \left\{\frac{\xi(t)}{t\sin^\beta t} M_n(t)\right\}^q dt\right]^{1/q}$$

$$= O\left(\frac{1}{(n+1)}\right)\left[\int_0^{\pi/(n+1)} O\left\{\frac{\xi(t)(n+1)}{t\sin^\beta t}\right\}^q dt\right]^{1/q}$$

$$= O\left[\left(\frac{\pi/(n+1)}{\sin\pi/(n+1)}\right)^{\beta q} \int_h^{\pi/(n+1)} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q} ; \quad h \to 0$$

$$= O\left[\int_h^{\pi/(n+1)} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q} ; \quad h \to 0$$

$$= O\left[\xi\left(\frac{\pi}{n+1}\right)\left(\int_h^{\pi/(n+1)} t^{-(1+\beta)q} dt\right)^{1/q}\right] ; \quad h \to 0$$

$$= O\left[\xi\left(\frac{1}{n+1}\right)\left(\int_h^{\pi/(n+1)} t^{-(1+\beta)q} dt\right)^{1/q}\right] ; \quad h \to 0$$

$$= O\left[(n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\right].$$

Now by Hölder's inequality, conditions (3.3), Lemma 3.2, the fact that $(\sin t)^{-1} \le \frac{\pi}{2t}$, *for* $0 < t \le \pi/2$, $p^{-1} + q^{-1} = 1$, we obtain

$$I_2 = \int_{\pi/(n+1)}^{\pi} \phi(t) M_n(t)\, dt$$

$$(3.7)\quad |I_2| \le \left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}\,|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{1/p} \left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t) M_n(t)}{t^{-\delta}\sin^\beta t}\right)^q dt\right\}^{1/q}$$

$$=O\left((n+1)^{\delta}\right)\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{\beta-\delta+1}}\right)^{q}dt\right\}^{1/q}$$

$$=O\left((n+1)^{\delta+1}\,\xi\left(\frac{\pi}{n+1}\right)\right)\left\{\int_{\pi/(n+1)}^{\pi}\frac{dt}{t^{(\beta-\delta)q}}\right\}^{1/q}$$

$$=O\left((n+1)^{\delta+1}\,\xi\left(\frac{1}{n+1}\right)\right)O\left((n+1)^{\beta-\delta-1/q}\right)$$

$$=O\left((n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\right).$$

Combining I_1 and I_2 yields

$$\left|(C\,E)_n^q(x)-f(x)\right|=O\left((n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\right). \tag{3.8}$$

Now, using the L_p-norm, we get

$$\left\|(C\,E)_n^q(x)-f(x)\right\|_{\mathrm{p}}=\left\{\int_0^{2\pi}\left|(C\,E)_n^q(x)-f(x)\right|^{\mathrm{p}}\mathrm{dx}\right\}^{1/\mathrm{p}}$$

$$=O\left\{\int_0^{2\pi}\left((n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\right)^{p}\mathrm{dx}\right\}^{1/\mathrm{p}}$$

$$=O\left\{(n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}\mathrm{dx}\right)^{1/\mathrm{p}}\right\}$$

$$=O\left((n+1)^{\beta+1/p}\,\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of our Theorem 3.1.

4. Applications

The following corollaries can be derived from our main Theorem 3.1.

Corollary 4.1. *If* $\beta=0$ *and* $\xi(t)=t^{\alpha}$, $0<\alpha\le 1$, *then the weighted* $W\left(L_p,\,\xi(t)\right),(p\ge 1)$*-class reduces to* $Lip(\alpha,p)$*-class and the degree of approximation of a function* $f(x)\in Lip(\alpha,p)$ *is given by*

(4.1) $$\left\| (C\,E)_n^q(x) - f(x) \right\|_p = O\left(\frac{1}{(n+1)^{\alpha - 1/p}} \right).$$

Proof. From our Theorem 3.1 for $\beta = 0$, we have

$$\left\| (C\,E)_n^q(x) - f(x) \right\|_p = \left(\int_0^{2\pi} \left| (C\,E)_n^q(x) - f(x) \right|^p dx \right)^{1/p}$$
$$= O\left((n+1)^{1/p}\, \xi\, (1/(n+1)) \right)$$
$$= O\left(\frac{1}{(n+1)^{\alpha - 1/p}} \right), \ \ p \geq 1.$$

This completes the proof of corollary 6.1.

Corollary 4.2. *If* $\xi(t) = t^{\alpha}$, $0 < \alpha < 1,$ *and* $p = \infty$ *in corollary 4.1, then* $f(x) \in Lip\,\alpha$ *and*

(4.2) $$\left\| (C\,E)_n^q(x) - f(x) \right\|_\infty = O(1/(n+1)^{\alpha}).$$

Proof. For $p = \infty$ in (4.1) we obtain

$$\left\| (C\,E)_n^q(x) - f(x) \right\|_\infty = \sup_{0 \leq x \leq 2\pi} \left| (C\,E)_n^q(x) - f(x) \right| = O\left(1/(n+1)^{\alpha} \right).$$

This completes the proof of corollary 4.2, which is Theorem 2.1.

Corollary 4.3. *If* $f : R \to R$ *is a* 2π *-periodic, Lebesgue integrable and belonging to weighted* $W(L_p, \xi(t))\,(p \geq 1)$ *-class, then the degree of approximation of* $f(x)$ *by* $(C,1)(E,1)$ *means of its Fourier series is given by*

$$\left\| (C\,E)_n^1 - f(x) \right\|_p = O\left((n+1)^{\beta + 1/p}\, \xi(1/n+1) \right) \quad \forall n > 0,$$

provided $\xi(t)$ *is positive increasing function of t satisfying the conditions (3.2), (3.3) uniformly in x, (3.4) and* $(C\,E)_n^1$ *are (C, 1) (E, 1) means of Fourier series (1.1).*

Proof. An independent proof of the corollary can be derived by taking $q = 1$ along the same lines as in our Theorem 3.1.

Note. If we put $\beta = 0$ in our corollary 4.3 then $f(x) \in Lip(\xi(t), p)$ and hence a theorem of Lal and Singh[14] becomes particular case of our Theorem 3.1.

5. Remarks

Example: Consider the infinite series

$$1 - 4\sum_{n=1}^{\infty} (-3)^{n-1}. \tag{5.1}$$

The nth partial sum of (7.1) is given by

$$s_n = 1 - 4\sum_{k=1}^{n} (-3)^{k-1} = (-3)^n,$$

and so

$$E_n^1 = 2^{-n}\sum_{k=0}^{n} \binom{n}{k} s_k = 2^{-n}\sum_{k=0}^{n} \binom{n}{k} (-3)^k = (-1)^n.$$

Therefore the series (4.1) is not (E, 1) summable. Also the series (5.1) is not (C, 1) summable. But since $\{(-1)^n\}$ is (C, 1) summable, the series (5.1) is (C, 1) (E,1) summable. Therefore the product summability (C, 1) (E, 1) is more powerful than the individual methods (C, 1) and (E, 1). Consequently (C, 1) (E, 1) mean gives better approximation than individual methods (C, 1) and (E, 1).

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