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On Trans-Sasakian Manifolds

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Abstract: In the present paper, pseudo projectively flat and M-projectively flat trans-Sasakian manifolds are studied. It is proved that a trans-Sasakian manifold cannot be pseudo projectively flat unless $(2n-1)(d\beta-\xi\beta\eta)+d\alpha o\phi=0$. If trans-Sasakian manifold is pseudo projectively flat then scalar curvature $r=\frac{-2n(2n+1)a}{(1-a)2nb-a^2}(\alpha^2-\beta^2-\xi\beta)$, where α and β are related by $(2n-1)(d\beta-\xi\beta\eta)+d\alpha o\phi=0$. It is also proved that a trans Sasakian manifold cannot be M-projectively flat unless $(2n-1)grad\beta-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$. If trans-Sasakian manifold is M-projectively flat then scalar curvature $r=2n(2n+1)(\alpha^2-\beta^2-\xi\beta)$, where α and β are related by $(2n-1)grad\beta-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$.

Keywords: Trans-Sasakian manifold, pseudo projectively flat manifold, M-projective curvature tensor.

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1. Introduction

J. A. Oubina[1] introduced a manifold which generalizes both α-Sasakian and β-Kenmotsu manifolds. Such manifold was called a trans-Sasakian manifold of type (α,β). Trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosympletic[2], α-Sasakian[3, 4] and β-Kenmotsu[3, 5] respectively. Concept of a nearly trans-Sasakian manifold was introduced by C. Gherghe[6].Thus, Sasakian, Kenmotsu and cosympletic manifold are particular cases of trans-Sasakian manifolds. J. C. Marrero[7] constructed three dimensional trans-Sasakian manifold. R. Prasad and V. Srivastava[8] obtained certain results on trans-Sasakian manifolds. Jeong- Sik kim et al.[9] studied a generalized Ricci-recurrent trans-Sasakian manifold.

In 2002, B. Prasad[10] defined a tensor field on a Riemannian manifold of dimension greater than 2 and he called it pseudo projective curvature tensor. This tensor is a generalisation of projective curvature tensor. Such tensor was studied on a φ-recurrent Kenmotsu manifold by Venkatesha and C. S. Bagewadi[11]. C. S. Bagewadi, Venkatesha and N. S. Basavarajappa[12] studied such tensor on LP-Sasakian manifold while H. G. Nagaraja and G. Somashekhara[13] studied such tensor on a Sasakian manifold.

The pseudo projective curvature tensor on an *n*-dimensional Riemannian manifold is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}b\left[S(Y,Z)X - S(X,Z)Y\right] - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)X - g(X,Z)Y\right], \tag{1.1}$$

where R, S, r and g are Riemannian curvature tensor, Ricci tensor, scalar curvature and metric of the Riemannian manifold respectively.

In 1971, G. P. Pokhariyal and R. S. Mishra[14] introduced a new curvature tensor in *n*-dimensional manifold denoted by *W* and defined by

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right]. \tag{1.2}$$

Such a tensor field *W* is known as *M*-projective curvature tensor. *M*-projective curvature tensor has been studied by J. P. Singh[15], S. K. Chaubey and R. H. Ojha[16], R. N. Singh and S. K. Pandey[17] and many others. In this paper we study pseudo-projectively flat and *M*-projectively flat trans-Sasakian manifolds.

2. Preliminaries

Let M be a $(2n+1)-$dimensional almost contact metric manifold[18] with almost contact metric structure (ϕ,ξ,η,g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form and g is a compatible Riemannian metric on M such that

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi)=1, \quad \phi\xi = 0, \tag{2.1}$$

(2.2) $$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

(2.3) $$g(\phi X,Y)=g(X,\phi Y),\qquad g(X,\xi)=\eta(X),$$

where $X,Y\in TM$.

An almost contact metric manifold is said to be contact manifold if

(2.4) $$d\eta(X,Y)=\varphi(X,Y)=g(X,\phi Y),$$

$\varphi(X,Y)$ is being called fundamental $2-$form on M .

An almost contact metric manifold M is called trans-Sasakian manifold if

(2.5) $$(\nabla_X\phi)Y=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\},$$

where ∇ is Levi-Civita connection of Riemannian metric g and α & β are smooth functions on *M*. From equations (2.1), (2.2) and (2.3), we get

(2.6) $$(\nabla_X\phi)\xi=-\alpha\phi X+\beta[X-\eta(X)\xi],$$

(2.7) $$(\nabla_X\eta)Y=-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y).$$

In a $(2n+1)-$dimensional trans-Sasakian manifold, we have

(2.8) $$\begin{aligned}R(X,Y)\xi=&\left(\alpha^2-\beta^2\right)\left[\eta(Y)X-\eta(X)Y\right]\\&+2\alpha\beta\left[\eta(Y)\phi X-\eta(X)\phi Y\right]+(Y\alpha)\phi X\\&-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y,\end{aligned}$$

(2.9) $$\begin{aligned}R(\xi,Y)X=&\left(\alpha^2-\beta^2\right)\left[g(X,Y)\xi-\eta(X)Y\right]+(X\alpha)\phi Y\\&+2\alpha\beta\left[g(\phi X,Y)\xi-\eta(X)\phi Y\right]+g(\phi X,Y)(grad\alpha)\\&+(X\beta)\left[Y-\eta(Y)\xi\right]-g(\phi X,\phi Y)(grad\beta),\end{aligned}$$

(2.10) $$\eta\left(R(\xi,Y)X\right)=g\left(R(\xi,Y)X,\xi\right)$$
$$=\left(\alpha^2-\beta^2-\xi\beta\right)\left[g\left(X,Y\right)-\eta(X)\eta\left(Y\right)\right],$$

(2.11) $$2\alpha\beta+\xi\alpha=0,$$

(2.12) $$S(X,\xi)=(2n(\alpha^2-\beta^2)-\xi\beta)\eta\left(X\right)-\left(2n-1\right)X\beta-\left(\phi X\right)\alpha$$

and

(2.13) $$Q\xi=(2n(\alpha^2-\beta^2)-\xi\beta)\xi-\left(2n-1\right)\left(grad\,\beta\right)+\phi\left(grad\,\alpha\right).$$

An almost contact metric manifold *M* is said to be η-Einstein if its Ricci-tensor *S* is of the form

$$S(X,Y)=ag(X,Y)+b\eta\left(X\right)\eta\left(Y\right),$$

where a and b are smooth functions on *M*. An η-Einstein manifold becomes Einstein manifold if $b=0$, i. e.

$$S(X,Y)=ag(X,Y).$$

If $\{e_1,e_2....,e_{2n},e_{2n+1}=\xi\}$ be a local ortho-normal basis of tangent space in a (2n+1)-dimensional almost contact manifold *M*, then we have

$$\sum_{i=1}^{2n+1} g(X,Y)=2n+1.$$

$$\sum_{i=1}^{2n+1} g(e_i,Y)S\left(X,e_i\right)=\sum_{i=1}^{2n+1} R\left(e_i,Y,X,e_i\right)$$
$$=S(X,Y).$$

3. Pseudo Projectively Flat and M-Projectively Flat Manifolds

Let *M* be a (2n+1)-dimensional Pseudo projectively flat manifold, then from equation (1.1) we have

(3.1) $$aR\left(X,Y\right)Z=b\left[S\left(X,Z\right)Y-S\left(Y,Z\right)X\right]$$

$$+\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\left[g\left(Y,Z\right)X-g\left(X,Z\right)Y\right].$$

Contracting equation (3.1) with U, we get

$$g(R\left(X,Y\right)Z,U)=\frac{b}{a}\left[S\left(X,Z\right)g\left(Y,U\right)-S\left(Y,Z\right)g\left(X,U\right)\right]$$
$$+\frac{r}{\left(2n+1\right)a}\left(\frac{a}{2n}+b\right)\left[g\left(Y,Z\right)g\left(X,U\right)\right.$$
$$\left.-g\left(X,Z\right)g\left(Y,U\right)\right]. \tag{3.2}$$

Let $\left\{e_1,e_2....,e_{2n},e_{2n+1}=\xi\right\}$ be a local ortho-normal basis of tangent space. Putting $Y=Z=e_i$, in equation (3.2), we get

$$S\left(X,U\right)=\frac{r}{2n+1}g\left(X,U\right), \tag{3.3}$$

for $a\neq b.$

Hence a pseudo projectively flat manifold is Einstein manifold if $a\neq b.$

If the manifold is *M*- projectively flat then from equation (1.2), we have

$$R\left(X,Y\right)Z=\frac{1}{4n}\left[S(Y,Z)X-S(X,Z)Y\right.$$
$$\left.+g(Y,Z)QX-g(X,Z)QY\right]. \tag{3.4}$$

Contracting equation (3.4) with *U*, we get

$$g(R\left(X,Y\right)Z,U)=\frac{1}{4n}\left[S(Y,Z)g\left(X,U\right)-S(X,Z)g\left(Y,U\right)\right.$$
$$\left.+g(Y,Z)S\left(X,U\right)-g(X,Z)S\left(Y,U\right)\right]. \tag{3.5}$$

Let $\left\{e_1,e_2....,e_{2n},e_{2n+1}=\xi\right\}$ be a local ortho-normal basis of tangent space. Putting $Y=Z=e_i$, in equation (3.5), we get

(3.6) $$S(X,U)=\frac{r}{2n+1}g(X,U).$$

Hence an *M*- projectively flat manifold is Einstein manifold.

4. Pseudo Projectively Flat Trans-Sasakian Manifold

If a trans-Sasakian manifold is Pseudo projectively flat then from equation (1.1), we have

(4.1) $$aR(X,Y)Z=b\left[S(X,Z)Y-S(Y,Z)X\right]+\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)X-g(X,Z)Y\right].$$

Contracting equation (4.1) with *U*, we get

(4.2) $$g(R(X,Y)Z,U)=\frac{b}{a}\left[S(X,Z)g(Y,U)-S(Y,Z)g(X,U)\right]+\frac{r}{(2n+1)a}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\right].$$

Putting $U=\xi$ and using equation (4.2), we get

(4.3) $$g(R(X,Y)Z,\xi)=\frac{b}{a}\left[S(X,Z)\eta(Y)-S(Y,Z)\eta(X)\right]+\frac{r}{(2n+1)a}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right].$$

Putting $X=\xi$ and using equations (2.3), (2.10) and (2.12), we get

(4.4) $$S(Y,Z)=\left[\frac{ra}{2n+1}\left(\frac{a}{2n}+b\right)-\frac{a}{b}\left(\alpha^2-\beta^2-\xi\beta\right)\right](Y,Z)+\left[\frac{a}{b}\left(\alpha^2-\beta^2-\xi\beta\right)-\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)-\frac{ra}{(2n+1)}\left(\frac{a}{2n}+b\right)\right]\eta(Y)\eta(Z)+(2n-1)Z\beta+(\phi Z)\alpha\eta(Y).$$

From equations (3.3) and (4.4), we have

(4.5) $$r=\frac{-2n(2n+1)a}{(1-a)2nb-a^2}\left(\alpha^2-\beta^2-\xi\beta\right)$$

and

(4.6) $$(2n-1)(d\beta-\xi\beta\eta)+d\alpha o\phi=0.$$

Theorem 4.1: *A trans-Sasakian manifold cannot be pseudo projectively flat unless* $(2n-1)(d\beta-\xi\beta\eta)+d\alpha o\phi=0.$

From equations (4.5) and (4.6), we get

Theorem 4.2: The *scalar curvature r of a pseudo projectively flat trans-Sasakian manifold satisfies*

(4.7) $$r=\frac{-2n(2n+1)a}{(1-a)2nb-a^2}\left(\alpha^2-\beta^2-\xi\beta\right),$$

where α *and* β *are related by* $(2n-1)(d\beta-\xi\beta\eta)+d\alpha o\phi=0.$

5. *M*-Projectively Flat Trans-Sasakian Manifolds

Let a trans-Sasakian manifold be *M*-projectively flat. From equation (1.2), we have

(5.1) $$R(X,Y)Z=\frac{1}{4n}\Big[S(Y,Z)X-S(X,Z)Y$$
$$+g(Y,Z)QX-g(X,Z)QY\Big].$$

Contracting equation (5.1) with *U*, we get

(5.2) $$g\left(R(X,Y)Z,U\right)=\frac{1}{4n}\Big[S(Y,Z)g(X,U)-S(X,Z)g(Y,U)$$
$$+g(Y,Z)S(X,U)-g(X,Z)S(Y,U)\Big].$$

Putting $U=\xi$ in equation (5.2) and using equations (2.3), we have

(5.3) $$g(R(X,Y)Z,\xi)=\frac{1}{4n}\big[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)$$
$$+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)\big].$$

Again putting $X=\xi$ in equation (5.3) and using equations (2.3), (2.10) and (2.12), we have

(5.4) $$4n(\alpha^2-\beta^2-\xi\beta)[g(Y,Z)-\eta(Y)\eta(Z)]=S(Y,Z)+\big\{2n(\alpha^2-\beta^2)$$
$$-\xi\beta\big\}\eta(Y)\eta(Z)-(2n-1)Z\beta\eta(Y)-(\phi Z)\alpha\eta(Y)$$
$$+2n(\alpha^2-\beta^2-\xi\beta)g(Y,Z)-\big\{2n(\alpha^2-\beta^2)$$
$$-\xi\beta\big\}\eta(Y)\eta(Z)+(2n-1)Y\beta\eta(Z)+(\phi Y)\alpha\eta(Z),$$

(5.5) $$S(Y,Z)=2n(\alpha^2-\beta^2-\xi\beta)g(Y,Z)-4n(\alpha^2-\beta^2-\xi\beta)\eta(Y)\eta(Z)$$
$$+(2n-1)Z\beta\eta(Y)+(\phi Z)\alpha\eta(Y)-(2n-1)Y\beta\eta(Z)$$
$$-(\phi Y)\alpha\eta(Z).$$

From equations (3.6) and (5.5), we get

(5.6) $$r=2n(2n+1)(\alpha^2-\beta^2-\xi\beta)$$

and

(5.7) $$(2n-1)grad\,\beta-\phi(grad\,\alpha)=(2n-1)(\xi\beta)\xi.$$

Theorem 5.1: *A trans-Sasakian manifold cannot be M-projectively flat unless* $(2n-1)grad\,\beta-\phi(grad\,\alpha)=(2n-1)(\xi\beta)\xi$.

From equations (5.6) and (5.7), we have

Theorem 5.2: *If a trans-Sasakian manifold is M-projectively flat then scalar curvature* $r=2n(2n+1)(\alpha^2-\beta^2-\xi\beta)$, *where* α *and* β *are related by* $(2n-1)grad\,\beta-\phi(grad\,\alpha)=(2n-1)(\xi\beta)\xi$.

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