ISSN 0974 - 9373

Vol. 24 No. 3 (2020) **Journal of International Academy of Physical Sciences** pp. 247- 262

On $\mathbb{R}$-Complex Finsler Space with Special *(α, β)* -Metric

S. M. Vanithalakshmi, S. K. Narasimhamurthy and M. K. Roopa

Department of Mathematics
Kuvempu University, Shankaraghatta, Shivamogga, Karnataka, India.
Email: vanithateju02@gmail.com, nmurthysk@gmail.com, roopamk1002@gmail.com

(Received June 24, 2020)

Abstract: In this paper, we determined the fundamental tensor fields g_{ij} and $g_{i\bar{j}}$ and its inverse of these tensor fields, their determinant. Further, we studied some properties of non-Hermitian $\mathbb{R}$-complex Finsler space with $(\alpha, \beta)-$metric.

Keywords: Special $(\alpha, \beta)-$metrics, Hermitian and non-Hermitian $\mathbb{R}$-Complex Finsler space.
2010 AMS Subject Classification: 53C25, 53C40, 53D15.

1. Introduction

The concept of $\mathbb{R}-$complex Finsler spaces is new in Finsler geometry. Munteanu and Purcuru[1] have extended the notion of complex Finsler spaces to $\mathbb{R}-$complex Finsler spaces by reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler spaces called $\mathbb{R}-$complex Finsler spaces. Nicolta Aldea and Gheorghe Munteanu[2] studied the $(\alpha, \beta)-$complex Finsler metric and also determined the fundamental metric tensor and some properties of Hermitian of the complex Randers metrics. Some important results on $\mathbb{R}-$complex Finsler spaces have been obtained by Purcuru, Shankar and Baby[3,4]. In the present paper, following the ideas from real Finsler spaces with class of $(\alpha, \beta)-$metrics, we introduce the notion of $\mathbb{R}-$complex Finsler space with $(\alpha, \beta)-$metric.

2. Preliminaries

Let M be a complex Finsler manifold, $\dim_C M = n$. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle TM and its conjugate $T''M$.

The bundle $T'M$ is in its turn a complex manifold, the local coordinate in a chart will be denoted by $\left(z^k, \eta^k\right)$ and these are changed by the rules,

$$z'^k = z'^k\left(z\right), \qquad \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{2.1}$$

The complexified tangent bundle of $T'M$ is decomposed as $T_C\left(T'M\right) = T'\left(T'M\right) \oplus T''M$. A natural local frame for $T'\left(T'M\right)$ is $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ which is changes by the rules obtained with Jacobi matrix of the above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains thesecond order partial derivatives. Non-linear connection briefly (c.n.c) is a supplementary distribution is spanned by $\frac{\partial}{\partial \eta^k}$ an adapted frame in $H\left(T'M\right)$ is $\frac{\partial}{\partial z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are coefficients of the c.n.c and an adapted frame in the coefficient of the c.n.c and they have a certain rule of changes at the equation (2.1), so that $\frac{\delta}{\delta z^k}$ transform like vectors onthe base manifold M. Next, we use the abbreviations $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and ∂_k, $\dot{\partial}_k$, δ_k for their conjugates. The dual adapted basis of δ_k, $\dot{\partial}_k$ are $\left\{dz^k, \delta\eta^k = d\eta^k + N_k^j dz^j\right\}$ and, $\left\{d\bar{z}^k, \delta\bar{\eta}^k\right\}$ their conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space[5-9] is with respect to complex scalars and the metric tensor of the space is a Hermitian one. The slightly changed the definition of complex Finsler spaces as

Definition[10] 2.1: *An $\mathbb{R}-$complex Finsler metric on M is continuous function $F: T'M \to \mathbb{R}$ satisifying*

(a) $L = F^2$ is a smooth on $T'M \setminus \{0\}$;

(b) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(c) $F(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = |\lambda| F(z,\eta,\bar{z},\bar{\eta})$ for all $\lambda \in \mathbb{R}$.

It follows that L is (2,0)-homogeneous with respect to the real scalar λ and is proved that the following identities are fulfilled;

(2.2) $$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i = 2L\,; \quad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j},$$

(2.3) $$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0\,; \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\bar{\eta}^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0,$$

where

$$2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j,$$

$$g_{ij} = \frac{\partial^2 L}{\eta^i\eta^j}\,; \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\eta^i\bar{\eta}^j}\,; \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j}.$$

Definition 2.2: *An $\mathbb{R}-$complex Finsler space (M,F) is called $(\alpha,\beta)-$ metric if the fundamental function $F(z,\eta,\bar{z},\bar{\eta})$ is $\mathbb{R}-$homogeneous by means of functions $\alpha(z,\eta,\bar{z},\bar{\eta})$ and $\beta(z,\eta,\bar{z},\bar{\eta})$ depends on z^i, η^i, $\bar{z}^i$ and $\bar{\eta}^i$ $(i = 1,2,....n)$ by means of $\alpha(z,\eta,\bar{z},\bar{\eta})$ and $\beta(z,\eta,\bar{z},\bar{\eta})$.*

(2.5) $$F(z,\eta,\bar{z},\bar{\eta}) = F(\alpha(z,\eta,\bar{z},\bar{\eta}), \beta(z,\eta,\bar{z},\bar{\eta})),$$

where,

$$\alpha^2(z,\eta,\bar{z},\bar{\eta}) = \frac{1}{2}\left(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\eta^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j\right),$$
$$= \mathrm{Re}\left\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\right\},$$

(2.6) $$\beta(z,\eta,\bar{z},\bar{\eta}) = \frac{1}{2}\left(b_i\eta^i + b_i\bar{\eta}^i\right) = \mathrm{Re}\left(b_i\eta^i\right),$$

with

$$a_{ij} = a_{ij}(z), \quad a_{i\bar{j}} = a_{i\bar{j}}(z), \quad b_i = b_i(z).$$

We denote,

(2.7) $$L\left(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})\right) = F^2\left(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})\right).$$

Definition 2.3: *An* $\mathbb{R}-$*complex Finsler space* (M, F) *is called Hermitian space, if the tensor* $g_{ij} = 0$ *and the Hermitian metric* $g_{i\bar{j}}$ *is invertible. An* $\mathbb{R}-$*complex Finsler space* (M, F) *is called non-Hermitian space if the metric tensor* $g_{i\bar{j}} = 0$ *and the Hermitian matric* g_{ij} *is invertible. Here,* g_{ij} *and* $g_{i\bar{j}}$ *are the metric tensors of the space and are given by* $g_{ij} = \frac{\partial}{\partial \eta^i}\frac{\partial}{\partial \eta^j} L$ *and* $g_{i\bar{j}} = \frac{\partial}{\partial \eta^i}\frac{\partial}{\partial \bar{\eta}^j} L$*.*

3. $\mathbb{R}-$Complex Space with Special (α, β)-metrics

Consider g_{ij} and $g_{i\bar{j}}$ are the metric tensors on $\mathbb{R}-$complex Finsler space. The Hermitian geometry $g_{i\bar{j}}$ invertible. We discussed Hermitian $\mathbb{R}-$complex Finsler spaces, if $\det\left(g_{i\bar{j}}\right) = 0$ and non-Hermitian $\mathbb{R}-$complex Finsler spaces, if $\det\left(g_{ij}\right) = 0$. In this section, we determine the fundamental metric tensor of complex space with Finsler metric that is $L = \left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)^2$ and obtained condition for property of non-Hermitian $\mathbb{R}-$complex Finsler spaces.

Consider $\mathbb{R}-$Complex Finsler space with special (α, β)-metric,

(3.1) $$L(\alpha, \beta) = \left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)^2.$$

Then, it follows that $F = \alpha + \beta + \frac{\beta^2}{\alpha}$.

Now, we find the following quantities on $\mathbb{R}-$complex Finsler spaces with the metric (3.1).
From the equations (2.2) and (2.3) with metric (3.1), we have

$$\alpha L_\alpha + \beta L_\beta = 2L \quad \text{and} \quad \alpha L_{\alpha\alpha} + \beta L_{\beta\beta} = L_\alpha\,, \tag{3.2}$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta\,,\quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L\,,$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}\,,\ L_\beta = \frac{\partial L}{\partial \beta},\ L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\,\partial\beta},\ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}\,,\ L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}\,. \tag{3.3}$$

$$L_\alpha = 2\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(1-\frac{\beta^2}{\alpha}\right), \tag{3.4}$$

$$L_\beta = 2\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(1+\frac{2\beta}{\alpha}\right), \tag{3.5}$$

$$L_{\alpha\alpha} = 2\left[\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(\frac{2\beta^2}{\alpha}\right)+\left(1-\frac{\beta^2}{\alpha}\right)\left(1-\frac{\beta^2}{\alpha}\right)\right], \tag{3.6}$$

$$L_{\beta\beta} = 2\left[\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(\frac{2}{\alpha}\right)+\left(1+\frac{2\beta}{\alpha}\right)\left(1+\frac{2\beta}{\alpha}\right)\right], \tag{3.7}$$

$$L_{\alpha\beta} = 2\left[\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(\frac{-2\beta^2}{\alpha^2}\right)+\left(1+\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{\beta^2}{\alpha}\right)\right], \tag{3.8}$$

$$\alpha L_\alpha + \beta L_\beta = \alpha\left[2\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(1-\frac{\beta^2}{\alpha^2}\right)\right]$$
$$+\beta\left[2\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(1+\frac{2\beta}{\alpha^2}\right)\right] = 2\left[\alpha+\beta+\frac{\beta^2}{\alpha}\right]^2 = 2L\,,$$

$$\alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = \alpha\left[2\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(\frac{2\beta^2}{\alpha^2}\right)+\left(1-\frac{\beta^2}{\alpha}\right)\left(1-\frac{\beta^2}{\alpha}\right)\right]$$

$$+\beta\left[\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)\left(\frac{-2\beta}{\alpha^2}\right)+\left(1+\frac{2\beta}{\alpha}\right)\left(1-\frac{\beta^2}{\alpha^2}\right)\right]$$
$$=2\left[\alpha+\beta-\frac{\beta^3}{\alpha^2}-\frac{\beta^4}{\alpha^3}\right]=L_\alpha \ .$$

Now, to determine the metric tensors of an $\mathbb{R}-$complex Finsler space using the following equations as:

$$g_{i\bar{j}}=\frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\,\partial\eta^j},\quad g_{i\bar{j}}=\frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\,\partial\bar{\eta}^j}.$$

Each of these being of interest in the following:
We consider,

$$\frac{\partial\alpha}{\partial\eta^i}=\frac{1}{2\alpha}\left(a_{ij}\eta^j+a_{i\bar{j}}\bar{\eta}^j\right)=\frac{1}{2\alpha l_i},\quad \frac{\partial\beta}{\partial\eta^i}=\frac{1}{2}b_i.$$

$$\frac{\partial\alpha}{\partial\bar{\eta}^i}=\frac{1}{2\alpha}\left(a_{\bar{i}\bar{j}}\bar{\eta}^j+a_{i\bar{j}}\eta^j\right)=\frac{\partial\beta}{\partial\bar{\eta}^i}=\frac{1}{2}b_i,$$

where,

$$l_i=\left(a_{ij}\eta^j+a_{i\bar{j}}\bar{\eta}^j\right),\quad l_{\bar{j}}=a_{\bar{i}\bar{j}}+a_{i\bar{j}}\eta^i.$$

We find immediately,

$$l_i\eta^i+l_{\bar{j}}\bar{\eta}^j=2\alpha^2.$$

We denote:

$$\eta^i=\frac{\partial L}{\partial\eta^i}=\frac{\partial}{\partial\eta^i}F^2=2F\frac{\partial}{\partial\eta^i}\left(\alpha+\frac{\beta^2}{\alpha}\right),$$

(3.9) $$\eta_i=\rho_0 l_i+\rho_1 b_i,$$

Where

(3.10) $$\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha$$

And

(3.11) $$\rho_1 = \frac{1}{2}L_\beta\,.$$

Differentiating ρ_0 and ρ_1 with respect to η^j and $\bar{\eta}^j$ respectively, which yields:

$$\frac{\partial\rho_0}{\partial\eta^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}$$

and

$$\frac{\partial\rho_0}{\partial\bar{\eta}^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}\,.$$

Similarly

$$\frac{\partial\rho_1}{\rho\eta^j} = \eta_{-1}l_i + \mu_0, \quad \frac{\partial\rho_1}{\partial\eta^j}l_{\bar{i}} + \mu_0 b_{\bar{i}}\,,$$

where,

(3.12) $$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}\,,\ \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}\,,\ \mu_0 = \frac{L_{\beta\beta}}{4}\,.$$

By direct computation, using equation (3.10),(3.11) and equation (3.12), we obtain the following result.

Theorem 3.1: *The invariants of* $\mathbb{R}-$*complex Finsler space with* (α, β)*-metric* $F = \alpha + \beta + \dfrac{\beta^2}{\alpha}$, *the quantities* ρ_0, ρ_1, ρ_2, ρ_{-1} *and* μ_0 *are given by;*

$$\rho_0 = \frac{\alpha^4 - \beta^4 + \alpha\beta\left(\alpha^2 - \beta^2\right)}{\alpha^4}\,,$$

$$\rho_1 = \alpha + 3\beta + \frac{\beta^2}{\alpha}\left(3 + \frac{2\beta}{\alpha}\right),$$

$$\rho_{-2} = \frac{\alpha^4 + \beta^4 - 2\alpha^2\beta^2\left(\alpha^2 + 2\beta + 2\beta^2\right)}{4\alpha^7},$$

$$\rho_{-1} = \frac{\alpha^3 - 2\beta^3 - \alpha\beta^2(3 - 2\beta)}{2\alpha^4}$$

and

$$\mu_0 = \frac{3\left(\alpha^2 + 2\beta^2 + 2\alpha\beta\right)}{2\alpha^2},$$

subscripts -2,-1,0,1 gives the degree of homogeneity of these invariants.

3.1 Fundamental tensor of $\mathbb{R}-$Complex Finsler space with metric: Consider the $\mathbb{R}-$complex Finsler space with special (α, β)-metric,

(3.13) $$L(\alpha, \beta) = \alpha + \beta + \frac{\beta^2}{\alpha}.$$

The fundamental metric tensors of $\mathbb{R}-$complex Finsler space with general (α,β)-metric are given by:

(3.14) $$g_{ij} = \rho_0 a_{ij} + \rho_{-2} l_i l_j + \mu_0 b_i b_j + \rho_{-2}\left(b_j l_i + b_i l_j\right).$$

Using invariants in theorem 3.1 in equation (3.12) then we have,

(3.15) $$g_{ij} = \left(\frac{\alpha^2 - \beta^4 + \alpha\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2}\right)a_{ij} + \left(\frac{\alpha^4 + \beta^4 - 2\alpha^2\beta^2 + 2\alpha^2\beta^2\left(\alpha^2 + 2\beta + 2\beta^2\right)}{4\alpha^7}\right)l_i l_j + \left(\frac{3\left(\alpha^2 + 2\beta^2 + 2\alpha\beta\right)}{2\alpha^2}\right)b_i b_j + \left(\frac{\alpha^3 - 2\beta^3 - \alpha\beta^2(3 - 2\beta)}{2\alpha^4}\right)$$

$$\times\left(b_j l_i + b_i l_j\right),$$

(3.16)
$$g_{i\bar{j}} = \left(\frac{\alpha^2 - \beta^4 + \alpha\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2}\right) a_{i\bar{j}}$$
$$+\left(\frac{\alpha^4 + \beta^4 - 2\alpha^2\beta^2 + 2\alpha^2\beta^2\left(\alpha^2 + 2\beta + 2\beta^2\right)}{4\alpha^7}\right) l_i l_{i\bar{j}}$$
$$+\left(\frac{3\left(\alpha^2 + 2\beta^2 + 2\alpha\beta\right)}{2\alpha^2}\right) b_i b_{i\bar{j}} + \left(\frac{\alpha^3 - 2\beta^3 - \alpha\beta^2\left(3 - 2\beta\right)}{2\alpha^4}\right)$$
$$\times\left(b_{\bar{j}} l_i + b_i l_{\bar{j}}\right).$$

The metric tensor g_{ij} and $g_{i\bar{j}}$ are reduced to

(3.17)
$$g_{ij} = \left[A a_{ij} + B l_i l_j + C b_i b_j + D\eta_i \eta_j\right],$$

(3.18)
$$g_{i\bar{j}} = \left[A a_{i\bar{j}} + B l_i l_{\bar{j}} + C b_i b_{\bar{j}} + D\eta_i \eta_{\bar{j}}\right],$$

where,

(3.19)
$$A = \frac{\alpha^4 - \beta^4 + \alpha\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2},$$

(3.20)
$$B = \frac{\alpha^4 + \beta^4 - 2\alpha^2\beta^2 + 2\alpha^2\beta^2\left(\alpha^2 + 2\beta + 2\beta^2\right)}{2\alpha^7},$$

(3.21)
$$C = \frac{3\left(\alpha^2 + 2\beta^2 + 2\alpha\beta\right)}{2\alpha^2},$$

(3.22)
$$D = \frac{\alpha^3 - 2\beta^3 - \alpha\beta^2\left(3 - 2\beta\right)}{2\alpha^4}.$$

We use the following proposition[11] for further calculations.

Proposition 3.2: *Suppose:*

(a) $\left(Q_{ij}\right)$ *a non-singular n × n complex matrix with inverse* $\left(Q^{ji}\right)$ *;*

(b) C_i *and* $C_{\bar{i}} = \bar{C}_i$, $i = 1, 2, ..., n$ *are complex numbers;*

(c) $C^i = Q^{ji} C_j$ *and its conjugates;* $C^2 = C^i C_i = \bar{C}^i C_{\bar{i}}$ *;* $H_{ij} = Q_{ij} \pm C_i C_j$.

Then,

i. $\det\left(H_{ij}\right) = \left(1 + C^2\right) \det\left(Q_{ij}\right)$,

ii. Whenever $\left(1 \pm C^2\right) \neq 0$*, the matrix* H_{ij} *is invertible and in this case its inverse is*

$$H^{ij} = Q^{ji} \pm \frac{1}{1 \pm C^2} C^i C^j.$$

We will find the inverse of fundamental metric tensor through the below theorem:

Theorem 3.3: *For a non-Hermitian* $\mathbb{R}-$*Complex Finsler space with* (α, β)*-metric* $F = \alpha + \beta + \dfrac{\beta^2}{\alpha}$*, then they have the following:*

i) The contravariant tensor g_{ij} *of the fundamental tensor* g_{ij} *is:*

$$g_{ij} = \Bigg[A a^{ji} + \left(\frac{B}{1 + B\gamma} - \frac{B^2 C \in^2}{\alpha (1 + B\gamma)^2} \right) \eta^i \eta^j + \frac{C}{\varphi} b^i b^j + \frac{BC \in}{\varphi (1 + B\gamma)} \left(b^i \eta^j + b^j \eta^i \right) \frac{P^2 \eta^i \eta^j + PQ\left(\eta^i b^j + \eta^j b^i \right) Q^2 b^i b^j}{\left(1 + \{P\gamma + Q \in\}\right)\sqrt{D}} \Bigg], \tag{3.23}$$

where,

$$P = \left[1 + \left(\frac{B}{1 + B\gamma} - \frac{B^2 C \in^2}{\alpha (1 + B\gamma)^2} \right) \right] \gamma + \frac{BC \in}{\varphi (1 + B\gamma)^3} \quad \text{and} \quad Q = \frac{Cq \in}{\varphi} + \frac{BC \in \gamma}{\varphi (1 + B\gamma)},$$

ii) $$\det\left(A a_{ij} + B l_i l_j + C b_i b_j + D \eta_i \eta_j\right) = \left[1 + \left(P\gamma + Q \in \right) \sqrt{D} \right] \times \left[1 + \omega + \frac{B \in^2}{1 + B\gamma} \right] (1 + B\gamma) \det\left(a_{ij}\right),$$

where,

$$D=\frac{\alpha^3-2\beta^3-\alpha\beta^2\left(3-2\beta\right)}{2\alpha^4}.$$

Proof: We claim of this theorem proved by following three steps:
We write g_{ij} from equation (3.14) in the form.

(3.24) $$g_{ij}=\left[Aa_{ij}+Bl_il_j+Cb_ib_j+D\eta_i\eta_j\right].$$

Step 1: We take $Q_{ij}=a_{ij}$ and $C_i=\sqrt{B}l_i$. By applying the proposition 3.2 we obtain

$$Q^{ij}=a^{ij},\ C^2=C_iC^i=\sqrt{B}l_i\times a^{ji}\times\sqrt{B}l_j=B\times l_ia^{ji}l_j=B\gamma,$$

and

$$1+C^2=\left(1+B\gamma\right).$$

So, the matrix $H_{ij}=a_{ij}-Bl_il_j$, is invertible with

$$H^{ij}=Aa^{ji}+\frac{1}{1+B\gamma}\eta^i\eta^j,$$

$$\det\left(a_{ij}+Bl_il_j\right)=\left(1+B\gamma\right)=\det\left(a_{ij}\right).$$

Step 2: Now, we consider

$$Q_{ji}=Aa_{ij}+Bl_il_j \quad \text{and} \quad C_i=Cb_i.$$

By applying the Proposition (3.2) we have

$$Q^{ji}=a^{ji}+\frac{P\eta^i\eta^j}{1+B\gamma},$$

$$C^2=C_iC^i=Q^{ji}\times C_j=\sqrt{C}b_i\left[a^{ji}+\frac{B\eta^i\eta^j}{1+B\gamma}\sqrt{C}b^j\right],$$

$$c^2 = C\left[\omega + \frac{B\in^2}{1+B}\right].$$

Therefore,

$$1 + C^2 = 1 + C\left[\omega + \frac{B\in^2}{1+B\gamma}\right] \neq 0,$$

where, $\in = b_j\eta^j$, $\omega = b_j b^j$.

Its result that the inverse of $H_{ij} = Aa_{ij} + Bl_i l_j + Cb_i b_j$ exists and it is

$$H^{ji} = Q^{ji} + \frac{1}{1+c^2}C^i C^j,$$

(3.25) $$H^{ji} = Aa^{ji} + \frac{B\eta^i\eta^j}{1+B\gamma} + \frac{c\left[b^i + \frac{B\in\eta^i}{1+B\gamma}\right]\left[b^j + \frac{B\in\eta^j}{1+B\gamma}\right]}{\varphi},$$

(3.26) $$H^{ji} = Aa^{ji} + \left(\frac{B}{1+B\gamma} + \frac{B^2C\in^2}{\varphi(1+B\gamma)^2}\right)\eta^i\eta^j + \frac{BC\in}{\varphi(1+B\gamma)}\left(b^i\eta^j + b^j\eta^i\right) + \frac{C}{\varphi}b^i b^j,$$

where

$$\varphi = 1 + C\left[\omega + \frac{B\in^2}{1+B\gamma}\right].$$

and

(3.27) $$\det\left[Aa_{ij} + Bl_i l_j + Cb_i b_j\right] = \left[1 + C\left(\omega + \frac{B\in^2}{1+B\gamma}\right)\right](1+b\gamma)\det\left(a_{ij}\right),$$

and

$$C_i = \sqrt{D}\eta_i.$$

Clearly, observe that and obtain

$$Q^{ji} = Aa^{ji} + \left(\frac{B}{1+B\gamma} + \frac{B^2 C \in^2}{\tau\left(1+B\gamma\right)^2} \right) \eta^i \eta^j$$
$$+ \frac{BC \in}{\left(1+B\gamma\right)} \left(b^i \eta^j + b^j \eta^i \right) + \frac{C}{1+B\gamma} b^i b^j ,$$

and

$$C_i = P\eta^i + Qb_i ,$$

where

(3.30) $$P = \left[1 + \left(\frac{B}{1+B\gamma} - \frac{B^2 C \in^2}{\alpha\left(1+B\gamma\right)^2} \right) \right] \gamma + \frac{BC \in}{\varphi\left(1+B\gamma\right)^3} ,$$

(3.31) $$Q = \frac{C}{\varphi} + \frac{BC \in \gamma}{\varphi\left(1+B\gamma\right)} ,$$

and

$$C^2 = \left(P\gamma + Q \in \right) \sqrt{D} ,$$
$$1 + C^2 = 1 + \left(P\gamma + Q \in \right) \sqrt{D} \neq 0 ,$$

clearly, the matrix H_{ij} is invertible

$$C^i = Aa^{ji} + \left\{ \frac{B\eta^i \eta^j}{1+B\gamma} + \frac{c\left[b^i + \frac{B \in \eta^i}{1+B\gamma} \right]\left[b^j + \frac{B \in \eta^j}{1+B\gamma} \right]}{\varphi} \right\} \eta_j ,$$

and

$$C^j = Aa^{ji} + \left\{ \frac{B\eta^i \eta^j}{1+B\gamma} + \frac{c\left[b^i + \frac{B \in \eta^i}{1+B\gamma} \right]\left[b^j + \frac{B \in \eta^j}{1+B\gamma} \right]}{\varphi} \right\} \eta_i ,$$

where

$$C_j = P^2\eta_i\eta_j + PQ\left(\eta_i b_j + \eta_j b_i\right) + Q^2 b_i b_j .$$

Again by applying proposition 3.2 we obtain the inverse of H_{ij} as:

$$(3.32) \qquad H^{ji} = Aa^{ji} + \left[\frac{B}{1+B\gamma} + \frac{B^2 C \in^2}{\varphi(1+B\gamma)^2}\right]\eta^i\eta^j + \frac{C}{\varphi} b^i b^j$$
$$+\frac{BC\in}{\varphi(1+B\gamma)}\left(b^i\eta^j + b^j\eta^i\right) + \frac{P^2\eta^i\eta^j + PQ\left(\eta^i b^j + \eta^j b^i\right)Q^2 b^i b^j}{\left(1+\{P\gamma + Q\in\}\right)\sqrt{D}},$$

$$(3.33) \qquad \det\left(Aa_{ij} + Bl_i l_j + Cb_i b_j + D\eta_i\eta_j\right) = \left[1+\left(P\gamma + Q\in\right)\sqrt{D}\right]$$
$$\times\left[1+\omega+\frac{B\in^2}{1+B\gamma}\right](1+B\gamma)\det\left(a_{ij}\right)$$

But $g_{ij} = \rho_0 H_{ij}$, with H_{ij} from last step. Thus

$$(3.34) \qquad g^{ji} = \frac{1}{A} H^{ji},$$

where,

$$A = \frac{\alpha^4 - \beta^4 + \alpha\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2}.$$

Therefore, from equation (3.32) in (3.34) and the equation (3.32), then we obtained the claims (i) and (ii). Here, we also observed the terms, γ, ε and δ from above theorem 3.1, then immediately wecan stated as;

Theorem 3.4: *Let* $F = \alpha + \beta + \dfrac{\beta^2}{\alpha}$ *be non-Hermitian* $\mathbb{R}-$*complex Finsler space with* (α, β)*-metric, then they have the following*

$$(3.35) \qquad \gamma + \bar{\gamma} = l_i\eta^i + l_{\bar{j}}\eta^{\bar{j}} = a_{ij}\eta^j\eta^i + a_{\bar{j}\bar{k}}\eta^{\bar{k}}\eta^{\bar{j}} = 2\alpha^2,$$

(3.36) $$\in + \bar{\in} = b_j \eta^j + b_{\bar{j}} \eta^{\bar{j}} = 2\beta, \quad \delta = \in,$$

where,

$$l_i = a_{ij}\eta^j,\ \eta_i = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3} a_{ij}\eta^i + \frac{\alpha^4}{(\alpha-\beta)^3} b_i,\ \gamma = a_{jk}\eta^j\eta^k = l_k\eta^k,\ \in = b_j\eta^j,$$

$$b^k = a^{jk} b_j,\ b_l = b^k a_{kl},\ \delta = a_{jk}\eta^j b^k = l_k b^k,\ l_j = a^{jl} l_l = \eta^j.$$

4. Conclusion

The $\mathbb{R}-$complex Finsler space is an important quantities in complex Finsler geometry and it has well known interrelation with the other quantities like $\mathbb{R}-$complex Finsler space with class of $(\alpha, \beta)-$metrics.

In this paper we determined the fundamental metric tensors g_{ij} and $g_{i\bar{j}}$ of $\mathbb{R}-$complex Finsler space with $(\alpha, \beta)-$metrics and also find their determinants. Finally, we studied the property of non-Hermitian $\mathbb{R}-$complex space with $(\alpha, \beta)-$metric.

References

1. G. Munteanu and M. Purcuru, On $\mathbb{R}-$complex Finsler Spaces, *Balkan J. Geom. Appl.*, **14** (2009), 15-59.
2. N. Aldea and G. Munteanu, (*α, β*)-Complex Finsler Metrics, *Proceedings of the 4th International Colloquium, Mathematics in Engineering and Numerical Physics*, October 6-8, Bucharest, Romania, (2006), 1-6.
3. M. Purcuru, On $\mathbb{R}-$complex Finsler Spaces with Kropina Metrics, *Bulletin of the Transilvania University of Brasov*, **4(53)(2)** (2011), 79-88.
4. G. Shankar and S. A. Baby, $\mathbb{R}-$Complex Finsler Spaces with Special *(α, β)*-metrics, *Int. J. of appl. Math.*, **29(5)** (2016), 609-619.
5. T. Aikau, On Finsler Geometry on Complex Vector Bundles, Riemann Finsler Spaces, *Indian J.*
6. M. Abate and G. Patrizio, *Finsler Metrics-A Global Approach*, Lecturer Notes in Math. 1591, Springer-Verlag, 1994.
7. R. Miron., General Randers Spaces, *Kluwer Acad Publ, FTPH*, **76** (1996), 126-140.
8. G. Munteanu, Complex Spaces in Finsler-Lagrange and Hamilton Geometries, *Kluwer Acad. Publ., FTPH*, **141(13)** (2004).

9. V. S. Sabau and H. Shimida, Remarkable Classes of *(α, β)*-metric Spaces, *Rep. on Math. Phys.*, **47(1)** (2001), 31-48.

10. S. M. Vanithalakshmi, S. K. Narasimhamurthy and M. K. Roopa, On $\mathbb{R}$ – Complex Finsler Space with Matsumoto Metric, *Int. J. of Math and. Appl.*, **7(1)** (2018), 23-30.

11. M. Matsumoto, On Theory of Finsler spaces with *(α,β)*-metric, *Rep. on Math.Phys.*, **31** (1991), 43-83.