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Hypersurface of a Special Finsler Space with Metric $\sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}$

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Abstract: In the present paper our study is confined to the hypersurface of a Finsler space with (α, β)-metric $\sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}$ .We have examined the hypersurface as a hyperplane of first, second or third kind.

Keywords: Finsler space, hypersurface, hyperplane and (α, β) -metric.

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1. Introduction

We consider an n-dimensional Finsler space $F^n = \left(M^n, L\right)$, i.e., a pair consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L. The concept of $(\alpha, \beta)-$metric $L(\alpha, \beta)$ was introduced first of all by M. Matsumoto[1] and has been studied by many authors[1-8]. A Finsler metric $L(x, y)$ is called and $(\alpha, \beta)-$ metric if L is positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M^n. Well known examples of $(\alpha, \beta)-$ metric are Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$, Matsumoto metric $\frac{\alpha^2}{(\alpha - \beta)}$ and generalized Kropina metric

$\frac{\alpha^{m+1}}{\beta^m}\left(m \neq 0,-1\right)$ whose studies have contributed a lot to the growth of Finsler geometry. Hypersurfaces of Finsler spaces with special metrics have also been studied by M. K. Gupta, P. N. Pandey and Vaishali Pandey[9-13].

2. Preliminaries

We forms on a special Finsler space $F^n = \left\{M^n, L\left(\alpha,\beta\right)\right\}$, where

$$L\left(\alpha,\beta\right) = \sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}. \tag{2.1}$$

Partial derivative of (2.1) w. r. t. α and β are given by

$$L_\alpha = \sum_{r=0}^{m} (1-n)\frac{\beta^r}{\alpha^r}, L_\beta = \sum_{r=0}^{m} r\frac{\beta^{r-1}}{\alpha^{r-1}},\ L_{\alpha\alpha} = \sum_{r=0}^{m} r\left(r-1\right)\frac{\beta^r}{\alpha^{r+1}},$$
$$L_{\beta\beta} = \sum_{r=0}^{m} r\left(r-1\right)\frac{\beta^{r-2}}{\alpha^{r-1}}, L_{\alpha\beta} = \sum_{n=0}^{m} r\left(n-1\right)\frac{\beta^{r-1}}{\alpha^r},$$

where $L_\alpha = \frac{\partial L}{\partial \alpha},\ L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}, L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta},\ L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the Finsler space $F^n = \left\{M^n, L\left(\alpha,\beta\right)\right\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor $h_{ij} = L^{-1}\dot{\partial}_i \dot{\partial}_j L$ are given by

$$\begin{cases} l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i \\ h_{ij} = \mu a_{ij} + \tau_0 b_i b_j + \tau_{-1}\left(b_i Y_j + b_j Y_i\right) + \tau_{-2} Y_i Y_j, \end{cases} \tag{2.1$'$}$$

where $Y_i = a_{ij} y^j$. For the fundamental function (2.1), the scalars are given by

$$\mu = L L_\alpha \alpha^{-1} = \sum_{r,s=0}^{m} \left(1-s\right)\frac{\beta^{r+s}}{\alpha^{r+s}},\ \tau_0 = L L_{\beta\beta} = \sum_{r,s=0}^{m} s\left(s-1\right)\frac{\beta^{r+s-2}}{\alpha^{r+s-2}} \tag{2.2}$$

$$\tau_{-1} = \sum_{r,s=0}^{m} s\left(s-1\right)\frac{\beta^{r+s-1}}{\alpha^{r+s}},\ \tau_{-2} = \sum_{r,s=0}^{m} \left(s^2-1\right)\frac{\beta^{r+s}}{\alpha^{r+s+2}}.$$

Fundamental metric tensor $g_{ij}=\frac{1}{2}\dot{\partial}_i\,\dot{\partial}_j\,L^2$ is given by

(2.3) $$g_{ij}=\mu a_{ij}+\mu_0 b_i\,b_j+\mu_{-1}\left(b_iY_j+b_jY_i\right)+\mu_{-2}Y_iY_j\,,$$

where

(2.4) $$\begin{cases}\mu_0=q_0+L_\beta^2=\sum\limits_{r,s=0}^{m}\left\{s^2+\left(r-1\right)r\right\}\dfrac{\beta^{r+s-2}}{\alpha^{r+s-2}},\\ \mu_{-1}=\tau_{-1}+L^{-1}\mu L_\beta=\sum\limits_{r,s,t=0}^{m}\left(s-1\right)\left(s-L^{-1}t\right)\dfrac{\beta^{r+s+t-1}}{\alpha^{r+s+t-1}},\\ \mu_{-2}=\tau_{-2}+\mu^2L^{-2}=\left[\sum\limits_{r,s=0}^{m}\dfrac{\left(s^2-1\right)}{\alpha^2}\dfrac{\beta^{r+s}}{\alpha^{r+s}}+L^{-2}\sum\limits_{r,s=0}^{m}\left(1-s\right)^2\left(\dfrac{\beta^{r+s}}{\alpha^{r+s}}\right)^2\right].\end{cases}$$

Moreover the reciprocal tensor g^{ij} of g_{ij} is given by[8],

(2.5) $$g^{ij}=\mu^{-1}a^{ij}-\sigma_0\,b^i\,b^j-\sigma_{-1}\left(b^iy^j+b^jy^i\right)-\sigma_{-2}\,y^i\,y^j\,,$$

where $b^i=a^{ij}\,b_j\,,b^2=a_{ij}\,b^i\,b^j$.

(2.6) $$\begin{cases}\sigma_0=\dfrac{1}{\theta\mu}\left\{\mu\mu_0+\left(\mu_0\,\mu_{-2}-\mu_{-1}^2\right)\alpha^2\right\},\\ \sigma_{-1}=\dfrac{1}{\theta\mu}\left\{\mu\mu_{-1}+\left(\mu_0\,\mu_{-2}-\mu_{-1}^2\right)\beta\right\},\\ \sigma_{-2}=\dfrac{1}{\theta\mu}\left\{\mu\mu_{-2}+\left(\mu_0\,\mu_{-2}-\mu_{-1}^2\right)b^2\right\},\\ \theta=\mu\left(\mu+\mu_0b^2+\mu_{-1}\beta\right)+\left(\mu_0\mu_{-2}-\mu_{-1}^2\right)\left(\alpha^2\,b^2-\beta^2\right).\end{cases}$$

The $hv-$torsion tensor $C_{ijk}=\frac{1}{2}\dot{\partial}_k\,g_{ij}$ is given by[8].

(2.7) $$2\mu C_{ijk}=\mu_{-1}\left(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j\right)+\rho m_i\,m_j\,m_k,$$

where

(2.8) $$\rho = \mu \frac{\partial \mu_0}{\partial \beta} - 3\mu_{-1}\tau_0\,,\; m_i = b_i - \alpha^{-2}\,\beta Y_i\,.$$

It is noted that the covariant vector m_i is a non-vanishing covariant vector orthogonal to the element of support y^i. We denote by ∇_k the covariant differentiation by x^k with respect to the associated Riemannian connection and $\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}$ are the Christoffel symbols of the associated Riemannian space R^n. We put $b_{ij} = \nabla_j b_i$ and

(2.9) $$2E_{ij} = b_{ij} + b_{ji}\,,\, 2F_{ij} = b_{ij} - b_{ji}\,,$$

where $b_{ij} = \nabla_j b_i$.

If we denote the Cartan's connection $C\Gamma$ as $\left(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk}\right)$ then the difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}$ of Matsumoto space is given by

(2.10) $$\begin{aligned} D^i_{jk} &= B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\ &\quad - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ &\quad + \lambda^s \left(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms} \right), \end{aligned}$$

where

(2.11) $$\begin{cases} B_k = \mu_0 b_k + \mu_{-1} Y_k\,,\; B^i = g^{ij} B_j\,,\; F_i^{\;k} = g^{kj} F_{ji}\,, \\ B_{ij} = \dfrac{\left\{ \mu_{-1}\left(a_{ij} - \alpha^{-2} Y_i Y_j\right) + \left(\partial\mu_0 / \partial\beta\right) m_i m_j \right\}}{2}, \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F_0^{\;m} + B_0 F_k^{\;m},\; B_i^k = g^{kj} B_{ji}\,, \\ \lambda^m = B^m E_{00} + 2B_0 F_0^{\;m}\, m,\; B_0 = B_i\, y^i\,, \end{cases}$$

And 0 denotes contraction with y^i except for the quantities μ_0, τ_0 and σ_0.

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i\left(u^\alpha\right)$ where $\alpha = 1,2,3,\ldots,(n-1)$. The $(n-1)$ tangent vectors to the hypersurface F^{n-1} are given by $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$. The element of support y^i of F^n is to be taken tangential[14] to F^{n-1}:

(3.1) $$y^i = B^i_\alpha(u)v^\alpha.$$

The metric tensor $g_{\alpha\beta}$ and $hv-$tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}\, B^i_\alpha\, B^j_\beta,\ C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha\, B^j_\beta\, B^k_\gamma$$

and at each point $\left(u^\alpha\right)$ of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}\left\{x(u,v), y(u,v)\right\} B^i_\alpha\, N^j = 0,\ g_{ij}\left\{x(u,v), y(u,v)\right\} N^i\, N^j = 1.$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface is given by

(3.2) $$h_{\alpha\beta} = h_{ij}\, B^i_\alpha\, B^j_\beta,\ h_{ij}\, B^i_\alpha\, N^j = 0,\ h_{ij}\, N^i\, N^j = 1.$$

The inverse of $\left(B^i_\alpha, N^i\right)$ is denoted by $\left(B^\alpha_i, N_i\right)$ and is given by

$$B^\alpha_i = g^{\alpha\beta} g_{ij}\, B^j_\beta,\ \ B^i_\alpha B^\beta_i = \delta^\beta_\alpha,\ \ B^\alpha_i N^i = 0,\ \ B^i_\alpha N_i = 0,$$
$$N_i = g_{ij} N^j,\ \ B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The induced connection $IC\Gamma = \left(\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma}\right)$ of F^{n-1} of the Cartan connection $C\Gamma = \left(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk}\right)$ is given by[14]:

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i}\left(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}\right) + M^{\alpha}_{\beta}\, H_{\gamma},$$

$$C^{\alpha}_{\beta} = B^{\alpha}_{i}\left(B^{i}_{0\beta} + \Gamma^{*i}_{0j} B^{j}_{\beta}\right), C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} C^{i}_{jk}\, B^{j}_{\beta}\, B^{k}_{\gamma},$$

where $M_{\beta\gamma} = N_i\, C^{i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}$, $M^{\alpha}_{\beta} = g^{\alpha\gamma} M_{\beta\gamma}$, $H_{\beta} = N_i\left(B^{i}_{0\beta} + \Gamma^{*i}_{0j} B^{j}_{\beta}\right)$ and $B_{\beta\gamma} = \dfrac{\partial B^{i}_{\beta}}{\partial u^{\gamma}}$, $B^{i}_{0\beta} = B^{i}_{\alpha\beta}\, v^{\alpha}$.

The quantities $M_{\beta\gamma}$ and H_{β} are called the second fundamental $v-$tensor and normal curvature vector respectively[14]. The second fundamental $h-$tensor $H_{\beta\gamma}$ is defined as[14]:

(3.3) $$H_{\beta\gamma} = N_i\left(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}\right) + M_{\beta} H_{\gamma},$$

where

(3.4) $$M_{\beta} = N_i\, C^{i}_{jk} B^{j}_{\beta} N^{k}.$$

The relative h and $v-$covariant derivatives of projection factor B^{i}_{α} with respect to $IC\Gamma$ are given by

(3.4)′ $$B^{i}_{\beta|\gamma} = H_{\alpha\beta}\, N^{i},\ B^{i}_{\alpha|\beta} = M_{\alpha\beta}\, N^{i}.$$

It is obvious from the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

(3.5) $$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta} H_{\gamma} - M_{\gamma} H_{\beta}.$$

The above equations yield:

(3.6) $$H_{0\gamma} = H_{\gamma},\ H_{\gamma 0} = H_{\gamma} + M_{\gamma} H_{0}.$$

We shall use following lemmas which are due to Matsumoto[14] in the coming section:

Lemma 3.1: *The normal curvature* $H_0 = H_{\beta} v^{\beta}$ *vanishes if and only if the normal curvature vector* H_{β} *vanishes.*

Lemma 3.2: *A hypersurface* F^{n-1} *is a hyperplane of the first kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$.

Lemma 3.3: *A hypersurface* F^{n-1} *is a hyperplane of the second kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$ *and* $Q_{\alpha\beta} = 0$, *where* $Q_{\alpha\beta} = C_{ijk|0} B^i_\alpha B^j_\beta N^k$ *and then* $H_{\alpha\beta} = 0$.

Lemma 3.4: *A hypersurface* F^{n-1} *is a hyperplane of the third kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$ *and* $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. Hypersurface $F^{n-1}(C)$ of the Special Finsler Space

Let us consider a Finsler spaces with the metric $L(\alpha,\beta) = \sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}$, where vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{n-1}(c)$ given by equation $b(x) = c$.

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = b_i B^i_\alpha = 0.$$

Above equation shows that $b_i(x)$ are covariant component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

(4.1) $$b_i y^i = 0, \quad \text{i.e.,} \ \beta = 0.$$

The induced metric $L(u,v)$ of $F^{n-1}(c)$ is given by

(4.2) $$L(u,v) = \sqrt{a_{\alpha\beta} v^\alpha v^\beta}, \ a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta.$$

Writing $\beta = 0$ in the equation (2.2), (2.3) and (2.5), we get for $F^{n-1}(c)$.

$$(4.3)\quad \begin{cases} \mu = 1, \mu_0 = 3, \mu_{-1} = \alpha^{-1}, \mu_{-2} = 0, \tau_0 = 2,\ \tau_{-1} = 0, \tau_{-2} = -\alpha^{-2}, \\ \theta = \left(1+2b^2\right), \sigma_0 = \dfrac{2}{\left(1+2b^2\right)}, \sigma_{-1} = \dfrac{1}{\alpha\left(1+2b^2\right)}, \sigma_{-2} = -\left\{\dfrac{b^2}{\alpha^2\left(1+2b^2\right)}\right\}. \end{cases}$$

From (2.4), (2.5) and (4.3) we get

$$(4.4)\qquad g^{ij} = a^{ij} - \frac{2}{\left(1+2b^2\right)} b^i b^j - \frac{1}{\left(1+2b^2\right)}\left(b^i y^j + b^j y^i\right) + \frac{b^2}{\left(1+2b^2\right)\alpha^2} y^i y^j.$$

Thus along $F^{n-1}(c)$, (4.4) and (4.1) lead to

$$g^{ij} b_i b_j = \frac{b^2}{\left(1+2b^2\right)},$$

so, we get

$$(4.5)\qquad b_i(x(u)) = \sqrt{\frac{b^2}{(1+2b^2)}} N_i,\ b^2 = a^{ij} b_i b_j,$$

where b is the length of the vector b^i. Again from (4.4) and (4.5), we get

$$(4.6)\qquad b^i = a^{ij} b_j = \sqrt{b^2\left(1+2b^2\right)} N^i + \frac{b^2}{\alpha} y^i.$$

Thus we have:

Theorem 4.1: *In a special Finsler hypersurface $F^{n-1}(c)$, the induced Riemannian metric is given by (4.2) and the vector field b_i is along the normal to $F^{n-1}(c)$.*

At the point of $F^{n-1}(c)$ the angular metric tensor h_{ij} and the metric tensor g_{ij} of F^n are given by

$$(4.7)\qquad h_{ij} = a_{ij} + 2b_i b_j - \alpha^{-2} Y_i Y_j \text{ and } g_{ij} = a_{ij} + 3b_i b_j + \alpha^{-1}\left(b_i Y_j + b_j Y_i\right),$$

which are obtained from (2.1)′, (2.3) and (4.3).

From (4.1), (4.7) and (3.2) it follows that if $h_{\alpha\beta}^{(a)}$ denotes the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}(c)$, $h_{\alpha\beta} = h_{\alpha\beta}^{(a)}$.

Thus along $F^{n-1}(c)$, $\frac{\partial \mu_0}{\partial \beta} = 0.$

From equation (2.6), we get $\rho = -\frac{6}{\alpha}$, $m_i = b_i$.

At the points of $F^{n-1}(c)$, $hv-$ torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}\left(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right) - \frac{3}{\alpha}b_i b_j b_k, \tag{4.8}$$

From relations (3.2), (3.3), (3.5), (4.1) and (4.8), we have

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{(1+2b^2)}}h_{\alpha\beta} \text{ and } M_\alpha = 0. \tag{4.9}$$

Therefore from equation (3.6) it follows that $H_{\alpha\beta}$ is symmetric. Thus, we have:

Theorem 4.2: *The second fundamental v-tensor of the special Finsler hypersurface $F^{n-1}(c)$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

From $b_i B_\alpha^i = 0.$ Then, we have

$$b_{i|\beta}B_\alpha^i + b_i B_{\alpha|\beta}^i = 0.$$

Therefore, from (3.4)′ and using $b_{i|\beta} = b_{i|j}B_\beta^j + b_{i|j}\, N^j\, H_\beta$ [14], we have

$$b_{i|j}B_\alpha^i B_\beta^j + b_{i|j}B_\alpha^i N^j H_\beta + b_i H_{\alpha\beta}N^i = 0. \tag{4.10}$$

Since, $b_i|_j = -b_h\, C_{ij}^h$, M_α and we get $b_{i|j}B_\alpha^i N^j = 0.$

Therefore from equation (4.10), we have

(4.11)
$$\sqrt{\frac{b^2}{(1+2b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0\,,$$

Because $b_{i|j}$ is symmetric. Now contracting (4.11) with v^β and using (3.1), (3.6), we get

(4.12)
$$\sqrt{\frac{b^2}{(1+2b^2)}}H_{\alpha} + b_{i|j}B^i_\alpha\, y^j = 0.$$

Again contracting equation (4.12) by v^α and using (3.1), we get

(4.13)
$$\sqrt{\frac{b^2}{(1+2b^2)}}H_0 + b_{i|j}y^i\, y^j = 0.$$

From Lemma (3.1) and (3.2), it is clear that the hypersurface $F^{n-1}(c)$ is a hyperplane of first kind if and only if $H_0 = 0.$ Thus from (4.13) it is obvious that $F^{n-1}(c)$ is a hyperplane of first kind if and only if $b_{i|j}y^i\, y^j = 0.$
This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n depends on y^i, but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\left\{ {i \atop j\,k} \right\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i.
We shall consider the difference $b_{ij} - b_{i|j}$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \left\{ {i \atop j\,k} \right\}$ is given by (2.10). Since b_i is a gradient vector from (2.9), we have

$$E_{ij} = b_{ij},\ F_{ij} = 0,\ F^i_j = 0.$$

Thus (2.10) reduces to

(4.14)
$$D^i_{jk} = B^i b_{jk} + B^i_j\, b_{0k} + B^i_k\, b_{0j} + \lambda^s \left(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms} \right)$$
$$- b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_s g^{is},$$

From (2.11) and (4.3) it follows that for $F^{n-1}(c)$, we have

$$(4.15)\qquad \begin{cases} B_k = 3b_k + \frac{1}{\alpha}Y_k,\ B^i = \frac{2}{(1+2b^2)}b^i + \left\{\frac{1+3b^2(1-\alpha)}{\alpha^2(1+2b^2)}\right\}y^i, \\ \lambda^m = B^m b_{00},\ B_{ij} = \frac{1}{2\alpha}(a_{ij} - \alpha^{-2}Y_iY_j), \\ B^i_j = \frac{1}{2\alpha}\left(\delta^i_j - \alpha^{-2}Y_j y^i\right) + \frac{1}{\alpha\left(1+2b^2\right)}b^i b_j - \frac{1}{2}\alpha^{-2}\frac{1}{\left(1+2b^2\right)}b_j y^i, \\ A^m_k = B^m_k b_{00} + B^m b_{k0}. \end{cases}$$

we have $B^i_0 = 0, B_{i0} = 0$ which leads to $A^m_0 = B^m b_{k0}$.

Now contracting (4.14) by y^k, we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again contracting the above equation with respect to y^j, we get

$$D^i_{00} = B^i b_{00} = \left[\frac{2}{\left(1+2b^2\right)}b^i + \left\{\frac{1+3b^2(1-\alpha)}{\alpha^2\left(1+2b^2\right)}\right\}y^i\right]b_{00}.$$

Paying attention to (4.1), along $F^{n-1}(c)$, we get

$$(4.16)\qquad b_i D^i_{j0} = \frac{2b^2}{\left(1+2b^2\right)}b_{j0} + \frac{3+2b^2}{2\alpha\left(1+2b^2\right)}b_j b_{00} + \frac{2}{\left(1+2b^2\right)}b_i b^m C^i_{jm} b_{00}.$$

Now, we contract (4.16) by y^j, we have

$$(4.17)\qquad b_i D^i_{00} = \frac{2b^2}{\left(1+2b^2\right)}b_{00}.$$

From (3.3), (4.5), (4.6), (4.9) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$, the equations (4.16) and (4.17) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{2b^2}{1+2b^2} b_{00}.$$

Consequently, (4.12) and (4.13) may be written as

$$(4.18) \qquad \begin{cases} \sqrt{\dfrac{b^2}{(1+2b^2)}} H_\alpha + \dfrac{2b^2}{1+2b^2} b_{i0} B^i_0 = 0, \\ \sqrt{\dfrac{b^2}{(1+2b^2)}} H_0 + \dfrac{2b^2}{1+2b^2} b_{00} = 0 \end{cases}$$

respectively.

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Using the fact $\beta = b_i y^i = 0$ on $F^{n-1}(c)$, the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j) = 01$ for some $c_j(x)$. Thus, we can write

$$(4.19) \qquad 2b_{ij} = b_i c_j + b_j c_i.$$

Now, from (4.1) and (4.19), we get

$$b_{00} = 0, b_{ij} B^i_\alpha B^j_\beta = 0, b_{ij} B^i_\alpha y^j = 0.$$

Hence, from (4.18) we get, $H_\alpha = 0$, Again, from (4.19) and (4.15), we get

$$b_{i0} b^i = \frac{c_0 b^2}{2}, \lambda^m = 0, A^i_j B^j_\beta = 0 \text{ and } B_{ij} B^i_\alpha B^j_\beta = \frac{2}{\alpha} h_{\alpha\beta}.$$

Now, we use equations (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14), to get

(4.20) $$b_r\, D^r_{ij} B^i_\alpha\, B^j_\beta = -\frac{c_0 b^2 \left(3b^2+2\right)}{2\alpha\left(1+2b^2\right)^2} h_{\alpha\beta}.$$

Thus the equation (4.11) reduces to

(4.21) $$\sqrt{\frac{b^2}{(1+2b^2)}} H_{\alpha\beta} + \frac{c_0 b^2 \left(3b^2+2\right)}{2\alpha\left(1+2b^2\right)^2} h_{\alpha\beta} = 0.$$

Hence, the hypersurface $F^{n-1}(c)$ is umbilical.

Theorem 4.3: *The necessary and sufficient condition for* $F^{n-1}(c)$ *to be hyperplane of first kind is that (4.21) holds good.*

In this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor. Hence the hypersurface $F^{n-1}(c)$ is umbilical. Now from lemma (3.3), $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $Q_{\alpha\beta} = 0,$ which implies that $H_{\alpha\beta} = 0.$ Thus, from (4.21), we get

$$c_0 = c_i(x) y^i = 0.$$

Therefore, there exists a function $\psi(x),$ such that

$$c_i(x) = \psi(x) b_i(x).$$

Therefore, from (4.19), we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x),$$

which gives

(4.22) $$b_{ij} = \psi(x) b_i b_j.$$

Thus, we have:

Theorem 4.4: *The necessary and sufficient condition for* $F^{n-1}(c)$ *to be hyperplane of the second kind is that (4.22) hold good.*

Again lemma (3.4) together with (4.9) and $M_\alpha = 0,$ shows that $F^{n-1}(c)$ does not become a hyperplane of the third kind.

Theorem 4.5: *The hypersurface* $F^{n-1}(c)$ *is not a hyperplane of the third kind.*

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