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Symmetries of the Generalized Lagrange Metric and Corresponding Finsler Metric

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1. Introduction

Applying the geometrical theory of the generalized Lagrange spaces Miron and Kawaguchi[1], Miron and Anastasiei[2,3] have studied the gravitational and electromagnetic fields in an optic medium endowed the Synge metric[4]

$$g_{ij}(x,V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x,V(x))}\right) y_i y_j, \tag{1.1}$$

where $\gamma_{ij}(x)$ is a Lorentz metric on the base manifold M, $x = (x^i)$ is a generic particle, $V^i(x)$ it's velocity and $\eta(x, V(x)) \geq 1$ is the refractive index.

Using the geometrical theory of the relativistic optics (Miron and Kawaguchi)[1] and the interesting properties for the Lie derivatives of the metric of the Generalized Lagrange spaces (Miron and Anastasiei[2,3] and Miron[5]) established by Yawata[6], the following results are proved by R. Miron, M. C. Chaki and B. Barua[7]

(a) Any symmetry of the Lorentz metric $\gamma_{ij}(x)$ and the refractive index $\eta(x, V(x))$ is symmetry of the Synge metric $g_{ij}(x,V(x))$.

(b) If the optic medium is non dispersive then the result (a) and it's converse are true.

In this paper we replace the Lorentz metric $\gamma_{ij}(x)$ with the Finsler metric $a_{ij}(x, y)$ and prove the same result as in (a) and (b) for the generalized Lagrange metric given by

$$g_{ij}(x,y) = a_{ij}(x) + \left(1 - \frac{1}{\eta^2(x,y)}\right) y_i y_j .$$

2. Preliminaries

Let M be an n-dimensional manifold, F(x, y) be a Finsler metric function on M then $F^n = (M, F)$ is called a Finsler space of dimension n. The metric tensor $a_{ij}(x, y)$ of the Finsler space F^n is given by

(2.1) $$a_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}.$$

Since F is positively homogeneous of degree one in y^i therefore $a_{ij}(x, y)$ is positively homgeneousof degree zero. Thus we have

(2.2) $$\frac{\partial a_{ij}}{\partial y^k} y^k = \frac{\partial a_{ij}}{\partial y^k} y^i = \frac{\partial a_{ij}}{\partial y^k} y^j = 0 \quad \text{and}$$

(2.3) $$F^2(x, y) = a_{ij}(x, y)\, y^i y^j.$$

We consider the generalized Lagrange metric given by

(2.4) $$g_{ij}(x,y) = a_{ij}(x) + \left(1 - \frac{1}{\eta^2(x,y)}\right) y_i y_j$$

wher y_i is the covariant vector field given by

(2.5) $$y_i = a_{ij}(x, y)\, y^j \quad \text{and} \quad \eta(x, y) \geq 1.$$

If $a_{ij}(x, y)$ in (2.4) become a pseudo Riemannian metric $\gamma_{ij}(x)$ and the dimension of the base manifold M is 4, the restriction of the metric (2.4) to the local section

(2.6) $S_v: M \rightarrow TM$ defined by

$S_v: x^i = x^i, \quad y^i = V^i(x)$

gives the Synge metric. Therefore in this case the d-tensor field $g_{ij}(x, y)$ given in (2.4) reduces to the metric (1.1) which has been called the Synge metric[7] on TM.

Some important properties related to the space $GL^n = (M, g_{ij}(x, y))$ where $g_{ij}(x, y)$ is given by (2.4) is given below:

(1) The space GL^n is not reducible to a Lagrange, a Finsler or a Riemannian space.

(2) The metric tensor g_{ij} is regular i.e.

(2.7) $$\operatorname{rank} \|g_{ij}(x, y)\| = n = \dim M$$

(3) The contravariant tensor $g^{ij}(x, y)$ corresponding to $g_{ij}(x, y)$ is given by

(2.8) $$g^{ij}(x,y) = a^{ij}(x) - \frac{1}{\sigma(x,y)}\left(1 - \frac{1}{\eta^2(x,y)}\right) y^i y^j ,$$

where

(2.9) $$\sigma(x,y) = 1 + \left(1 - \frac{1}{\eta^2(x,y)}\right) F^2 .$$

Let $v^i(x)$ be a local vector field in M. Then it defines an infinitesimal transformation T_v on the tangent bundle TM given by

(2.10) $$T_v\colon\ \bar{x}^i + v^i(x)dt \quad \text{and} \quad T_v\colon\ \bar{y}^i + y^j \partial_j v^i dt$$

where $\partial_j = \dfrac{\partial}{\partial x^j}$ and *dt* is an infinitesimal constant.

Definition (2.1): The Lie derivative of the tensor field K^i_j of type (1, 1) in the manifold M is defined by (Yawata[6], H.Rund[8])

(2.11) $$L_v K^i_j = \partial_r K^i_j v^r + \dot{\partial}_h K^i_j \partial_r v^h y^r - K^r_j \partial_r v^i + K^i_r \partial_j v^r$$

where $$\dot{\partial}_j = \frac{\partial}{\partial y^j} .$$

Regarding the transformation (2.10) the following properties has been established by Yawata[6]

The transformation T_v which preserve a d-tensor field $g_{ij}(x, y)$ are given by the equation

(2.12) $$L_v g_{ij}(x, y) = 0 \quad \text{and}$$

(2.13) $$L_v g_{ij}(x, y) = \theta_v g_{ij}(x, y) + g_{hj}\frac{\partial v^h}{\partial x^i} + g_{ih}\frac{\partial v^h}{\partial x^j}$$

The operator θ_v is defined by

(2.14) $$\theta_v = v^h \partial_h + y^h \frac{\partial v^i}{\partial x^h}\frac{\partial}{\partial y^i} .$$

It is to be noted that Lv possesses the following well known properties of Lie derivative:

(i) It is a R-linear operator;

(ii) It satisfies the Libnitz rule with respect to a tensor product;

(iii) It commutes the operation of contraction;

(iv) It commutes the operation of partial derivatives with respect to y^k i.e.

(2.15) $$L_v(\dot{\partial}_k K^i_j) = \dot{\partial}_k(L_v K^i_j) ,$$

(v) for a scalar field η(x, y) it satisfies;

(2.16) $$L_v\, \eta(x, y) = \theta_v\, \eta(x, y)$$

(vi) and lastly

(2.16) $$L_v y^i = 0.$$

3. Remarkable Symmetries of the space GL^n

In this section the symmetries of the generalized Lagrange space GL^n endowed with the metric (2.4) has been studied.

Difinition(3.1): An infinitesimal transformation T_v on TM is called a symmetry of a geometric object Ω(x, y) if T_v is an automorphism of this object.

Applying the result of Yawata[6] we can therefore say that T_v preserve a geometric object Ω(x, y) if and only if LvΩ(x, y) vanishes. Hence we have:

Theorem (3.1): *The infinitesimal transformation T_v on TM is a symmetry of the generalized Lagrange metric $g_{ij}(x, y)$ if and only if*

(3.1) $$L_v\, g_{ij}(x, y) = \theta v\, g_{ij}(x, y) + g_{hj}\frac{\partial v^h}{\partial x^i} + g_{ih}\frac{\partial v^h}{\partial x^j} = 0.$$

Some remarkable symmetries are given in the following:

Corollary 1: *Any symmetric of the generalized Lagrange metric $g_{ij}(x, y)$ is symmetry of the contravariant tensor $g^{ij}(x, y)$.*

Corollary 2: *Any infinitesimal transformation T_v is a symmetry of the vector field y^i.*

Corollary 3: *If an infinitesimal transformation T_v is a symmetry of the metric $g_{ij}(x, y)$*

Then it is asymmetry of the absolute energy E(x, y) defined by

(3.2) $$E(x, y) = g_{ij}(x, y)\, y^i y^j.$$

From (2.5) and (2.17) we have

(3.3) $L_v y_i = (L_v a_{ij})\, y^j$

hence from (2.4) we have

$$(3.3)\qquad L_v\, g_{ij}(x,y) = L_v\, a_{ij}(x,y) - \left(Lv \frac{1}{\eta^2(x,y)}\right) y_i y_j + \left(1 - \frac{1}{\eta^2(x,y)}\right) Lv(y_i y_j).$$

Putting

$$(3.5)\qquad u^2(x,y) = \frac{1}{\eta^2(x,y)}$$

and taking account of (3.3), (3.4) and (3.5) we have the following:

Theorem (3.2): *The Lie derivative of the generalized Lagrange metric can be expressed as*

$$(3.6)\qquad L_v\, g_{ij}(x,y) = L_v\, a_{ij}(x,y) - (L_v u^2(x,y)) y_i y_j + (1- u^2(x,y)) \{(L_v\, a_{ih}) y_j + (L_v\, a_{jh})\, y_i\} y_h.$$

In virtue of (3.5) and (3.6) we have the following:

Theorem (3.3): *Any symmetry T_v of the Finsler metric $a_{ij}(x, y)$ and of the refractive index $\eta(x, y)$ is also a symmetry of the generalized Lagrange metric $g_{ij}(x, y)$.*

The converse of the above theorem is not in general true. For converse, let us assume that the medium be non dispersive (Miron and Kawaguchi)[1] i.e. the refractive index $\eta(x, y)$ does not depend on the directional variable y^i so that $\dot{\partial}_j \eta(x, y) = 0$.

We shall state and prove the main result:

Theorem (3.4): *Any infinitesimal transformation T_v on TM is a symmetry of the generalized Lagrange metric $g_{ij}(x, y)$ of a nondispersive medium M if and only if T_v is a symmetry of the Finsler metric $a_{ij}(x, y)$ and of the refractive index $\eta(x)$.*

Proof: First suppose that $L_v\, a_{ij}(x, y) = 0$ and $L_v\, (\eta(x)) = 0$. Then from theorem (3.3) it follows that $L_v\, g_{ij}(x, y) = 0$.

To prove the converse, put

(3.7) $L_v\, a_{ij}(x, y) = \alpha_{ij}(x, y)$, $\alpha_{ij}(x, y)\, yiyj = \alpha^2(x, y)$, $L_v u^2(x) = b(x)$.

Equation (2.2), the commutation formula (2.15) and equation (3.7) give the following:

(3.7)′ $$\frac{\partial \alpha_{ij}}{\partial y^k} y^k = \frac{\partial \alpha_{ij}}{\partial y^k} y^i = \frac{\partial \alpha_{ij}}{\partial y^k} y^j = 0.$$

Then by virtue of the Killing equation $L_v\, g_{ij}(x, y) = 0$ the equation (3.6) takes the form

(3.8) $$\alpha_{ij}(x, y) - b(x)\, y_i\, y_j + (1-u^2)\, [y_j\, \alpha_{ih}(x, y) + y_i\, \alpha_{jh}(x, y)]\, y^h = 0.$$

Contracting (3.8) by $y^i y^j$, we get

(3.9) $$\alpha^2 - bF^4 + 2(1-u^2)\, \alpha^2 F^2 = 0.$$

Again contracting (3.8) with $a^{ij}(x, y)$ we get

(3.10) $$c^2(x, y) - bF^2 + 2(1-u^2)\, \alpha^2 = 0, \quad c^2 = a^{ij}\, \alpha_{ij}.$$

On differentiation of (2.5) with respect to y^k and use of (2.2) give $\frac{\partial y_i}{\partial y^k} = a_{ik}$.

Hence differentiating (3.8) with respect to y^k we get

(3.11) $$\frac{\partial \alpha_{ij}}{\partial y^k} - b\, (a_{ik}\, y_j + a_{jk}\, y_i) + (1-u^2)\, [a_{jk}\, \alpha_{ih} + a_{ik}\, \alpha_{jh}]\, y^h + (1-u^2)$$

$$[\, y_j\, \alpha_{ik} + y_i\, \alpha_{jk}\,] + (1-u^2)\, [y_j \frac{\partial \alpha_{ih}}{\partial y^k} + y_i \frac{\partial \alpha_{jk}}{\partial y^k}]\, y^h = 0.$$

Contracting above with $y^i y^j y^k$ and using (2.3), (3.7) and (3.7)′ we get

(3.11) $$bF^2 - 2(1-u^2)\, \alpha^2 = 0.$$

From (3.9), (3.10) and (3.12) it follows:

(3.12) $$\alpha(x, y) = 0, \quad b(x) = 0 \text{ and } c^2 = 0.$$

Hence using above in (3.11) we get

(3.14) $$\frac{\partial \alpha_{ij}}{\partial y^k} + (1-u^2)\, [a_{jk}\, \alpha_{ih} + a_{ik}\, \alpha_{jh}]\, y^h + (1-u^2)\, [\, y_j\, \alpha_{ik} + y_i\, \alpha_{jk}\,]$$

$$+ (1-u^2)\, [y_j \frac{\partial \alpha_{ih}}{\partial y^k} + y_i \frac{\partial \alpha_{jk}}{\partial y^k}]\, y^h = 0.$$

Contracting (3.14) by y^k and using (3.7)′ we have

(3.15) $$[y_j\, \alpha_{ih} + y_i\, \alpha_{jh}\,]\, y^h = 0.$$

Again contracting (3.15) by y^j and using (3.13) we have

$$\alpha_{ih}\, y^h = 0.$$

Differentiating this equation with respect to y^k and using (3.7)′ we get

$$\alpha_{ik} = 0.$$

Hence $L_v g_{ij} = 0$, gives $L_v a_{ij} = 0$ and $L_v u^2(x) = \frac{1}{\eta^2(x)}$

hence the theorem.

We can conclude that for non dispersive media the symmetries of the generalized Lagrange space GL^n endowed with the generalized Lagrange metric can be studied only by symmetries of the Finsler metric $a_{ij}(x, y)$ and of the refractive index.

Similarly we can prove:

Theorem 3.5: *For a non dispersive medium any symmetry of the absolute energy E(x, y) if a symmetry of the Finsler metric function* $F^2 = a_{ij}(x, y) y^i y^j$ *and for the refractive index and conversely*.

References

1. R. Miron, and T. Kawaguchi, Relativistic Geometrical Optics, *Int. Journ. Theo. Phys.,* **30(11)**(1991)1521.
2. R. Miron, and Anastesiei, Vecror bundles, Lagrange spaces with Application to Relativity, *Geometry Balkanpress.* **1**(1997).
3. R. Miron, and Anastesiei, The geometry of Lagrange spaces: *Theory and Application. Kluwar Acad., Publ.,* 59, FTPH (1994).
4. J. L. Synge, Relativity. The General Theory. *North Holland Publ. Com., Amsterdam.*
5. R. Miron, The geometry of higher order Lagrange spaces. *App. to mechanics and Physics, Kluwar Acad., Publ.,* 82, FTPH (1997).
6. M. Yawata, Killing equations in tangent bundle in the Volume Lagrange and Finsler Geometry. *Kluwar Acad., Publ.,* **76,189**, FTPH (1997).
7. R. Miron, M. C. Chaki, and B. Barua, Symmetries of the Synge metric in the relativistic optics, *Bull. Cal. Math. Soc.* **92 (3**)(2000) 219-224.
8. H. Rund, The Differential Geometry Of Finsler spaces. *Springer Verlag* (1959).
9. M. C. Chaki, Tensor Calculus, *2nd Ed. Calcutta Publishers, Calcutta*.