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On Submanifolds of Almost r – Contact Structure Manifolds

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Abstract: Almost r – contact structure was defined and studied by Vanzura[1] and several other geometers including Mishra, Pandey[2] and Imai[3]. Recently, Das, Ram Nivas, S. Ali and M. Ahmad[4] have studied quarter symmetric connections and have obtained some interesting results. In this paper, authors have studied submanifolds of an almost r – contact structure manifold. Quarter symmetric (*F, G*) – connection has also been defined and submanifolds of a manifolds with such connection have been studied. Study of (*F, G*) geodesic and (*F, G*) umbilical submanifolds is also the subject matter of this paper.

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1. Subamnifolds of Codimension 2*r*

Let M^{n+r} be on $(n + r)$ – dimensional differentiable manifold of class c^{∞}. Suppose there exists on M^{n+r} a tensor field ϕ of type (1, 1), $r(c^{\infty})$ contravariant vector fields $\xi_1, \xi_2, \ldots, \xi_r$ and $r(c^{\infty})$ 1 – forms $\eta_1, \eta_2, \ldots, \eta_r$ such that

(1.1a) $$\phi^2 = -I + \sum_{l=1}^{r} \eta_1 \otimes \xi_1,$$

(1.1b) $$\phi\xi_1 = 0,$$

(1.1c) $$\eta_l \circ \phi = 0 \quad \text{and} \quad \eta_1(\xi_m) = \delta_{lm}$$

where l, m = 1, 2, …, r and δ_{lm} denotes the Kronecker delta. Then M^{n+r} satisfying above equations (1.1) will be called an almost r – contact structure manifold[1]. If M^{n+r} is endowed with a positive definite Riemannian metric g such that

$$g(\phi X,\phi Y)=g(X,Y)-\sum_{l=1}^{r}\eta_l(X)\eta_l(Y),$$

we say that M^{n+r} admits almost r – contact metric structure.

Let M^n be an n – dimensional submanifold of the almost r – contact structure manifold M^{n+r} such that the vector fields ξ_l l = 1, 2, …, ξ_r are always tangent to M^n. Throughout this paper we will assume that the vector fields ξ_1, ξ_2, …, ξ_r are always tangent to M^n. Thus there exist r mutually orthogonal unit normals N_1, N_2, …, N_r such that if X is in the tangent space of M^n, the transformations for ϕX and ϕN_l can be written as[5]

$$\phi X = fX+\sum_{l=1}^{r}\alpha_l(X)N_l \tag{1.2}$$

where α_l, l = 1, 2, …, r are 1 – forms and f is a tensor field of type (1, 1) on the submanifold M^n. Also

$$\phi N_l=-A_l,\qquad l=1,2,\ldots,r. \tag{1.3}$$

Here A_l, l = 1, 2, …, r are c^∞ vector fields on the submanifold M^n and tangential to M^n.

Operating (1.2) by ϕ and making use of (1.2) itself and also the equations (1.1a) and (1.3), we get

$$-X+\sum_{l=1}^{r}\eta_l(X)\xi_l=f^2X+\sum_{l=1}^{r}\alpha_l(fX)N_l+\sum_{l=1}^{r}\alpha_l(X)(-A_l)$$

Comparison of vector fields tangential and normal to M^n yields

$$f^2=-I+\sum_{l=1}^{r}\{\eta_l\otimes\xi_l+\alpha_l\otimes A_l\}$$

and

$$\alpha_l\circ f=0$$

In view of the equation (1.2), we have

$$(\eta_m\circ\phi)(X)=(\eta_m\circ f)(X)+\sum_{l=1}^{r}\alpha_l(X)\eta_m(N_l)$$

Taking $\eta_m(N_l)=0$ we get

$$\eta_m\circ f=0$$

Again, in view of the equation (1.2), we have

$$\varphi(A_m) = f(A_m) + \sum_{l=1}^{r} \alpha_l(A_m) N_l .$$

Taking $\phi A_m = N_m$ and $\alpha_l(A_m) = \delta_{lm}$, we get

$$f(A_m) = 0,\ m = 1, 2, \ldots, r.$$

Again by virtue of the equation (1.2), we have

$$\phi(\xi_m) = f(\xi_m) + \sum_{l=1}^{r} \alpha_l(\xi_m) N_l .$$

Taking $\alpha_l(\xi_m) = 0$, we get

$$f(\xi_m) = 0.$$

Thus the submanifold M^n of almost r – contact structure manifold M^{n+r} admits a structure satisfying

(1.4a) $$f^2 = -I + \sum_{l=1}^{r} \{\eta_l \otimes \xi_l + \alpha_l \otimes A_l\}$$

(1.4b) $$\alpha_l \circ f = \eta_m \circ f = 0,$$

(1.4c) $$\phi\xi_l = f\xi_l = 0,$$

(1.4d) $$\alpha_l(\xi_m) = \eta_l(A_m) = 0,$$

(1.4e) $$f(A_l) = 0$$

(1.4f) $$\alpha_l(\xi_m) = \eta_l(\xi_m) = \delta_{lm};\quad l, m = 1, 2, \ldots, r.$$

We have

Theorem 1.1. *The submanifold M^n of codimension r of almost r – contact structure manifold M^{n+r} such that the vector fields ξ_l and A_l are tangents to M^n admits a structure given by the equation (1.4).*

Now let us define a (1, 1) tensor field $\tilde{f}$ on M^n as

(1.5) $$\tilde{f} = f + \sum_{l=1}^{r} \eta_l \otimes A_l .$$

Then in view of the equation (1.4), it is easy to show that

(1.6) $$\tilde{f}^2 = f^2.$$

Hence by virtue of (1.4), (1.5) and (1.6), it can be easily shown that

(1.7a) $$\tilde{f}^2 = -I + \sum_{l=1}^{r} \{\eta_l \otimes \xi_l + \alpha_l \otimes A_l\},$$

(1.7b) $$\alpha_l \circ \tilde{f} = \eta_l,$$

(1.7c) $$\eta_l \circ \tilde{f} = 0,$$

(1.7d) $\tilde{f}(\xi_l) = A_l$

(1.7e) $\tilde{f}(A_l) = 0.$

Thus we have

Theorem 1.2. *The (1, 1) tensor field* $\tilde{f}$ *defined on the submanifold* M^n *of the almost r – contact structure manifold* M^{n+r} *defines a structure on* M^n *given by the equation (1.7).*

2. Quarter Symmetric (*F, G*) Connection

As in the previous section, M^n is the submanifold of codimension r immersed differentiably in the r – contact structure manifold M^{n+r}. Let ζ be the immersion $M^n \rightarrow M^{n+r}$ and $B = d\zeta$. Hence the vector field X in the tangent space of M^n corresponds to a vector field BX tangential to M^{n+r} If $\tilde{g}$ be the Riemannian metric on M^{n+r} and g the induced metric on M^n, we have

(2.8) $\tilde{g}\ (BX, BY) = g(X, Y)$ for all X, Y tangents to M^n.

As N_x, $x = 1, 2, \ldots, r$ are mutually orthogonal unit normals to M^n

(2.9a) $\tilde{g}\ (BX, N_x) = 0$

and

(2.9b) $\tilde{g}\ (N_x, N_y) = \delta_{xy},\ x, y = 1, 2, \ldots, r$

where δ_{xy} is the Kronecker delta.

Suppose M^{n+r} admits a connection $\tilde{\nabla}$ given by

(2.10) $\tilde{\nabla}_{BX} BY = \overset{\circ}{\tilde{\nabla}}_{BX}\ BY + \tilde{\pi}(BY)\tilde{F}(BX) - \tilde{\pi}(BX)\tilde{G}(BY) - \tilde{g}(BX, BY)\tilde{P}$

where $\overset{\circ}{\tilde{\nabla}}$ is the Levi-civita connection and $\tilde{F}, \tilde{G}$ are tensor fields of type (1, 1) on M^{n+r}. We call the connection $\tilde{\nabla}$ as quarter symmetric (*F, G*) connection on M^{n+r}. Also $\tilde{P}$ is a vector field and $\tilde{\pi}$ 1 – form on the enveloping manifold M^{n+r}.

Theorem 2.1. *The connection induced on the submanifold* M^n *from the quarter symmetric (F, G) connection* $\tilde{\nabla}$ *on* M^{n+r} *is also the quarter symmetric (F, G) connection.*

Proof. Let us put

(2.11a) $\tilde{P} = B\ P + \lambda^x N_x$

(2.11b) $$\tilde{\nabla}_{BX} BY = B\nabla_X Y + \sum_{x=1}^{r} h^x(X,Y)N_x$$

(2.11c) $$\overset{\circ}{\tilde{\nabla}}_{BX}\, BY = B(\overset{\circ}{\nabla}_X\, Y) + \sum_{x=1}^{r} m^x(X,Y)N_x$$

where h^x (X, Y) and m^x (X, Y) are tensor fields of type (1, 2) and P the vector field tangential to M^n. We can also write

(2.11d) $$\tilde{F}(BX) = BFX + \sum_{x=1}^{r} \mu^x N_x$$

(2.11e) $$\tilde{G}BY = BGY + \sum_{x=1}^{r} \gamma^x N_x$$

where λ^x, μ^x, γ^x are scalars. Hence (2.10) takes the form

(2.12) $$B\nabla_X Y + \sum_{x=1}^{r} h^x(X,Y)N_x = B\overset{\circ}{\nabla}_X\, Y + \sum_{x=1}^{r} m^x(X,Y)N_x + \pi(Y)\{BFX + \sum_{x=1}^{r} \mu^x N_x\}$$
$$- \pi(X)\{BGY + \sum_{x=1}^{r} \nu^x N_x\} - g(X,Y)\{BP + \sum_{x=1}^{r} \lambda^x N_x\}.$$

Comparison of vector fields tangential to M^{n+r} gives

$$\nabla_X Y = \overset{\circ}{\nabla}_X\, Y + \pi(Y)FX - \pi(X)G(Y) - g(X,Y)P$$

which shows that ∇ is quarter symmetric (*F, G*) connection on M^n.

3. (*F, G*) Geodesic and (*F, G*) Umbilical Submanifolds

Let $\{X_1, X_2, \ldots, X_n\}$ be the set of orthonormal vector fields on M^n. We call

$$\frac{1}{nr}\sum_{x=1}^{r}\sum_{i=1}^{n} h^x(X_i, X_i)$$

the mean curvature of M^n with respect to connection ∇ and

$$\frac{1}{nr}\sum_{x=1}^{r}\sum_{i=1}^{n} m^x(X_i, X_i)$$

the mean curvature of M^n with respect to connection $\overset{\circ}{\nabla}$.

Definition 3.1. *We say that the submanifold M^n is (F, G) geodesic with respect to quarter symmetric (F, G) connection ∇ if h^x (X, Y) = 0, x =* 1, 2, *…, r.*

Definition 3.2. *We say that the submanifold M^n is (F, G) umbilical with respect to connection ∇ if h^x (X, Y) are proportional to the metric tensor g.*

Definition 3.3. *The submanifold M^n will be (F, G) geodesic or (F, G) umbilical with respect to Riemannian connection $\overset{\circ}{\nabla}$ according as m^x (X, Y) vanish or proportional to the metric tensor g.*

We now prove the following theorem.

Theorem 3.1. *In order that the mean curvature of M^n with respect to quarter-symmetric (F,G) connection ∇ may coincide with that of M^n with respect to Riemannian connection $\overset{\circ}{\nabla}$, it is necessary and sufficient that the vector fields $\tilde{F}BX_i$, $\tilde{G}BX_i$ and $\tilde{P}$ are tangential to M^n.*

Proof. In view of the equation (2.12), comparison of vector fields normal to M^n yields

$$h^x(X_i, X_i) = m^x(X_i, X_i) + \pi(X_i)\mu^x - \pi(X_i)\nu^x - \lambda^x g(X_i, X_i).$$

Hence

$$\frac{1}{nr}\sum_{x=1}^{r}\sum_{i=1}^{n} h^x(X_i, X_i) = \frac{1}{nr}\sum_{x=1}^{r}\sum_{i=1}^{n} m^x(X_i, X_i)$$

if and only if $\mu^x = \nu^x = \lambda^x = 0$ for $x = 1, 2, \ldots, r$. Hence the vector fields $\tilde{F}BX_i$, $\tilde{G}BX_i$ and $\tilde{P}$ are tangential to M^n.

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