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On a Smooth Riemannian Space with Generalized 2–Recurrent Curvature Tensors*

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Abstract: M. C. Chaki and A. K. Roy introduced the notions of generalized 2-recurrent Riemannian manifold Later De and Konar, De and Pathak made further studied in this direction. In this paper, we have defined a type of generalized 2-recurrent Riemannian manifold and investigated some results. We have also studies the nature of the recurrence parameter on generalized 2-recurrent Riemannian manifold. We have defined some interesting results on generalized conformally 2-recurrent, generalized conharmonically 2-recurrent and generalized projectively 2-recurrent Riemannian manifold.

Keywords: 2-recurrent smooth Riemannian manifold, conformally 2-recurrent Riemannian manifold, concircularly 2-recurrent Riemannian manifold.

1. Introduction

Recurrent manifolds have been of great interest and were studied by Walker[1], Prakasn[2] and many other. De and Guha[3] introduced and studied a type of non - flat smooth Riemannian manifold whose curvature tensor R(X,Y,Z) satisfies condition **:**

(1.1) $$(D_U R)(X,Y,Z) = A(U)\, R(X,Y,Z) + B(U)[<Y,Z>X - <X,Z>Y],$$

where A, B are 1- forms, B is non- zero and D denotes the operator of covariant differentiation with respect to the metric tensor $<,>$. Such a Riemannian manifold has been called generalized recurrent Riemannian manifold. Here B is called its associated 1-form. If the 1-form B is zero in equation (1.1), then smooth Riemannian manifold reduces to recurrent smooth Riemannian manifold according to Walker[1]
Contracting equation (1.1) with respect to X, we get

(1.2) $$(D_U Ric)(Y,Z) = A(U) Ric\,(Y,Z) + (n-1)\, B(U) <Y,Z>.$$

In this case; the smooth Riemannian manifold M_n is called generalized

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Ricci recurrent smooth Riemannian manifold.

In 1972 A. K. Roy[4] generalized the notion of 2- recurrent smooth Riemannian manifold. A non- flat smooth Riemannian manifold (M_n, <, >) is called generalized 2 - recurrent if the Riemannian tensor curvature satisfies the condition

(1.3) $$(D_V D_U R)(X,Y,Z) = A(V)(D_U R)(X,Y,Z) + B(U,V)R(X,Y,Z),$$

where A is non- zero 1-form and B is non- zero (0,2) tensor.
A smooth Riemannain manifold is called generalized Ricci 2-recurrent smooth Riemannian manifold, if Ricci tensor is non- zero and satisfy the condition,

(1.4) $$(D_V D_U Ric)(X,Y) = A(V)(D_U Ric)(X,Y) + B(U,V)Ric(X,Y).$$

Such a smooth Riemannian manifold is denoted by $G({}^2R_n)$. Let A and B be 1-forms and 'U', 'V' be vector field on smooth Riemannian manifold M_n. Then the 1-form A is closed if[5]

(1.5) $$(D_V A)(U) - (D_U A)(V) = 0,$$

and the 1-form A is collinear with the 1-form B and 1-form F if[6]

(1.6) $$A(V)B(U) - A(U)B(V) = 0,$$

(1.7) $$A(V)F(U) - F(V)A(U) = 0.$$

The Weyl conformal curvature tensor C is defined by Singh & Khan[7].

(1.8) $$C(X,Y,Z) = R(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y + \langle Y,Z\rangle K(X) - \langle X,Z\rangle K(Y)] + \frac{r}{(n-1)(n-2)}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}.$$

where R is the curvature tensor of type (1,3), Ric is the Ricci tensor of type (0,2), *r* is the scalar curvature and K is the Ricci tensor of type (1,1) defined by

(1.9) $$Ric(X,Y) = \langle K(X), Y\rangle,$$

An n-dimension Riemannian manifold M_n is called Einstein manifold if for all $X,Y \in \mathcal{X}(M_n)$

(1.10) $$Ric(X,Y) = K\langle X,Y\rangle,$$

where $k : M_n \rightarrow R$ is a real valed function. From equation (1.10), we get

(1.11) $K(X) = k(X)$,

which on contraction, gives

(1.12) $r = rk$,

where k is constant.

The conharmonic curvature tensor N (X, Y, Z), projectively curvature tensor P(X, Y, Z) and concircular curvature tensor W(X, Y, Z) are given by[2]

(1.13) $$N(X,Y,Z) = R(X,Y,Z) - \frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y + <Y,Z> K(X) - <X,Z> K(Y)\},$$

(1.14) $$P(X,Y,Z) = R(X,Y,Z) - \frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y\},$$

(1.15) $$W(X,Y,Z) = R(X,Y,Z) - \frac{r}{(n-2)}\{<(Y,Z> X - <X,Z> Y\}.$$

In section second and third of this paper, we have studied generalized 2-recurrent smooth Riemannian manifold and obtained a few results on it.

2. A Type of Generalized 2-Recurrent Smooth Riemannian Manifold

On using equation (1.1) in equation (1.3), we get

(2.1) $$(D_V D_U R)(X,Y,Z) = \{A(V)B(U) + E(U,V)\}R(X,Y,Z) + F(U)A(V)\{<Y,Z> X - <X,Z> Y\},$$

where A, B, F are 1-forms ; E is non- zero 2- forms and $D_V D_U$ denotes the operator of bi - covariant differentiation w. r. to 'U' and 'V'.

Definition 2.1: *A smooth Riemannian manifold is called generalized 2-recurrent smooth Riemannain manifold, if it satisfies equation* (2.1) *and is denoted by* $G(^2R_n)$.

On contracting equation (2.1) with respect to 'X', we get

(2.1) $$(D_V D_U Ric)(Y,Z) = \{A(V)B(U) + E(U,V)\}Ric(Y,Z) + F(U)A(V)(n-1) <Y,Z>.$$

Definition 2.2: *A smooth Riemannian manifold* Mn *is called generalized Ricci* 2*-recurrent smooth Riemannain manifold if it satisfy the equation* (2.2).

Here F is called associated 1-form. If the 1-form F is zero in equation (2.1), then the manifold reduces to a 2- recurrent manifold[1]. Here, E has the following property:

(2.3) $$E(K(U),V) = r < U, V >,$$

where *r* is the scalar curvature .
Again, contracting equation (2.2), with respect to ´Z` , we get

(2.4) $$(D_V D_U K)(Y) = \{A(V)B(U) + E(U,V)\}K(Y) + (n-1)Y\ F(U)A(V).$$

If the 1-form F(U) becomes zero in equation (2.2), then the generalized Ricci bi- recurrent Riemannian manifold reduces to a Ricci 2- recurrent Riemannian manifold. In this paper, we have considered a non- flat *n*-dimensional smooth Riemannian manifold in which the conformal curvature tensor C satisfies the condition:

(2.5) $$(D_V D_U C)\ (X,Y,Z) = \{\ A(V)\ B(U)\ +\ E(U,V)\}\ C(X,Y,Z) + A(V)\ F(U)\ \{\ < Y,Z > X\ -\ < X,Z > Y\},$$

where A, B, F and E are stated earlier , F is non-zero.
If the conharmonic curvature tensor (N), Projective curvature tensor (P) and concircular curvature tensor (W) satisfy the condition:

(2.6) $$(D_V D_U Q)(X,Y,Z) = \{A(V)B(U) + E(U,V)\}\ Q(X,Y,Z) + A(V)F(U)\{\ < Y,Z > X\ -\ < X,Z > Y\},$$

where Q stands for N, P, W and A, B, F are 1-forms, then smooth Riemannian manifold is known as generalized conhormonicaly 2- recurrent smooth Riemannian manifold, generalized protectively 2-recurrent smooth Riemannian manifold and a generalized concircularly 2- recurrent smooth Riemannian manifold, respectively.

3. Nature of the 1-Forms A, B and F on A Generalized 2- Recurrent Riemannian Manifold

Taking bi- covariant derivative of equation (1.9) with respect to ´U` and ´V`, we have

(3.1) $$< (D_U D_V K)(X),\ Y > \ = \ (D_U D_V Ric)\ (X,Y).$$

On using equation (2.2) in equation (3.1), we have

$$< (D_U D_V K)(X) , Y >= \{ A(V)B(U) + E(U,V) \} Ric (X,Y) + (n-1) < X,Y > F(U)A(V),$$

which yields

$$(3.2) \quad (D_U D_V K)(X) = \{A(V)B(U) + E(U,V)\}K(X) + (n-1)\, X\, F(U)A(V) .$$

Contracting equation (3.2) with respect to ´X`, we get

$$(3.3) \quad (D_V U)r + UVr = \{A(V)B(U) + E(U,V)\}\, r + n(n-1)\, F(U)A(V),$$

where r is the scalar curvature .
First, we consider the case when the scalar curvature r is a constant and is different from zero. Then from equation (3.3), we have

$$(3.4) \quad \{A(V)B(U) + E(U,V)\}r + n(n-1)F(U)A(V) - (D_V U)r - UVr = 0.$$

Taking covariant derivative of equation (3.4) with respect to 'W' , we get

$$\begin{aligned}(3.5) \quad & \{(D_W A)(V) - (D_V A)(W)\}B(U) + A(V)(D_W B)(U) \\ & - A(W)(D_V B)(U) + \{(D_W E)(U,V) - (D_V E)(U,W)\}r \\ & + n(n-1)\{(D_W F)(U)A(V) - (D_V F)(U)A(W)\} + F(U)\{(D_W A)(V) \\ & - (D_V A)(W)\} - (D_W D_V U)r - (D_W U)Vr - U(D_W V)r - UVWr = 0.\end{aligned}$$

Thus, we have

Theorem 3.1: *In a generalized* 2- *recurrent smooth Riemannian manifold of non- zero constant scalar curvature r, the* 1- *form* A *and* 2- *forms are closed if and only if* A *is collinear with* B *and* F.

Next, we consider the case when the scalar curvature r is not constant. Taking covariant derivation of equation (3.3), with respect to 'W' it follows that

$$\begin{aligned}(3.6) \quad (D_W D_V U)r + (D_V U)Wr + U(D_W V)r + UVWr & \\ = \{(D_W A) (V)\, B(U) + A(V)\, (D_W E)\, (U,V)\}r & \\ + \{A(V)B(U) + E(U,V)\}Wr + n(n-1)\, \{(D_W F)(U)A(V) & \\ + F(U)(D_W A)(V)\}. &\end{aligned}$$

Inter changing W and V in Equation (3.6) and then subtracting, we get

$$\begin{aligned}(3.7) \quad & (D_W D_V U - D_V D_W U)r + \{(D_V A)(W) - (D_W A)(V) \}B(U) \\ & + \{A(W)(D_V B)(U) + A(V)(D_W B)(U)r \\ & + \{ (D_V E)\, (U,W) - (D_W E)(U,V)\}r \\ & + \{A(W)\, V - A(V)W\}r + n(n-1)\, [\{(D_V F)(U)A(W) \\ & - (D_W F)(U)A(V)\} + F(U)\, \{(D_V A)(W) - (D_W A)(U)\}] = 0.\end{aligned}$$

$$-(D_W F)(U)A(V)\} + F(U)\{(D_V A)(W) - (D_W A)(U)\}] = 0.$$

Thus, we state the following theorem:

Theorem 3.2: *In a generalized* 2*-recurrent space of non– zero constant scalar curvature r, the 1- forms* A, B *and* 2*– forms* E *cannot be closed unless the* 1*- form A is collinear with the* 1*- forms B and* F.

4. Generalized conformally 2- recurrent, conharmonicaly 2 recurrent and projectively 2-recurrent smooth Riemannian manifold

Let M_n be generalized 2-recurrent smooth Riemannian manifold of n-dimension .Taking bi- covariant derivative of equation (1.8) with respect to ´U ` and ´V`, we get

$$(4.1)\quad (D_U D_V C)(X,Y,Z) = (D_U D_V R)(X,Y,Z) - \frac{1}{(n-2)}[(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Y)Z + \langle Y,Z\rangle (D_V D_U K)(X) - \langle X,Z\rangle (D_V D_U K)(Y)] - \frac{UVr}{(n-1)(n-2)}(\langle Y,Z\rangle X - \langle X,Z\rangle Y].$$

On

Using equations (2.2), (2.3) and (2.4) in equation (4.1), we get

$$(4.2)\quad (D_V D_U C)(X,Y,Z) = \{A(V)\, B(U) + E(U,V)\}R(X,Y,Z) + A(V)F(U) \{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} - \frac{1}{(n-2)}[\{A(V)B(U) + E(U,V)\}Ric\,(Y,Z)X + (n-1)F(U)A(V)\langle Z,X\rangle Y - \{A(V)B(U) + E(U,V)\}Ric(X,Z)Y - (n-1)F(U)A(V)\langle X,Z\rangle Y + \langle Y,Z\rangle (\{A(V)B(U) + E(U,V)\}K(X) + (n-1)A(V)F(U)X) - \langle X,Z\rangle (\{A(V)B(U) + E(U,V)\}K(Y) + (n-1)A(V)F(U)Y)] + \frac{1}{(n-1)(n-2)}[\{A(V)B(U) + E(U,V)\}r + n(n-1) \{A(V)F(U)r + (DvU)r\}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}].$$

On using equation (2.4) and (2.5) in equation (4.2) , we have

$$(D_V D_U C)(X,Y,Z) = \{A(V)B(U) + E(U,V)\}C(X,Y,Z),$$

which shows the condition of conformally 2- recurrent smooth Riemannian manifold.

Thus, we get the following theorem.

Theorem 4.1: *A generalized* 2- *recurrent smooth Riemannian manifold is conformally* 2- *recurrent for the same recurrence parameter.*

From equations (2.1), (2.6) and (1.13) in generalized 2-recurrent smooth Riemannian manifold, it follows that

$$(4.3)\quad (D_V D_U R)(X,Y,Z) - \{A(V)B(U)+E(U,V)\}R(X,Y,Z) = \frac{1}{(n-2)}[(D_V D_U Ric)(Y,Z)X$$
$$- (D_V D_U Ric)(X,Z)Y + <Y,Z>(D_V D_U K)(X) - <X,Z>(D_V D_U K)(Y)$$
$$- \{A(V)B(U)+E(U,V)\}\{Ric(Y,Z)X - Ric(X,Z)Y + <Y,Z>K(X)$$
$$- <X,Z>K(Y)\}],$$

Permuting equation (4.3) with respect to U, X, Y; adding the three equations and using Bianchi's second identity in modified form , we get

$$(4.4)\quad \{A(V)B(U) + E(U,V)\}R(X,Y,Z) + \{A(V)B(X) +$$
$$E(X,V)\}R(Y,U,Z) + \{A(V)B(Y)+E(Y,V)\}R(U,X,Z)$$
$$+\frac{1}{(n-2)}[(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Z)Y$$
$$+ <Y,Z>(D_V D_U K)(X) - <X,Z>(D_V D_U K)(Y) + (D_V D_X Ric)(U,Z)Y$$
$$- (D_V D_X Ric)(Y,Z)U + <U,Z>(D_V D_X K)(Y) - <Y,Z>(D_V D_X K)(U)$$
$$+ (D_V D_Y Ric)(X,Z)U - (D_V D_Y Ric)(U,Z)X + <X,Z>(D_V D_Y K)(U)$$
$$- <U,Z>(D_V D_Y K)(X) + \{A(V)B(U) + E(U,V)\}\{Ric(Y,Z)X$$
$$- Ric(X,Z)Y + <Y,Z>K(X) - <X,Z>K(Y)\} + \{A(V)B(X)$$
$$+ E(X,V)\}\{Ric(U,Z)Y - Ric(Y,Z)U + <U,Z>K(Y) - <Y,Z>K(U)\}$$
$$- \{A(V)B(Y) + E(Y,V)\}\{Ric(X,Z)U - Ric(U,Z)X + <X,Z>K(U)$$
$$- <U,Z>K(X)\}] = 0 .$$

Contracting equation (4.4), with respect to '*X*', we get

$$\{A(V)B(U) + E(U,V)\}Ric(Y,Z) - \{A(V)B(Y) + E(Y,V)\}Ric(U,Z)$$
$$+ R(Y,V,U,Z,\rho) + \frac{1}{(n-2)}[(n-1)(D_V D_U Ric)(Y,Z) + <Y,Z>D_V(Ur)$$
$$- <(D_V D_U K)(Y),Z> + (1-n)(D_V D_Y Ric)(U,Z) + <(D_V D_Y K(U),Z)>$$
$$<U,Z>D_V(Yr) + (D_V D_Y Ric)(U,Z) - (D_V D_U Ric)(Y,Z)$$
$$+\frac{1}{2}<U,Z>D_V(Yr) - <Y,Z>D_V(Ur)$$

$$- (n-1)\{A(V)B(U) + E(U,V)\}\{Ric\ (Y,Z) - r < Y,Z > + < K(Y),Z >\}$$
$$- \{A(V)B(Y) + E(Y,V)\}\ Ric\ (U,Z) - \{A(V)B(U) + E(U,V)\}Ric\ (Y,Z)$$
$$- \{A(V)B(K(Y)) + E(K(Y),V)\} < U,Z > + \{A(V)B(K(U))$$
$$+ E(K(U),V)\} < Y,Z > - \{A(V)\ B(Y) + E(Y,U)\}Ric\ (U,Z)$$
$$+ n\{A(V)B(Y) + E(Y,V)\}\ Ric\ (U,Z) - \{A(V)B(Y) + E(Y,V)\} < K(U),Z >$$
$$+ \{A(V)B(Y) + E(Y,V)\} < U,Z > r] = 0\ ,$$

or

$$\begin{aligned}(4.5)\qquad & (n-2)\ R(Y,V,U,Z,\ \rho) + (n-2)\ (D_V D_U Ric)(Y,Z) + \frac{1}{2} < Y,Z > (D_V (Ur)) \\ & - < (D_V D_U K)(Y),Z > + (2-n)\ (D_V D_U Ric)(U,Z) + < (D_V D_Y K)(U),Z > \\ & - \frac{1}{2} < U,Z > D_V (Yr) + \{A(V)B(U) + E(U,V)\ \}\{< K(Y),Z >\} - < Y,Z > r\} \\ & - \{A(V)B(K(Y)) + E(K(Y),V)\} < U,Z > + \{A(V)B(K(U)) \\ & + E(K(U),V)\} < Y,Z > + \{A(V)B(Y) + E(Y,V)\} < K(U),Z > r = 0,\end{aligned}$$

where ρ is a vector field defined by $<(X,Y),\rho> = E(X,Y)$.
Factorizing 'Z' of equation (4.5), we get

$$\begin{aligned}(4.6)\qquad & (n-2)R(Y,V,U,\ ) + (n-3)(D_V D_U K)(Y) + (3-n)(D_V D_Y K)(U) \\ & + \frac{1}{2} Y(D_V (Ur)) - \frac{1}{2}\ U\ D_V (Yr) + \{A(V)B(U) + E(U,V\}(-Yr + K(Y)) \\ & - U\{A(V)B(K(Y)) + E(K(Y),V) + \{A(V)B(K(U)) + E(K(U),V)\}Y \\ & - \{A(V)B(Y) + E(Y,V)K(V) + \{A(V)B(Y) + E(Y,V)\}Ur = 0.\end{aligned}$$

On using equation (2.3) in equation (4.6), we get

$$\begin{aligned}(4.7)\qquad & (n-2)R(Y,V,U,\rho\ ) + (n-3)(D_V D_U K)(Y) + (3-n)(D_V D_Y K)(U) \\ & + \frac{1}{2}\ r\{Y(D_V U) - U(D_V Y)\} + \{A(U)B(V) + E(U,V)\}K(Y) \\ & - \{A(Y)B(V) + E(Y,V)\}K(U) = 0\ .\end{aligned}$$

Contracting equation (4.7) with respect to '*Y*', we get

$$(n-2)Ric(U,V) + (n-3)D_V (Ur) + \frac{(3-n)}{2}\ D_V (Ur) + \frac{r}{2}\{n(D_V U) - V(Ur)\} = 0,$$

or

$$(4.8)\qquad (2n-4)Ric(U,V)\ \rho + (2n-3)(D_V U)r + UVr\ (n-r-3) = 0.$$

On using equation (1.10) in equation (4.8), we get

$$(4.9)\qquad (2n-3)(D_V U)r + UVr\ (n-r-3) = 0.$$

Thus, we have the following.

Theorem 4.2: *The necessary and sufficient condition for the scalar curvature r of generalized conformally 2-recurrent n- dimensional (n>3) smooth Riemannian manifold to be constant is that*

$$(2n-3)(D_V U) + UV(n-r-3) = 0.$$

Taking bi- covariant derivative of equation (1.13) with respect to 'U' and 'V', we get

$$\text{(4.10)} \quad (D_V D_U N)(X,Y,Z) = (D_U D_V R)(X,Y,Z) - \frac{1}{(n-2)}[(D_U D_V Ric)(Y,Z)X$$

$$- (D_V D_U Ric)(X,Z)Y + \langle Y,Z\rangle (D_U D_V K)(X) - \langle X,Z\rangle (D_U D_V K)(Y)].$$

Let *Mn* be a generalized Ricci 2- recurrent smooth Riemannian manifold. Then from equations (2.3), (2.4) and (4.10) , we have

$$\begin{aligned}(D_V D_U N)(X, Y, Z) = (D_V D_U R)(X, Y, Z) - \frac{1}{(n-2)}[\{A(V)B(U)+E(U,V)\}Ric\ (Y,Z)X \\ +(n-1)X\ F(U)\ A(V) \langle Y,Z\rangle - \{A(V)B(U)+E(U,V)\}Ric\ (X,Z)Y \\ -(n-1)YF(U)\ A(V) \langle X,Z\rangle + \langle Y,Z\rangle \{A(V)B(U)+E(U,V)\}K(X) \\ +(n-1)\ X\ F(U)A(V) \langle Y,Z\rangle - \langle X,Z\rangle \{A(V)B(U) \\ + E(U,V)\}Ric\ K(Y) - (n-1)YF(U)A(U) \langle X,Z\rangle],\end{aligned}$$

or

$$\begin{aligned}\text{(4.11)} \quad (D_V D_U N)(X, Y, Z) = (D_V D_U R)(X, Y, Z) - \frac{1}{(n-2)}[\{A(V)B(U) \\ + E(U,V)\}Ric\ (Y,Z)X - Ric\ (X,Z)Y\} \\ +\{\langle Y,Z\rangle K(X) - \langle X,Z\rangle K\ (Y)\}A(V)B(U) \\ +2(n-1)\ F(U)A(V)\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\},\end{aligned}$$

From equation (4.11), it is evident that if any one of the equations (2.1), (2.2) hold, then (2.6) also holds. This leads us to the following.

Theorem 4.3: *A necessary and sufficient condition for an n-dimensional* (*n*>3) *generalized Ricci 2–recurrent manifold to be a generalized conharmonicaly* 2-*recurrent smooth Riemannian manifold is that equation* (4.11) *holds the same recurrence parameter.*

We now prove the following:

Theorem 4.4: *In an Einstein smooth Riemannian manifold* M_n *the conharmonic curvature tensor satisfies the following identity.*

$$(D_V D_U N)(X,Y,Z) + (D_V D_X N)(Y,U,Z) + (D_V D_Y N)(U,X,Z) = 0.$$

Proof: On using equations (1.10), (1.11) in equation (1.13), it follows that

$$(4.12) \qquad N(X,Y,Z) = R(X,Y,Z) - \frac{2k}{(n-2)}[<Y,Z>X - <X,Z>Y].$$

Taking covariant derivative of equation (4.12), with respect to 'U', we get

$$(4.13) \qquad (D_U N)(X,Y,Z) = (D_U R)(X,Y,Z).$$

Permuting equation (4.13), with respect to U, X, Y; adding the three equations using Bianchi second identity and after taking covariant derivative w. r. to 'v' we get, required result.
Now from equations (1.13)and (1.14) we have, the following relation in the *N(X, Y, Z)* and *P(X, Y, Z)*.

$$(4.14) \qquad N(X,Y,Z) = P(X,Y,Z) - \frac{1}{(n-2)}[\frac{1}{(n-1)}\{Ric(Y,Z)X - Ric(X,Z)Y\} + \{<Y,Z> K(X) - <X,Z> K(Y)\}].$$

Taking bi-covariant derivative with respect to 'U' and 'V', we get

$$(4.15) \qquad (D_V D_U N)(X,Y,Z) = (D_V D_U P)(X,Y,Z) - \{(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Z)Y\} + <Y,Z>(D_V D_U K)(X) - <X,Z>(D_V D_U K)(Y)].$$

Let M_n be generalized Ricci 2- recurrent smooth Riemannian manifold. Then from equations (2.2) ,(2.4) and (4.15), it follows that

$$\begin{aligned}(D_V D_U N)(X,Y,Z) = (D_V D_U P)(X,Y,Z) - \frac{1}{(n-2)}[\frac{1}{(n-2)}\{A(V)B(U) + E(U,V)\}Ric(Y,Z)X \\ + (n-1) <Y,Z> X\, F(U)A(V) - \{A(V)B(U) + E(U,V)\}Ric(X,Z)Y \\ -(n-1) <X,Z> Y\, F(U)A(V) + <Y,Z>\{A(V)\,B(U) + E(U,V)\}K(X) \\ + (n-1)X\, F(U)A(V) <Y,Z> - <X,Z>\{A(V)B(U) + E(U,V)\}K(Y) \\ - (n-1) <X,Z> Y\, F(U)A(V)],\end{aligned}$$

or

$$\begin{aligned}(D_V D_U N)(X,Y,Z) = (D_V D_U P)(X,Y,Z) - \frac{1}{(n-2)}[\frac{1}{(n-1)}\{A(V)B(U) + E(U,V)\}\{Ric(Y,Z)X \\ - Ric(X,Z)Y\} + (n-1)F(U)A(V)\{<Y,Z>X - <X,Z>Y\} \\ - \{A(V)B(U) + E(U,V)\}\{<Y,Z> K(X) - <X,Z> K(Y)\} \\ + (n-1)F(U)A(V)\{X <Y,Z> - Y <X,Z>\}],\end{aligned}$$

or

$$(4.16)\qquad (D_V D_U N)(X,Y,Z) = (D_V D_U P)(X,Y,Z) - \frac{1}{(n-2)}\Big[\frac{1}{(n-1)}\{A(V)B(U) + E(U,V\}\{Ric(Y,Z)X - Ric(X,Z)Y\} + \{A(V)B(U) + E(U,V)\}\{<Y,Z>K(X) - <X,Z>K(Y)\} + 2(n-1)F(U)A(V)\{<Y,Z>X - <X,Z>Y\}\Big].$$

With the help of equations (4.12), (4.16) and (2.6), we have the following

Theorem 4.5: *Let* M_n *be an* n- *dimensional (n>3) generalized Ricci 2- recurrent smooth Riemannian manifold. Then* M_n *is generalized conharmoniclly* 2-*recurrent manifold if and only if it is generalized projectively* 2- *recurrent manifold for the same recurrence parameter.*

The relations among curvature tensor (R), conformal curvature tensor (C), conharmonic curvature tensor (N) and concircular curvature tensor (W) are given by

$$(4.17)\qquad C(X,Y,Z) = N(X,Y,Z) + \frac{n}{(n-2)}[R(X,Y,Z) - W(X,Y,Z)].$$

Taking bi–covariant derivative of equation (4.17) with respect to 'U' and 'V', we get

$$(4.18)\qquad (D_V D_U C)(X,Y,Z) = (D_V D_U N)(X,Y,Z) + \frac{n}{(n-2)}[(D_V D_U R)(X,Y,Z) - (D_V D_U W)(X,Y,Z)].$$

Now, let M_n be generalized 2- recurrent smooth Riemannian manifold. Then from equations (4.13) and (4.18), we have

$$(4.19)\qquad (D_V D_U C)(X,Y,Z) = (D_V D_U N)(X,Y,Z) + \frac{n}{(n-2)}[\{A(V)B(U) + E(U,V)\}R(X,Y,Z) + F(U)A(V)(<Y,Z>X - <X,Z>Y) - (D_V D_U W)(X,Y,Z)].$$

Then, we have the following:

Theorem 4.6: *For an n- dimensional* (*n*>3) *generalized* 2- *recurrent smooth Riemannian manifold if any two of the following properties hold, and then the third also hold.*

(a) It is generalized conformally 2- recurrent manifold,

(b) It is generalized conharmonically 2- *recurrent manifold,*

(c) It is generalized concircularly 2- *recurrent manifold,*

for the same recurrence parameter.

Now, we deduce a necessary and sufficient condition for a generalized projectively 2- recurrent Riemannian manifold to a 2- recurrent Riemannian manifold in the following:

Theorem 4.7: *A necessary and sufficient condition for a generalized projectively* 2*– recurrent smooth Riemannian manifold to be generalized* 2*-recurrent smooth Riemannian manifold is that the Riemannian manifold is a Ricci* 2*- recurrent manifold.*

Proof: First, we assume that a G(P^2 (R_n)) is G(^2R_n). Then, from equations (2.6) and (1.8), we have

$$\begin{aligned}(D_V D_U R)\ (X,Y,Z) - \{(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)\ (X,Z)Y\} \\ = \{A(V)B(U) + E(U,V)\}P(X,Y,Z) + A(V)F(U)[<Y,Z>X - <X,Z>Y] \\ = \{A(V)B(U) + E(U,V)\}[R(X,Y,Z) - \frac{1}{(n-1)}\{Ric\ (Y,Z)X \\ - Ric(X,Z)Y\} + A(V)F(U)\{<Y,Z>X - <X,Z>Y\}],\end{aligned}$$

or

$$\begin{aligned}(4.20)\qquad (D_V D_U R)\ (X,Y,Z) - \{A(V)B(U) + E(U,V)\}R(X,Y,Z) \\ - A(V)\ F(U)\{<Y,Z>X - <X,Z>Y\} \\ = \frac{1}{(n-1)}[\{(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Z)Y\} \\ - \{A(V)B(U) + E(U,V)\}\{Ric\ (Y,Z)X - Ric(X,Z)Y\}].\end{aligned}$$

Consequently, using equation (2.2), we obtain

$$\begin{aligned}(4.21)\qquad \{(D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Z)Y\} \\ - \{A(V)B(U) + E(U,V)\}\{Ric(Y,Z)X - Ric(X,Z)Y\} = 0.\end{aligned}$$

Contracting with respect to 'X' of equation (4.21), we get

$$(4.22)\qquad (D_V D_U Ric)(Y,Z) - Ric\ (Y,Z)\{A(V)B(U) + E(U,V)\} = 0.$$

That is, the smooth Riemannian manifold is Ricci 2-recurrent smooth Riemannian manifold.

Conversely, we suppose that if the condition (4.22), holds then from equation (4.21), it follows that the smooth Riemannian manifold is generalized projectively 2- recurrent smooth Riemannian manifold. This completes the proof.

Theorem 4.8: *Every generalized projectively 2-recurrent smooth Riemannian manifold* ($n>3$) *is conformally 2-recurrent smooth Riemannian manifold.*

Proof: Consider a generalized projectively 2-recurrent smooth Riemannian manifold. Then putting $Y = Z = e_i$, where $\{e_i\}$, $i = 1,2,3,4,5\ldots\ldots n$ is an orthonormal basis of the tangent space at point of the manifold and taking summation over i, $1 \le i \le n$, we get

(4.23) $$D_V D_U r - \{A(V)B(U) + E(U,V)\}r = 0.$$

Also, from equation (4.22), we have

(4.24) $$(D_V D_U S)(Y) - \{A(V)\, B(U) + E(U,V)\}S(Y) = 0\,,$$

where S is the Ricci operator defined by $Ric(X,Y) = \langle SX,Y\rangle$.
On using equations (1.3) and (1.8), we obtain

(4.25) $$(D_V D_U C)(X,Y,Z) = (D_V D_U R)(X,Y,Z) - \frac{1}{(n-1)}[\langle Y,Z\rangle (D_V D_U Q)X - \langle X,Z\rangle (D_V D_U Q)Y + (D_V D_U Ric)(Y,Z)X - (D_V D_U Ric)(X,Z)Y] + \frac{D_U D_V r}{(n-1)(n-2)}[\langle Y,Z\rangle X - \langle X,Z\rangle Y].$$

With the help equations (1.8), (4.24) and (4.25); we get from the above equation

$$(D_V D_U C)(X,Y,Z) = \{A(V)B(U) + E(U,V)\}\, C(X,Y,Z).$$

That is, the smooth Riemannian manifold in conformally 2-recurrent.

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