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On $\tilde{K}$ -Curvature Inheritance in a Finsler Space

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Abstract: $\tilde{K}$-Curvature inheritance and Projective $\tilde{K}$-Curvature inheritance have been discussed by S. P. Singh and J. K. Gatoto[1,2]. They obtained several theorems on such transformations, especially generated by contra and concurrent vector fields. The aim of the present paper is to discuss $\tilde{K}$-Curvature inheritance and Projective $\tilde{K}$-Curvature inheritance in a Finsler Space and to generalize the theorems of S. P. Singh and J. K. Gatoto.

Keywords: $\tilde{K}$-Curvature inheritance, Projective $\tilde{K}$-Curvature inheritance, contra and concurrent vector fields.

Mathematics Subject Classification 2000: 53B40.

1. Preliminaries

Let F_n be an n-dimensional Finsler space equipped with symmetric connection coefficients $\Gamma^i_{jk}(x,\xi)$. The covariant derivative of a tensor field T^i_j with respect to connection coefficients Γ^i_{jk} is given by

(1.1) $$T^i_{j;k} = \partial_k T^i_j + \left(\dot{\partial}_h T^i_j\right)\partial_k \xi^h + T^h_j \Gamma^i_{hk} - T^i_h \Gamma^h_{jk},$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and $\dot{\partial}_k = \dfrac{\partial}{\partial \dot{x}^k}$.

The commutation formula for such covariant derivative is given by

(1.2) $$X^i_{;hk} - X^i_{;kh} = \tilde{K}^i_{jhk}(x,\xi) X^j,$$

where

(1.3) $$\tilde{K}^i_{jhk}(x,\xi) = \left(\partial_k \Gamma^i_{jh} + (\dot{\partial}_l \Gamma^i_{jh})\,\partial_k \xi^l\right) - \left(\partial_h \Gamma^i_{jk} + (\dot{\partial}_l \Gamma^i_{jk})\partial_h \xi^l\right) + \Gamma^i_{mk}\Gamma^m_{jh} - \Gamma^i_{mh}\Gamma^m_{jk}.$$

This tensor is called relative curvature tensor, since it depends on partial derivatives of the field $\xi^m(x^k)$ with respect to x^h [Rund[3]]. The relative curvature tensor $\tilde{K}^i_{jhk}(x,\xi)$ satisfies the following:

(1.4) $(a)\quad \tilde{K}^i_{jhk} = -\tilde{K}^i_{jkh} \qquad (b)\ \tilde{K}^i_{jhk;m} + \tilde{K}^i_{jmh;k} + \tilde{K}^i_{jkm;h} = 0$.

The associate tensor of the relative curvature tensor is defined as

(1.5) $$g_{jm}\,\tilde{K}^m_{ikh} = \tilde{K}_{ijkh} \quad .$$

This tensor satisfies

(1.6) $$\tilde{K}_{jihk}\dot{x}^i = -\tilde{K}_{ijhk}\dot{x}^i \ ,$$

The Lie-derivative of a tensor field T^i_j with respect to the infinitesimal transformation

(1.7) $$\bar{x}^i = x^i + \varepsilon\, v^i(x^j) \quad ,$$

where $v^i(x^j)$ is a contravariant vector field which depends upon position co-ordinates only and ε is an infinitesimal constant, is given by

(1.8) $$\pounds\, T^i_j = T^i_{j;h} v^h - T^h_j v^i_{;h} + T^i_h v^h_{;j} \ ,$$

The Lie-derivative of the connection coefficients $\Gamma^i_{jk}(x,\xi)$ is given by

(1.9) $$\pounds\,\Gamma^i_{jk} = v^i_{;jk} + \tilde{K}^i_{jkh} v^h \ ,$$

The commutation formulae for Lie-differentiation and covariant differentiation are given by

(1.10) $$\pounds(T^i_{j;k}) - \left(\pounds T^i_j\right)_{;k} = T^h_j\,\pounds\Gamma^i_{hk} - T^i_h\,\pounds\Gamma^h_{jk} \ ,$$

and

(1.11) $$(\pounds\,\Gamma^i_{jk})_{;h} - \left(\pounds\,\Gamma^i_{jh}\right)_{;k} = \pounds\,\tilde{K}^i_{jkh} \ ,$$

2. $\tilde{K}$-Curvature inheritance

An infinitesimal transformation

(2.1) $$\bar{x}^i = x^i + \varepsilon\, v^i(x^j) \quad ,$$

is called a $\tilde{K}$-Curvature inheritance if the Lie-derivative of the relative curvature tensor $\tilde{K}^i_{jhk}$ is proportional to itself , i.e.

(2.2) $$£\tilde{K}^i_{jhk} = \alpha(x)\tilde{K}^i_{jhk} \quad ,$$

where $\alpha(x)$ is a non-zero scalar field[1,2] depending on x^i. If we use the terminology of P. N. Pandey[4], $\tilde{K}$-Curvature inheritance may be called as $\tilde{K}$- Lie-recurrence. Obviously, in the above definition, the relative curvature tensor is assumed to be non-zero.

The infinitesimal transformation (2.1) is called a $\tilde{K}$-Curvature collineation if

(2.3) $$£\tilde{K}^i_{jkh} = 0 \; .$$

Thus, we see that a $\tilde{K}$-Curvature inheritance cannot be a curvature collineation.

In other words, we may say that the sets of $\tilde{K}$-Curvature inheritances and $\tilde{K}$-Curvature collineations are disjoint.

The necessary and sufficient condition for the infinitesimal transformation (2.1) to be an affine motion is given by

(2.4) $$£\,\Gamma^i_{jk} = 0 \; .$$

In view of (1.11), (2.4) gives (2.3). Thus an affine motion is a $\tilde{K}$-Curvature collineation. Hence an affine motion cannot be a $\tilde{K}$-Curvature inheritance. J. K. Gatoto and S. P. Singh[2] studied a $\tilde{K}$-Curvature inheritance which is also an affine motion. Obviously, there exists no such $\tilde{K}$-Curvature inheritance, and therefore Theorem (1.1) and Lemma (1.1) of J. K. Gatoto and S. P. Singh[2] are meaningless. Since every homothetic transformation is an affine motion, a homothetic transformation cannot be a $\tilde{K}$-Curvature inheritance. Thus, Theorem (1.3) of Gatoto and Singh[2] is misleading.

Let us consider a recurrent space characterised by

(2.5) $$\tilde{K}^i_{jkh;m} = \lambda_m \tilde{K}^i_{jkh} ,$$

where λ_m is a non- zero covariant vector field and $\tilde{K}^i_{jkm} \neq 0$. The vector field λ_m is called the recurrence vector. If it admits the $\tilde{K}$-Curvature inheritance (2.1), we have (2.2).

Differentiating (2.2) covariantly with respect to x^m, we have

(2.6) $$(£\,\tilde{K}^i_{jkh})_{;m} = (\alpha_{;m} + \alpha\,\lambda_m)\,\tilde{K}^i_{jkh} \; ,$$

while operating £ on both sides of (2.5), we get

(2.7) $$£(\tilde{K}^i_{jkh;m}) = (£\lambda_m + \alpha\lambda_m)\tilde{K}^i_{jkh}.$$

From (2.6) and (2.7), we find

(2.8) $$(£\tilde{K}^i_{jkh})_{;m} - £(\tilde{K}^i_{jkh;m}) = (\alpha_{;m} - £\lambda_m)\tilde{K}^i_{jkh}.$$

From (2.8), we conclude

Theorem 2.1: *In a recurrent space, $\tilde{K}$-Curvature inheritance and covariant differentiation for the connection $\Gamma^i_{jk}(x,\xi)$ commute if and only if* $£\lambda_m = \alpha_{;m}$.

In view of (1.2), we have

(2.9) $$\tilde{K}^i_{jkh;ml} - \tilde{K}^i_{jkh;lm} = \tilde{K}^r_{jkh}\tilde{K}^i_{rml} - \tilde{K}^i_{rkh}\tilde{K}^r_{jml} - \tilde{K}^i_{jrh}\tilde{K}^r_{kml} - \tilde{K}^i_{jkr}\tilde{K}^r_{hml}.$$

From (2.5), we have

(2.10) $$A_{ml}\tilde{K}^i_{jkh} = \tilde{K}^r_{jkh}\tilde{K}^i_{rml} - \tilde{K}^i_{rkh}\tilde{K}^r_{jml} - \tilde{K}^i_{jrh}\tilde{K}^r_{kml} - \tilde{K}^i_{jkr}\tilde{K}^r_{hml},$$

where $A_{ml} = \lambda_{m;l} - \lambda_{l;m}$.

Differentiating (2.10) covariantly and using (2.5), we get

$$(A_{ml;p} + \lambda_p A_{ml})\tilde{K}^i_{jkh} = 2\lambda_p(\tilde{K}^r_{jkh}\tilde{K}^i_{rml} - \tilde{K}^i_{rkh}\tilde{K}^r_{jml} - \tilde{K}^i_{jrh}\tilde{K}^r_{kml} - \tilde{K}^i_{jkr}\tilde{K}^r_{hml}),$$

which in view of (2.10), gives

(2.11) $$A_{lm;p} = \lambda_p A_{lm}.$$

From (1.4 b) and (2.5), we have

(2.12) $$\lambda_m\tilde{K}^i_{jkh} + \lambda_h\tilde{K}^i_{jmk} + \lambda_k\tilde{K}^i_{jhm} = 0.$$

Multiplying (2.10) by λ_p and taking skew-symmetric part with respect to the indices *l, m* and *p,* we have

(2.13) $$\lambda_p A_{lm} + \lambda_m A_{pl} + \lambda_l A_{mp} = 0,$$

due to (2.12).

Thus, we have

Theorem 2.2: *The tensor $A_{ml}\ (=\lambda_{m;l} - \lambda_{l;m})$ is recurrent in a $\tilde{K}$-recurrent Finsler space and satisfies the identity (2.13).*

Operating (2.10) by the operator of Lie-differentiation and using (2.2), we get

(2.14) $$\pounds\, A_{lm} = \alpha\, A_{lm} \ .$$

This leads to

Theorem 2.3: *The tensor* A_{lm} *is Lie-recurrent with respect to a* $\tilde{K}$-*Curvature inheritance.*

J. K. Gatoto and S. P. Singh[2] assumed $\pounds\, A_{lm} = -\,\alpha\, A_{lm}$ in proving Theorem (1.4), which cannot be true in view of the above theorem. Theorem (1.5) of Gatoto and Singh[2] states that a general recurrent Finsler space does not admit a $\tilde{K}$-Curvature inheritance if it becomes an affine motion. In fact, there is no Finsler space admitting $\tilde{K}$-Curvature inheritance which is an affine motion.

3. Projective $\tilde{K}$-Curvature inheritance

A $\tilde{K}$-Curvature inheritance is called a Projective $\tilde{K}$-Curvature inheritance if it is also a Projective motion[1].

The necessary and sufficient condition for an infinitesimal transformation to be a Projective motion is given by

(3.1) $$\pounds\, \Gamma^{i}_{jk} = \delta^{i}_{j}\, p_k + \delta^{i}_{k}\, p_j \ .$$

Using condition (3.1) in equation (1.11), we have

$$\delta^{i}_{j}\, p_{h;k} + \delta^{i}_{h}\, p_{j;k} - \delta^{i}_{j}\, p_{k;h} - \delta^{i}_{k}\, p_{j;h} = \alpha(x)\, \tilde{K}^{i}_{jhk} \ ,$$

$$\delta^{i}_{j}\, (p_{h;k} - p_{k;h}) + \delta^{i}_{h}\, p_{j;k} - \delta^{i}_{k}\, p_{j;h} = \alpha(x)\, \tilde{K}^{i}_{jhk} \ ,$$

(3.2) $$2\,\delta^{i}_{j}\, p_{[h;k]} + 2\,\delta^{i}_{[h}\, p_{j;k]} = \alpha(x)\, \tilde{K}^{i}_{jhk} \ ,$$

where each square bracket denotes the skew-symmetric part with respect to the indices enclosed in it.

Therefore we may state

Theorem 3.1: *In a Finsler space* F_n *admitting a Projective* $\tilde{K}$-*Curvature inheritance, the relative curvature tensor* $\tilde{K}^{i}_{jhk}$ *can be expressed in terms of derivatives of* p_j *in the form (3.2).*

Gatoto and Singh proved three theorems assuming a Projective $\tilde{K}$-Curvature inheritance as a motion, homothetic transformation and an affine motion. These theorems are misleading because a Projective $\tilde{K}$-Curvature inheritance cannot be a motion, homothetic transformation or an affine motion.

4. Special $\tilde{K}$-Curvature inheritance

In this section, we study the $\tilde{K}$-Curvature inheritance generated by a contra vector field and a concurrent vector field. These vector fields are respectively characterised by

(4.1) $$v_{i\,;\,j} = 0$$

and

(4.2) $$v^i_{\;;\,j} = \rho\, \delta^i_j\ ,$$

where ρ is a constant.

These types of inheritances were discussed by Gatoto and Singh[1]. P. N. Pandey showed that the above types of vector fields generate an affine motion. Therefore the question of $\tilde{K}$-Curvature inheritance or Projective $\tilde{K}$-Curvature inheritance in a Finsler space does not arise.

References

1. J. K. Gatoto and S. P. Singh, Projective $\tilde{K}$ -Curvature inheritance in a Finsler space, *Tensor, N. S.,* **70**(2008) 1-7.
2. J. K. Gatoto and S. P. Singh, $\tilde{K}$ - Curvature inheritance in Finsler space, *Tensor, N. S.,* **70**(2008) 8-15.
3. H. Rund, *The differential geometry of Finsler spaces*, Springer - Verlag, Berlin, 1959.
4. P. N. Pandey, On Lie Recurrent Finsler manifold, *Indian Journal of Mathematics*, **24** (1982) 135-143.
5. P. N. Pandey, CA- collineation in a birecurrent Finsler manifold, *Tamkang J. Math.*, **9** (1978) 79-81.
6. P. N. Pandey, A note on recurrence vector, *Proc. Nat. Acad. Sci.* **51(A)** (1981) 6-8.
7. P. N. Pandey, On a Finsler space of zero projective curvature, *Acta Math. Acad. Sci. Hunger.*, **39** (1982) 387-388.
8. P. N. Pandey, On some Finsler spaces of scalar curvature, *Prog of Maths.*, **18 (1)** (1984) 41-48.
9. K. Yano, *Theory of Lie derivative and its applications*, North-Holland Publishing co., Amsterdam, 1957.