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Approximate Solution of Rectangular Profile Fin with Temperature Dependent Thermal Conductivity by Spline Collocation Method*

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Abstract: Spline collocation approach has been widely accepted to solve boundary value problems. In this article, Spline collocation method has been used to evaluate the efficiency of Fins with temperature dependent thermal conductivity for rectangular profile. On comparison with solutions obtained by analytical method and differential transform method, spline solution obtained shows good agreement, suggesting that Spline method is very effective and convenient.

Keywords: Fins, Variable thermal conductivity, rectangular profile, heat transfer co-efficient, spline collocation method.

2000 Mathematics Subject Classification: 65D07.

1. Introduction

The class of differential equations is too wide and obtaining their solutions is an important task in the study of such differential equations. In case, where an analytical function as the solution of a differential equation is not possible or rather too difficult to obtain, one has to go for obtaining approximate or numerical solutions to the equation, which requires minimum number of steps, consumes the shortest computing time and yet does not produce any excessive errors.

Fins are widely used to enhance heat transfer between primary surface and the environment in many industrial units such as air conditioning units, processor/ microprocessor cooling systems, refrigeration systems, heat exchangers, gas turbine blades, car radiators etc. Thus, the study of fin profiles is gaining momentum due to its multiple applications.

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Heat transfer analysis of fin profiles has been studied by many researchers. The analysis of extended surface heat transfer is extensively studied by Kraus et al[1]. Arslanturk[2] used decomposition method to evaluate the temperature distribution and analytical expression for the fin efficiency. Razelos and Imre[3] considered the variation of the convective heat transfer coefficient from the base of a convicting fin to its tip. A method of temperature-correlated profiles is used to obtain the solution of optimum convective fin when the thermal conductivity and heat transfer co-efficient are functions of temperature. Coskun and Atay[4] used variation iteration method to analyze convective straight and radial fins with temperature dependent thermal conductivity. Sharqawy and Zubaie[5] carried out an analysis to study the efficiency of straight fins with different configurations when subjected to simultaneous heat and mass transfer mechanisms. Rajabi[6] obtained efficiency and fin temperature distribution by ADM and the HPM wit temperature dependent thermal conductivity. Lin and Lee[7] investigated boiling on a straight fin with linearly varying thermal conductivity.

In this paper, we apply the spline collocation method, to construct approximate solutions of the governing equations of the straight fins for rectangular profile and temperature dependent thermal conductivity. The idea of existence and uniqueness of particular bicubic splines was first established by Carl de Boor[8]. Bickley[9] brought forward a useful aspect of spline functions in light that can be employed to solve a linear two-point boundary value problem. Similarly, Blue[10] discussed the applicability of spline functions to nonlinear differential equations. The problem was also discussed by A. Moradi and H. Ahmadikia[11], they obtained the solution using DTM method and compared it with the exact solution.

2. Description and Formulation of Fin Profile Problem

Heat transfer through fin surfaces is widely used in many industrial applications. Fins are physically designed to increase the surface area in contact with the cooling fluid surrounding the purposed medium. Heat conduction analysis of fin is of great interest, since they are used in many industrial applications, such as air conditioning units, processor/ microprocessor cooling systems, refrigeration systems, heat exchangers, gas turbine blades and car radiators. For most actual situations with a high temperature difference between the fin base and its tip, the variation of the thermal conductivity of the fin material with temperature should be taken into account. For instance, the heat pipe/fin space radiators have fins with temperature-dependent thermal conductivity. Variable thermal conductivity introduces non-linearity, which directly affects the energy balance equation and its solution.

The heat transfer co-efficient is a function of temperature profiles in some important modes of heat transfer like laminar and turbulent natural convection, condensation, radiation, and especially boiling. The dependence of the heat transfer co-efficient on temperature is usually expressed in a power-law-type form, in which its power depends on a heat transfer mode.

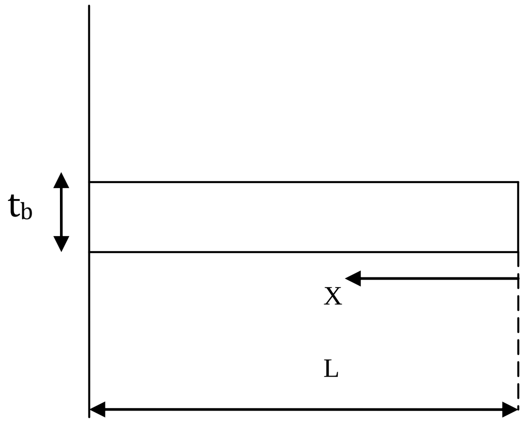

Figure1: Rectangular Profile

Consider a straight fin of the length L, with a cross-section area $A(x)$. Fin surface is exposed to a convective environment at temperature T_∞. The local heat transfer coefficient h along the fin surface is constant, and the thermal conductivity varies with the temperature linearly. The one- dimensional energy equation can be expressed as,

(2.1) $$\frac{d}{dx}\left[k(T)A(x)\frac{dT}{dx}\right]-ph(T-T\infty)=0,$$

where p is the periphery of the fin, T_∞ is the ambient temperature and $k(T)$ is defined as,

(2.2) $$k(T)=k_b\left[1+\lambda\left(T-T_\infty\right)\right],$$

where k_b is the fin thermal conductivity at ambient temperature and λ is a constant.

Straight fin can be classified according to its profile as shown in Figure 1. The fin profile is defined according to variation of the fin thickness along its extended length. For example, the cross- section area of the fin may vary as,

(2.3) $$A(x) = bt(x),$$

where b is the width of the fin, $t(x)$ is the fin thickness along the length. By employing the following dimensionless parameters,

(2.4) $$\theta = \frac{T - T_\infty}{T - T_b}, \quad X = \frac{x}{L}, \quad N = \left(\frac{hpL^2}{K_b A_b}\right)^{\frac{1}{2}},$$

where, A_b is the base area, thus, the energy equation for two profiles are reduced to,

(2.5) $$(1 + \beta\theta)\frac{d^2\theta}{dX^2} + \beta\left(\frac{d\theta}{dX}\right)^2 - N^2\theta = 0,$$

where $\beta = \lambda\left(T_b - T_\infty\right)$ in which T_b is the base temperature and fin tip is insulated. Therefore, boundary conditions for this problem are defined as follows,

(2.6) $$x = 0, \quad \theta' = 0,$$

(2.7) $$x = 1, \quad \theta = 1.$$

3. Basic Principles of Spline Collocation

Use of Spline functions with moments for the solution of nonlinear differential equation was suggested by Blue[10]
Consider a linear two-point boundary value problem

(3.1) $$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

Subject to the boundary conditions

(3.2)
$$G_1\left[y(a),\ y'(a)\right]=0 \quad at \quad x=a$$
$$G_2\left[y(b),\ y'(b)\right]=0 \quad at \quad x=b$$

Since $s(x)$ is a cubic Spline interpolating $y(x)$ given by equation, we have $s(x_i)=y(x_i)$, and also $s''(x)$ is a linear function. Let us define this $s''(x)$ in the subinterval $[x_i, x_{i+1}]$ of $[a,b]$ as follows:

(3.3)
$$s''(x)=y''_{i+1}\frac{x-x_i}{h_{i+1}}+y''_i\frac{x_{i+1}-x}{h_{i+1}},\quad i=0,1,2,....n-1,$$

where $h_{i+1}=x_{i+1}-x_i$

Two successive integrations produce a cubic Spline $s(x)$ in $[x_i, x_{i+1}]$ which has the form

(3.4)
$$s(x)=y''_{i+1}\frac{(x-x_i)^3}{6h_{i+1}}+y''_i\frac{(x_{i+1}-x)^3}{6h_{i+1}}+\frac{A(x_{i+1}-x)}{h_{i+1}}+\frac{B(x-x_i)}{h_{i+1}},$$

with A and B are constants to be determined. Interpolation condition provides their evaluation.

Equation (3.4) is given due to these constants as follows:

(3.5)
$$s(x)=y''_{i+1}\frac{(x-x_i)^3}{6h_{i+1}}+y''_i\frac{(x_{i+1}-x)^3}{6h_{i+1}}+\left(y_i-\frac{h_{i+1}^2}{6}y''_i\right)\frac{(x_{i+1}-x)}{h_{i+1}}$$
$$+\left(y_{i+1}-\frac{h_i^2}{6}y''_{i+1}\right)\frac{(x-x_i)}{h_{i+1}}.$$

Similarly $s(x)$ can be obtained in the interval $[x_{i-1},\ x_i]$ as

(3.6)
$$s(x)=y''_i\frac{(x-x_{i-1})^3}{6h_i}+y''_{i-1}\frac{(x_i-x)^3}{6h_i}+\left(y_i-\frac{h_i^2}{6}y''_i\right)\frac{(x-x_{i-1})}{h_i}$$
$$+\left(y_{i-1}-\frac{h_i^2}{6}y''_{i-1}\right)\frac{(x_i-x)}{h_i}.$$

Deriving $s'(x)$ at $x=x_i$ from the equation (3.6) which is denoted by $s'(x_{i+})$, we have

(3.7) $$s'(x_{i+}) = -\frac{h_{i+1}}{3} y_i'' - \frac{h_{i+1}}{6} y_{i+1}'' + \frac{y_{i+1} - y_i}{h_{i+1}}, \quad i = 0,1,2,....n-1,$$

and in the same way

(3.8) $$s'(x_{i-}) = \frac{h_i}{6} y_{i-1}'' - \frac{h_i}{3} y_i'' + \frac{y_i - y_{i-1}}{h_i}, \quad i = 1,2,....n.$$

Continuity of $s''(x)$ at $x = x_i$ requires that $s''(x_{i+}) = s''(x_{i-})$
So that

(3.9) $$h_i y_{i+1} - (h_i + h_{i+1}) y_i + h_{i+1} y_{i-1} = h_i h_{i+1} \left(\frac{h_i}{6} y_{i-1}'' + \frac{h_i + h_{i+1}}{6} y_i'' + \frac{h_{i+1}}{6} y_{i+1}'' \right),$$

where $i = 1,2,....n-1.$

This equation gives a system of $(n-1)$ equations in $(n+1)$ variables y_i, $i = 0,1,2,.....n$ to be determined. Therefore, the moments y_i'', $i = 0,1,2,.....n$ can be obtained from relation (3.8) if a curve is initially fitted to the data and two extra conditions are considered to complete the system (3.9). Here our objective is to solve the differential equation (2.5) with the help of relation (3.9). Let us express the equation (2.5) in the form

(3.10) $$y''(x) = f(x, y, y').$$

Subject to the boundary conditions (2.6) and (2.7). From these boundary conditions, it is seen that there are four pairs of boundary conditions as possible combinations viz.

(i) $y(a) = K;\ y(b) = L$

(ii) $y(a) = K;\ y'(b) = L$

(iii) $y'(a) = K;\ y(b) = L$

(iv) $y'(a) = K;\ y'(b) = L.$

The relation (3.9) will assumes the form for case (iii) as

(3.11) $$h_1 y_2 - \left(h_1 + h_2\right) y_1 + h_2 y_0 = \frac{h_1 h_2}{6}\left(\frac{h_1}{6} y_0'' + \frac{h_1 + h_2}{3} y_1'' + \frac{h_2}{6} y_2''\right) h_i y_{i+1}$$

$$-(h_i + h_{i+1}) y_i + h_{i+1} y_{i-1}$$

$$= \frac{h_i h_{i+1}}{6}\left(\frac{h_i}{6} y_{i-1}'' + \frac{h_i + h_{i+1}}{3} y_i'' + \frac{h_{i+1}}{6} y_{i+1}''\right), i = 2, 3, \ldots. n-1.$$

$$h_{n-1} y_n - \left(h_n + h_{n-1}\right) y_{n-1} + h_n y_{n-2} = \frac{h_n h_{n+1}}{6}\left(\frac{h_n}{6} y_{n-2}'' + \frac{h_{n-1} + h_n}{3} y_{n-1}'' + \frac{h_{n-1}}{6} y_n''\right).$$

Similarly, we are able to deal with remaining cases. It is observed for any case that on the left-hand side the equations are obtained for which the coefficient matrix, for in the matrix form is upper tridiagonal one.Now in order to obtain a solution to equation (3.9) with the boundary conditions given in equations (3.2), we fit a straight-line $y(x) = mx + c$ through the boundary points, which is of course an initial guess, the moments y_i'' are calculated from the relation (3.5) at the nodal points $x = x_i$ as

(3.12) $$y_i'' = f(x,\ y_i,\ y_i'),\quad i = 0,\ 1,\ 2\ \ldots n.$$

These moments are now utilized to evaluate $y_i,\ i = 0, 1, 2, \ldots. n,$ through the relation (3.9) along with the two additional equations from boundary conditions. This can be furnished by solving merely a tridiagonal system of equations. The results so obtained can again be improve by continuing the same process till the desired solutions are found or two successive iterations produce the same values.

4. Iterative Solution of Rectangular Profile Fin

For finding out the Spline approximation $s(x)$ of $\phi(x)$ described in equation (2.5) satisfying the boundary condition, a line $\phi(x) = ax + b$ is assumed to be the first approximation to start with the iterative scheme. The straight line $\phi(x) = 1$ can be fitted through these points $x = 0$ and $x = 1$. The calculation of $\phi_i, i = 0, 1, 2, \ldots N$ is to be carried out through the solution of tridiagonal system of $(N+1)$ equations. The solution after obtaining the tridiagonal system of $(N+1)$ equation is given below

Table 1: Comparison Between Exact and Spline Results for Rectangular Profile Fin for $\beta = 0$ and $N = 1$ and $N = 0.5$

$N = 0.5,\ \beta = 0$,			$N = 1,\ \beta = 0$		
X	Spline Collocation	DTM	Spine Collocation	Exact Solution	DTM
0	0.88681	0.886819	0.648111	0.648054	0.648054
0.2	0.891248	0.891257	0.66111	0.661059	0.661059
0.4	0.904606	0.904615	0.700628	0.700594	0.700594
0.6	0.927019	0.927026	0.76826	0.768246	0.768246
0.8	0.958711	0.958716	0.866728	0.86673	0.86673
1	1	1	1	1	1

Table 2: Iterative Spline Solution of different values of β and $N = 1$ for Rectangular Profile Fin Problem.

X	$\beta = 1$	$\beta = 0.5$	$\beta = 0.3$	$\beta = 0.1$	$\beta = -0.1$	$\beta = -0.2$
0	0.781854	0.727982	0.702513	0.672916	0.642921	0.611112
0.2	0.790628	0.738677	0.713994	0.685163	0.655590	0.624682
0.4	0.816958	0.770892	0.748679	0.722335	0.694352	0.666266
0.6	0.860769	0.824913	0.80721	0.785688	0.761565	0.738684
0.8	0.921877	0.901129	0.890561	0.877287	0.861373	0.847135
1	1	1	1	1	1	1

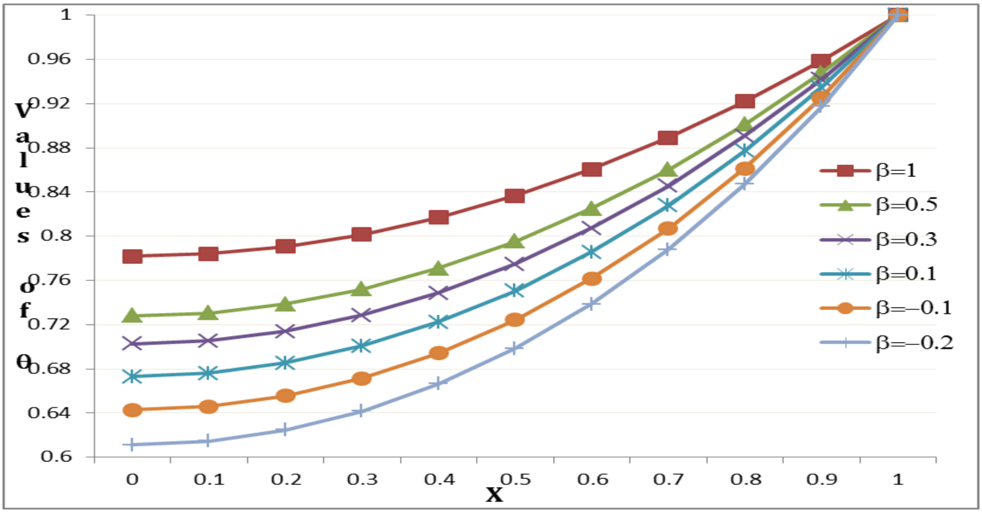

Figure 2 : Temperature Distribution of Rectangular Profile Fin at different values of β at N=1.

5. Concluding Remarks

The comparison between the exact solution, solution of differential transform method (DTM) and results of spline collocation method at $\beta = 0$ and $N = 1$ and $N = 0.5$ for rectangular profile fin is shown in table 1. The comparison indicates that solutions of spline collocation are very close to the exact analytical solution.

The differential equation remains linear in the absence of parameter; on the other hand, the little contribution of this parameter leads to the non-linear differential equation. The convergent nature of the iterative method is visualized through aspects of representing result table 2. Temperature distribution for different values of β for $N = 1$ is presented for rectangular profile in figures2. Result of spline collocation method shows a good agreement with differential transform method. From the result, it can be concluded that, with decreasing β, the fin base temperature decreases for any profile in straight fin.

An approximate method is suggested to determine the temperature and efficiency of rectangular fin with temperature dependent thermal conductivity. The analysis is presented without doing any linearization for solving highly non-linear governing equations. To validate the present system, the results determined by the above analysis has been compared with solution obtained by differential transform method and exact solution. The results indicate that spline collocation method gives the best approximation for the linear and non-linear engineering problems without any assumption and linearization.

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