ISSN 0974 - 9373

Vol. 14 No.3 (2010) **Journal of International Academy of Physical Sciences** pp. 337-343

On Contact Metric Manifold*

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(Received February 20, 2010)

Abstract: D.E.Blair[1] introduced contact metric manifold in 2001.In the present paper some important results in contact metric manifols have been investigated. Nijenhuis tensors have been studied with a new light. Some results have also been investigated in K-contact and sasakian manifolds[1, 2].

Key words: Nijenhuis tensors, K-contact, sasakian manifolds

1. Introduction

Definition(1.1). *By contact manifold we mean a C^∞ manifold M^{2n+1} together with 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. In particular $\eta \wedge (d\eta)^n \neq 0$ is a volume element on M so that a contact manifold in orientable.*

Note that on a contact manifold M we have a characteristic vector field or Reeb vector field of the contact structure η satisfying[1,2].

(1.1) $d\eta(\xi, X) = 0, \eta(\xi) = 1.$

We have[1]

(1.2) $L_\xi \eta = 0, \quad L_\xi d\eta = 0.$

Definition (1.2). Almost contact metric structure: *A C^∞-manifold M^{2n+1} is called almost contact metric manifold with structure (φ, ξ, η, g) satisfying*

(1.3) (a) $\varphi^2 = -I + \eta \otimes \xi$, (b) $\eta(\xi) = 1$, (c) $\varphi(\xi) = 0$

(d) $\eta(\varphi X) = 0$, (e) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$

where and in the following X,Y,Z,W… etc are vector fields; unless otherwise stated[2].

*Presented at CONIAPS XI, University of Allahabad, Feb. 20-22, 2010.

Let us define following four tensors[2]

(1.4) $N^{(1)}(X,Y) = N_{\varphi}(X,Y) + 2d\eta(X,Y)\xi,$

where N_{φ} is Nijenhuis tensor corresponding to φ

(1.5) $N^{(2)}(X,Y) = (L_{\varphi X}\eta)(Y) - (L_{\varphi Y}\eta)(X),$

(1.6) $N^{(3)}(X) = (L_{\xi}\varphi)(X),$

(1.7) $N^{(4)}(X) = (L_{\xi}\eta)(X).$

Definition(1.3). *Almost contact metric structure* (φ, ξ, η, g) *is normal if and only if these four tensors vanish*[2].

It is known[1] that vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ so that the normality condition is simply

$$N_{\varphi}(X,Y) + 2\,d\eta(X,Y)\,\xi = 0.$$

Definition(1.4). *An almost contact metric manifold* (ϕ,ξ,η,g) *is called contact metric manifold if* $g(X,\phi Y) = d\eta(X,Y)$ *and structure* (ϕ,ξ,η,g) *is called contact metric structure*[1].

Note(1.1). Contact metric structure is also known as "Contact Riemannian Structure"

Remark(1.1). In Contact metric manifold following results hold :

(i) $d\eta(X,\xi) = 0,$

(ii) $\varphi[X,\xi] - [\varphi X,\xi] = (L_{\xi}\varphi)(X) = N^{(3)}(X),$

(iii) $\eta[\xi, \varphi X] = 0,$

(iv) $\varphi N_{\varphi}(X,\xi) = -N^{(3)},$

(v) $(L_{\xi}\eta)(\varphi X) = 0$,

(vi) $(L_{\xi}\eta)(\xi) = 0$,

(vii) $(L_{\xi}\eta)(X) = (L_{\xi}g)(X,\xi) = 0 = N^{4},$

(viii) $(L_{\xi X}\eta)(Y) = 2d\eta(\varphi X,Y) = 2g(\varphi X,\varphi Y) \Leftrightarrow N^{2}$ vanishes,

(ix) $$(L_{\xi X}\eta)(Y) = (L_{\varphi Y}\eta)(X) = (L_{\varphi X}\,g)(Y,\xi) - (L_{\varphi Y}\,g)(X,\xi) + g\,(Y,[\varphi X,\xi] - g\,(X)\,[\varphi Y,\xi]),$$

(x) $(L_{\varphi X}\eta)(\xi) = \eta([\varphi X,\xi]) = 0,$

see[1,2].

Remark (1.2). For a contact metric manifold (φ, ξ, η, g), $N^{(2)}$ and $N^{(4)}$ vanish.

Remark (1.3). For contact metric manifold $N^{(3)}=0 \Leftrightarrow \xi$ is Killing.

Proof. For contact metric manifold

$$\left(L_\xi d\eta\right)(X,Y)=0 \Rightarrow \left(L_\xi g\right)(X,\varphi Y)+g\left(X,\left(L_\xi\varphi\right)(Y)\right)=0$$

$$\left(L_\xi g\right)=0 \Rightarrow \xi \textit{ is Killing} \Leftrightarrow \left(L_\xi\varphi\right)(Y)=0 \Leftrightarrow N^{(3)}=0.$$

Remark (1.4). On contact metric manifold the interval curve of ξ geodesic i.e. $\nabla_\xi\xi=0$ and $\nabla_\xi\phi=0$.

Remark (1.5). On contact metric manifold we have, (see[1])

$$(\nabla_\xi h)(X)=\phi X-h^2\phi X-\phi R(X,\xi,\xi)----(1)$$

and $$\frac{1}{2}(R((\xi,X,\xi)-\phi R(\xi,\phi X,\xi))=h^2X+\phi^2X----(2)$$

from (1) we have $R(\xi,X,\xi)=h^2X+\phi^2X-\phi(\nabla_\xi h)X$

and $$\phi R(\xi,\phi X,\xi)=-h^2X-\phi^2X-\phi(\nabla_\xi h)X,$$

where $h=\frac{1}{2}L_\xi\,\varphi.$

Definition (1.5)K-Contact Structure: *A K-contact structure is a contact metric structure for which the vector field ξ is killing i.e. the symmetric operator $h=½\, L_\xi\varphi=0$.*

Definition (1.6)(Sasakian manifold): *Almost contact metric manifold (φ,ξ,η,g) is a Sasakian if and only if $\left(\nabla_X\varphi\right)(Y)=g(X,Y)\,\xi-\eta(Y)\,X$.*

In the following $R(X,Y,Z)=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$ is curvature tensor, where ∇ is covariant differentiation.

R(X,Y) will stand for Ricci tensor and $R(X,Y)=g(R(X),Y)$ where R(X) in a tensor of type (1.1). Contraction on X in R(X)gives 'r', V(X,Y,Z) stands for conformal curvature tensor given by:

$$\begin{aligned}V(X,Y,Z)=R(X,Y,Z)-\frac{1}{2n-1}\{R(Y,Z)X-R(X,Z)Y\}\\ -g(X,Z)\,R(Y)+g(X,Z)\,R(X)\}\\ +\frac{r}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned}$$

2. Some Propositions on Contact Manifold

In the following M^{2n+1} stands for contact metric manifold.

Let $h = \frac{1}{2} L_\xi \varphi$. It is known that

(2.1) $$\nabla_X \xi = -\varphi X - \varphi h X$$

where ∇ is Levicivita connection.

Proposition (2.1). *On contact metric manifold we have*

(2.2) $$2hX = [\xi, \varphi X] - \varphi[\xi, X],$$

where [,], stands for Lie bracket.

Proof. $L_\xi \varphi X = (L_\xi \varphi)\, X + \varphi(L_\xi X)$

or $[\xi, \varphi X] = 2hX + \varphi[\xi, X]$

which gives (2.2).

Proposition (2.2). *Let* N_φ *be Nijenhuis tensor in* M^{2n+1}*, then*

(2.3) $$N_\varphi(\xi, \varphi X) = -\varphi h \varphi X.$$

Proof. $N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$

$$\begin{aligned} N_\varphi(\xi, \varphi X) &= -\varphi\left[\xi, \varphi^2 X\right] + \varphi^2[\xi, \varphi X] \\ &= -\varphi\left\{\left[\xi, \varphi^2 X\right] - \varphi[\xi, \varphi X]\right\} \\ &= -\varphi\, h\, \varphi X\ . \end{aligned}$$

Hence proved.

Proposition(2.3). *On* M^{2n+1}*, we have*

(2.4) $$(\nabla_\xi N)(\xi, \bar{X}) = \varphi^2(\nabla_\xi h)\, X - \phi h\, \nabla_\xi \bar{X}\ .$$

Proof. We have

$$N(\xi, \bar{X}) = -\varphi h\, \bar{X}$$

$$\therefore (\nabla_\xi N)(\xi, \bar{X}) + N(\nabla_\xi \xi, \bar{X}) + N(\xi, \nabla_X \bar{X})$$
$$= -(\nabla_\xi \varphi)\, h\, \bar{X} - \varphi(\nabla_\xi h)(\bar{X}) - \varphi\, h\, \nabla_\xi \bar{X}$$

or $(\nabla_\xi N)(\xi, \bar{X}) + N(\xi, \nabla_\xi \bar{X}) = -(\nabla_\xi \varphi)\, h\, \bar{X}$
$$-\varphi\left\{\varphi \bar{X} - h^2\, \varphi \bar{X} - \varphi R(\bar{X}, \xi, \xi)\right\} - \varphi h\, \nabla_\xi \bar{X}\ .$$

or $(\nabla_\xi N)(\xi, \bar{X}) + N(\nabla_\xi \xi, \bar{X}) = -(\nabla_\xi \varphi)\, h\, \bar{X}$
$$-\left\{-\varphi X + h^2 \varphi X - \varphi^2 R(\bar{X}, \xi, \xi)\right\} - \varphi h\, \nabla_\xi \bar{X}$$
$$= -(\nabla_\xi \varphi)\, h\, \bar{X} - \left\{-\varphi X + h^2 \varphi X + \varphi^2 R(\xi, \bar{X}, \xi)\right\} - \varphi h\, \nabla_\xi \bar{X}\ .$$

Recalling $\nabla_\xi \varphi = 0$ and h $\varphi + \varphi h = 0$, we have

$$\left(\nabla_\xi N\right)\left(\xi, \bar{X}\right) + N\left(\xi, \nabla_X \bar{X}\right) = \varphi X - h^2 \varphi X - \varphi\left(-h^2 X - \varphi^2 X\right)$$
$$- \varphi\left(\left(\nabla_\xi h\right) X\right) - \varphi\, h\, \nabla_\xi \bar{X}$$
$$= - \varphi^2 \left(\nabla_\xi h\right)\left(X\right) - \varphi\, h\, \nabla_\xi \bar{X} .$$

Proposition (2.4). *On contact metric manifold M^{2n+1}, we have*

(2.4) (A) $$\varphi N\left(X, \xi\right) = 2\varphi^2 hX .$$

Proof. We have

$$N\left(X, \xi\right) = - \varphi\left[\varphi X, \xi\right] + \varphi^2 \left[X, \xi\right]$$
$$\varphi N\left(X, \xi\right) = \varphi^2 \left[\varphi X, \xi\right] - \varphi\left[X, \xi\right]$$
$$= -\varphi^2 \left(\nabla_{\varphi X} \xi - \nabla_\xi \varphi X\right) - \varphi\left(\nabla_X \xi - \nabla_\xi X\right)$$
$$= - \varphi^2 \left(-\varphi^2 X - \varphi h \varphi X - \left(\nabla_\xi \varphi\right)\left(X\right) - \varphi \nabla_\xi X\right)$$
$$- \varphi\left(-\varphi X - \varphi h X - \nabla_\xi X\right)$$
$$= - \varphi^2 X + \varphi^3 h \varphi X + \varphi^3 \nabla_\xi X + \varphi^2 X + \varphi^2 hX + \varphi \nabla_\xi X$$
$$= 2\varphi^2\, hX ,$$

which is (2.4) (A).

Propostion (2.5). *On contact metric manifold M^{2n+1}, we have*

(2.5) $$-\varphi N\left(X, \xi\right) + 2R\left(\xi, hX, \xi\right) = 2h^3 X - 2\varphi\left(\nabla_\xi h\right) hX .$$

Proof. Recall that[1]

$$R\left(\xi, X, \xi\right) = h^2 X + \varphi^2 X - \varphi\left(\nabla_\xi h\right) X .$$

(2.6) $$2R\left(\xi, hX, \xi\right) = + 2h^3 X + 2\varphi^2 hX - 2\varphi\left(\nabla_\xi h\right) hX .$$

From (2.4) (A) and (2.6), we have

$$2R\left(\xi, hX, \xi\right) = + 2h^3 X + \varphi N\left(X, \xi\right) - 2\varphi\left(\nabla_\xi h\right) hX .$$

which is (2.5).

Theorem (2.6). *On contact metric manifold we have*

(2.7) $$\varphi N\left(X, \xi\right) + 2\varphi R\left(\xi, \varphi hX, \xi\right) = - 2h^3 X - 2\varphi\left(\nabla_\xi h\right) hX .$$

Proof. Recall that

$$\varphi R\left(\xi, \varphi X, \xi\right) = - h^2 X - \varphi^2 X - \varphi\left(\nabla_\xi h\right) X$$

which gives

(2.8) $$2\varphi R\left(\xi,\varphi hX,\xi\right)=-2h^{3}X-2\varphi^{2}hX-2\varphi\left(\nabla_{\xi}h\right)hX\ .$$

Adding (2.4) (A) and (2.8) we get (2.7).

Corollary (2.1). *On contact metric manifold M^{2n+1}, we have*

(2.9) $$R\left(\xi,hX,\xi\right)+\varphi R\left(\xi,\varphi hX,\xi\right)=-2\varphi\left(\nabla_{\xi}h\right)hX$$

or

(2.10) $$`R\left(\xi,hX,\xi,Z\right)+`\varphi\left(R\left(\xi,\varphi hX,\xi\right),Z\right)=-2`\varphi\left(\left(\nabla_{\xi}h\right)hX,Z\right)$$

where $`R\left(X,Y,Z,W\right)=g\left(R\left(X,Y,Z\right)W\right)$ and $`\varphi\left(X,Y\right)=g\left(\varphi X,Y\right)$.

Proof. Adding (2.5)(a) and (2.7), we get (2.9), equation (2.10) immediate from (2.9).

3. Propositions on K-Contact Manifold

Let us define on M^{2n+1}, the conformal curvature tensor

$$\begin{aligned}V\left(X,Y,Z\right)=K\left(X,Y,Z\right)-\frac{1}{2n-1}&\left\{R\left(Y,Z\right)X-R\left(X,Z\right)Y\right.\\&\left.-g\left(X,Z\right)R\left(Y\right)+g\left(Y,Z\right)R\left(X\right)\right\}\\&+\frac{r}{2n\left(2n-1\right)}\left\{g\left(Y,Z\right)X-g\left(X,Z\right)Y\right\}.\end{aligned}$$

Let X be orthogonal to ξ, then on K-contact manifold $Ric\left(\xi,\xi\right)=2n$, $R\left(\xi,X,\xi\right)=\varphi^{2}X$, $g\left(X,\xi\right)=0$, or $\eta\left(X\right)=0$, $R\left(\xi\right)=2n\xi$, then

$$\begin{aligned}V\left(\xi,X,\xi\right)&=\varphi^{2}X-\frac{1}{2n-1}\left\{Ric\left(X,\xi\right)\xi-2nX-R\left(X\right)\right\}\\&\quad+\frac{r}{2n\left(2n-1\right)}\left\{-X\right\}\\&=-X+\eta\left(X\right)\xi-\frac{1}{2n-1}\left\{Ric\left(X,\xi\right)\xi-2nX-R\left(X\right)\right\}-\frac{rX}{2n\left(2n-1\right)}.\end{aligned}$$

(3.1) $$V\left(\xi,X,\xi\right)=\frac{X-Ric\left(X,\xi\right)\xi+R\left(X\right)}{2n-1}-\frac{rX}{2n\left(2n-1\right)}$$

Thus, we have

Theorem(3.1). *On a K-contact metric manifold M^{2n+1}, with a vector field ‘X’ orthogonal to ξ, we have*

$$V(\xi, X, \xi) = \frac{(2n-r)X + 2nR(X) - 2n\, Ric(X,\xi)\xi}{2n(2n-1)}$$

and

$$`V(\xi, X, \xi, \xi) = 0$$

where $`V(\xi, X, \xi, \xi) = g(V(\xi, X, \xi)\xi)$.

Corollary(3.1). *On K-contact manifold M^{2n+1}, with a vector field X, orthogonal to ξ, conformal curvature $V(\xi, X, \xi)$ vanishes if and only if*

$$Ric(X,\xi)\xi = R(X) + \frac{2n-r}{2n} X .$$

Corollary(3.2). *On K-contact metric manifold we have*

$$\phi N(X,\xi) \text{ or } `N(X,\xi,\varphi Z) = 0$$

where $$`N(X,Y,Z) = g(N(X,Y),Z).$$

Proof. Since in K-contact metric manifold h=0, we get from (2.7) that $\varphi N(X,\xi)=0$.

Theorem (3.2). *On Sasakian manifold we have* $V(Y,\xi)=0$.

Proof. Recall that an almost contact metric manifold becomes Sasakian iff $(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X$.

In a Sasakian manifold, we have

$$K(X,Y,\xi) = \eta(Y)X - \eta(X)Y,$$

$$Ric(Y,\xi) = 2n\eta(Y),$$

$$R(Y) = 2nY,$$

$$R(\xi) = Ric(\xi,\xi) = 2n\xi,$$

$$V(Y,Z) = C_X V(X,Y,Z).$$

Then by easy calculations, we get

$$V(Y,\xi) = 0 .$$

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