ISSN 0974 - 9373

Vol. 14 No.3 (2010) **Journal of International Academy of Physical Sciences** pp. 323-329

A Note on Certain Sequence Space of a Non-Absolute Type*

Tanweer Jalal
Department of Mathematics
National Institute of Technology, Srinagar-190006
E-mail : tjalal@rediffmail.com

Z. U. Ahmad
Sidrah, 4-389, Noor Bagh, Dodhpur, Civil Lines, Aligarh-202001
Email: zafaruddinahmad@yahoo.com

(Received February 20, 2010)

Abstract: The main purpose of this note is to define the sequence space $l_A(p,s)$ and characterize the classes of matrices $(l_A(p,s), l_\infty)$ and $(l_A(p,s), c)$, where A is any arbitrary lower triangular matrix.
Keywords: Cesaro sequence spaces of non-absolute type, matrix transformations.
AMS Subject Classification (2000): 40C05, 46A45.

1. Introduction

Let ω be the space of all sequences real or complex. Nung[1] introduced Cesaro sequence spaces of non-absolute type, denoted by X_p and is defined as

$$X_p = \left\{ x \in w : \left(\sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{k=0}^{n} x_k \right|^p \right)^{\frac{1}{p}} < \infty \right\}, \ (1 \le p < \infty)$$

The following sequence spaces are well known:

$$l_\infty = \left\{ x : \sup_k |x_k| < \infty \right\}, \qquad c = \left\{ x : \lim_k x_k \ \ exists \right\},$$

$$c_0 = \left\{ x : \lim_k x_k = 0 \right\}, \qquad l^p = \left\{ x : \sum_k |x_k|^p < \infty \right\},$$

*Presented at CONIAPS XI, University of Allahabad, Feb. 20-22, 2010.

$$l(p)=\left\{x:\sum_{k}|x_k|^{p_k}<\infty\right\},\ p=\{p_k\}\, is\, such\, that\ p_k>0$$

$$l(p,s)=\left\{x:\sum_{k}k^{-s}|x_k|^{p_k}<\infty,\ s>0,p_k>0\right\}\quad see^{2}.$$

When s = 0, $p_k = p$ for all k, then $l(p, s)$ is the same as l^p, of course $l(p, s) = l(p)$ for s = 0.

An infinite matrix $A=(a_{nk})$ is called a factorable matrix if the non-zero entries are of the form $a_{nk} = c_k d_n$, where $\{c_k\}$ and $\{d_n\}$ are arbitrary real or complex sequences. An infinite matrix $A=(a_{nk})$ is called a lower triangular if $a_{nk} = 0$ for each $k > n$, and $a_{nn} \neq 0$ for each n.

Let $B = (b_{nk})$ be an infinite matrix of complex numbers b_{nk} $(n,k=1,2,\ldots)$ and X, Y be two subsets of the space of complex sequences. We say that the matrix B defines a matrix transformation from X into Y and denote it by $B \in (X,Y)$, if for every $x \in X$, the sequence $Bx=\{B_n(x)\} \in Y$,

where $B_n(x) = \sum_{k=0}^{\infty} b_{nk}x_k$, provided the series on the right converges.

In an attempt to generalize the space X_p and also the results of Nung[3], mentioned at the outset, Khan and Rehman[4] recently defined the following new sequence space,

$$l_A^p=\left\{x\in w: Ax=\{A_n(x)\}\in l^p; 1<p<\infty\right\},$$

where A is a lower triangular factorable infinite matrix.

We further extend this to the sequence space:

$$l_A(p,s)=\{x\in w: Ax\in l(p,s)\},$$

for any lower triangular infinite matrix $A=(a_{nk})$ such that $a_{nk} \neq 0$ $(k \leq n)$. This reduces to l_A^p when $s=0$, $p_k = p$ for all k and A a factorable matrix.

For a space X of complex sequences, $x=\{x_k\}$, the generalized Köthe-Toeplitz dual denoted by X^+ is defined by,

$$X^+=\left\{\{\alpha_k\}:\sum_{k=0}^{\infty}\alpha_k x_k\ converges\ for\ every\ x\in X\right\}.$$

In this paper we determine the generalized Köthe-Toeplitz dual of the space $l_A(p,s)$ where $A=(a_{nk})$ is any arbitrary lower triangular matrix with non-zero entries and characterize the matrices of classes $(l_A(p,s), l_\infty)$ and $(l_A(p,s), c)$. These results yield the results of Khan and Rehman[4] as a special case when the matrix A is factorable, $s = 0$ and $p_k = p$ for all k and hence also generalize and extend the results of Nung[4].

The following results are pertinent for the proof of our theorems:

Lemma[2] 1: *i) Let* $1 < p_k \le \sup p_k = H < \infty$ *for every* $k \in N$, *then* $B \in (l(p,s), l_\infty)$ *if and only if there exists an integer D > 1 such that*

$$\text{(1.1)} \qquad \sup_n \sum_{k=1}^{\infty} \left| b_{nk} \right|^{q_k} D^{-q_k}\, k^{s\,(q_k - 1)} < \infty$$

where $\dfrac{1}{p_k} + \dfrac{1}{q_k} = 1$

ii) If $0 < m = \inf_k p_k \le p_k \le 1$ *for each* $k \in N$, *then* $B \in (l_A(p,s), l_\infty)$ *if and only if*

$$\text{(1.2)} \qquad K = \sup_{n,k} \left\{ \left| b_{nk} \right|^{p_k} k^s \right\} < \infty$$

Lemma[2] 2 : *i) Let* $1 < p_k \le \sup_k p_k = H < \infty$, *for every* $k \in N$, *then* $B \in (l(p,s), c)$ *if and only if together with (1.1) the condition*

$$\text{(1.3)} \qquad b_{nk} \to \beta_k \ (n \to \infty, k \text{ fixed}) \text{ holds.}$$

ii) Let $0 < m = \inf_k p_k \le p_k \le 1$ *for each* $k \in N$, *then* $B \in (l(p,s), c)$ *if and only if the conditions (1.2) and (1.3) hold.*

2. Köthe-Toeplitz dual of $l_A(p,s)$

In the sequel we assume that $A = (a_{nk})$ is a lower triangular with non-zero entries.

Suppose that $x = (x_k) \in l_A(p,s)$ and $y = (y_k) \in l_A^+(p,s)$, where $p_k > 0$. By Abel's transformation, we have

$$\sum_{k=0}^{\infty} x_k y_k = \sum_{k=0}^{n-1}\left(\sum_{i=0}^{k} a_{ni} x_i\right)\left(\frac{y_k}{a_{nk}} - \frac{y_{k+1}}{a_{n,k+1}}\right) + \left(\sum_{i=0}^{n} a_{ni} x_i\right)\frac{y_n}{a_{nn}} = \sum_{k=0}^{n} b_{nk} t_k, \ say$$

where,

$$b_{nk} = \begin{cases} \dfrac{y_k}{a_{nk}} - \dfrac{y_{k+1}}{a_{n,k+1}}, \ when \ 0 \le k \le (n-1) \\ \dfrac{y_n}{a_{nn}}, \qquad when \ k = n \\ 0, \qquad when \ k > n \end{cases}$$

and

$$t_k = \sum_{i=1}^{k} a_{ni} x_i .$$

Let $B = (b_{nk})$. Then for every $x \in l_A(p,s)$ and $t = t_k \in l(p,s)$, we have Bt $\in c$. Then by using Lemma 2, B: $l(p,s) \to c$ if and only if conditions (1.1) together with (1.3) hold for $1 < p_k \le \sup_k p_k = H < \infty$, for every $k \in N$, or conditions (1.2) and (1.3) hold for $0 < p_k \le 1$.

Thus we have

Theorem 1: (a) *Let* $1 < p_k \le \sup_k p_k = H < \infty$, *for every* $k \in N$ *Then the Köthe- Toeplitz dual* $[l_A(p,s)]^+$ *of* $l_A(p,s)$ *is the space of all sequences y such that*

$$\text{(1.4)} \qquad \sup_n \left\{ \sum_{k=0}^{n-1} \left| \left(\frac{y_k}{a_{nk}} - \frac{y_{k+1}}{a_{n,k+1}} \right) \right|^{q_k} D^{-q_k} k^{s(q_k - 1)} + \left| \frac{y_n}{a_{nn}} \right|^{q_n} D^{-q_n} n^{s(q_n - 1)}; n \ge 1 \right\} \le \infty,$$

where $\dfrac{1}{p_k} + \dfrac{1}{q_k} = 1.$

(b) Let $0 < m < \inf_k p_k \leq p_k \leq 1$*, for every* $k \in N$*. Then the Köthe -Toeplitz dual* $[l_A(p,s)]^+$ *of* $l_A(p,s)$ *is the space of all sequences y such that*

$$(1.5) \qquad \sup_{n,k}\left\{\left|\left(\frac{y_k}{a_{nk}}-\frac{y_{k+1}}{a_{n,k+1}}\right)\right|^{p_k} k^{-s}\right\} < \infty.$$

Remark 1: We may split the condition (1.4) into two as follows:

$$(1.6) \qquad \sup_n\left\{\left|\frac{y_n}{a_{nn}}\right|^{q_n} D^{-q_n} n^{s(q_n-1)};\, n \geq 0\right\} < \infty,$$

and

$$(1.7) \qquad \sup_n\left\{\sum_{k=0}^{n-1}\left|\left(\frac{y_k}{a_{nk}}-\frac{y_{k+1}}{a_{n,k+1}}\right)\right|^{q_k} D^{-q_k} k^{s\,(q_k-1)};\, n \geq 1\right\} < \infty.$$

1. Characterization of the matrix classes $(l_A(p,s), l_\infty)$ and $(l_A(p,s), c)$

Theorem 2: *Let* $1 < p_k < \infty$ *and* $\frac{1}{p_k}+\frac{1}{q_k}=1$*. Then M = ($m_{nk}$)* $\in$ $(l_A(p,s), l_\infty)$ *if and only if there exists an integer D >1 such that*

$$(3.1) \qquad \sup_n\left\{\sum_{k=0}^{\infty}\left|\left(\frac{m_{nk}}{a_{n,k}}-\frac{m_{n,k+1}}{a_{n,k+1}}\right)\right|^{q_k} D^{-q_k} k^{s\,(q_k-1)}\right\} < \infty$$

and

$$(3.2) \qquad \sup_{n,k}\left\{\left|\left(\frac{m_{nk}}{a_{kk}}\right)\right|^{q_k} D^{-q_k} k^{s(q_k-1)}\right\} < \infty.$$

Theorem 3: *Let* $1 < p_k < \infty$ *and* $\frac{1}{p_k}+\frac{1}{q_k}=1$*. Then M = ($m_{nk}$)* $\in$ $(l_A(p,s), c)$ *if and only if there exists an integer D>1 such that (3.1), (3.2) and (3.3) holds.*

(3.3) $\qquad \lim_n m_{nk}$ *exists for each k.*

Proof of Theorem 2: Since M = (m_{nk}) ϵ $\left(l_A(p,s), l_\infty\right)$ for each n=0,1,2,……,m_{nk} is in the Köthe-Toeplitz dual of $l_A(p,s) \Leftrightarrow B \in l(p,s), l_\infty \Leftrightarrow (1.1)$ holds. Thus the proof follows immediately by substituting $y_k = m_{n,k}$ in b_{nk} and observing that the condition can be split into two.

Proof of Theorem 3: Conditions (3.1) and (3.2) follow from Theorem 2 and (3.3) follows from (1.1).

Remark 2: When the matrix A = (a_{nk}) is a triangular factorable matrix i.e. $a_{nk} = c_k d_n$, then the matrix b_{nk} takes the form

$$b_{nk} = \begin{cases} \dfrac{1}{d_k}\left(\dfrac{y_k}{c_k} - \dfrac{y_{k+1}}{c_{k+1}}\right), & when\ 0 \le k \le (n-1) \\ \dfrac{y_n}{d_n c_n}, & when\ k = n \\ 0, & when\ k > n \end{cases}$$

and the conditions (3.1) and (3.2) of Theorem 2 and 3 will become

$$\sup_n \left\{ \sum_{k=0}^{\infty} \left| \frac{1}{d_k}\left(\frac{m_{nk}}{c_k} - \frac{m_{n,k+1}}{c_{k+1}} \right) \right|^{q_k} D^{-q_k} k^{s(q_k-1)} \right\} < \infty, \tag{3.4}$$

and

$$\sup_{n,k} \left| \frac{m_{nk}}{d_k c_k} \right| D^{-q_k} k^{s(q_k-1)} < \infty. \tag{3.5}$$

Remark 3: If s = 0, $p_k = p$ for all p, $c_k = t_k$ and $d_n = \dfrac{1}{Q_n}$, $Q_n = \sum_{k=0}^{n} t_k$, then we obtain corresponding results for weighted means, which is the same as given in Remark1 of Khan and Rehman[4].

Remark 4: If $c_k = 1, k = 0,1,2,....$ and $d_n = \dfrac{1}{n+1}, n = 0,1,.....$ Then we get results more general than Theorems 1 and 2 of Khan and Rehman[4].

References

1. Ng Peng – Nung : On modular space of a non – absolute type, *Nanta Math*., **2** (1978) 84 – 98.
2. E. Bulut and Ö. Çakar, The sequence space $l(p,s)$ and related matrix transformations, communications de la Faculte des Sciences de l'universite', *Ankara*, **28** (1979) 33 – 44.
3. Ng Peng – Nung : Matrix Transformations on Cesaro sequence spaces of a non – absolute type, *Tamkang J. Math*., **10 (2)** (1979) 214 – 221.
4. F. M. Khan and M. F. Rahman : Matrix Transformations on Cesaro sequence spaces of a non – absolute type, *J. Analysis*, **4** (1996) 97 – 101.