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Second Order Parallel Tensors on LP-Sasakian Manifolds

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Abstract: Levy[1] has proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma[2] has proved that a second order symmetric parallel tensor in a Kaehlar space of constant holomorphic sectional curvature is a linear combination (with constant coefficients) of Kaehlarian metric and the fundamental 2-form. In this paper, we show that on an LP-Sasakian manifold, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor and there is no non-zero parallel 2-form.
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1. Introduction

The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto[3] (1989). Later on several authors studied Lorentzian almost paracontact manifolds and their different classes, viz. LP- Sasakian and LSP- Sasakian manifolds (cf. Matsumoto & Mihai[4] (1988), Mihai & Rosca[5] (1992), Matsumoto, Mihai & Rosca[6] (1995), Pokhariyal (1996), Mihai, Saikh & De[7] (1999), Mishra & Ojha[8] (2000), Saikh & De (2000)). In 1923, Eisenhart[9] proved that if a positive definite Riemannian manifold admits a second order parallel symmetric tensor other than a constant multiple of metric tensor, then it is reducible. In 1926, Levy[1] proved that a second order parallel symmetric non-singular (with non-vanishing determinant) tensor in a space of constant curvature is proportional to the metric tensor. Recently Sharma[2] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an n-dimensional (n >2) space of constant curvature is a constant multiple of metric tensor. Sharma[10] has also proved that there is no non-zero parallel 2-form on a Sasakian manifold. In this paper we prove that a second order symmetric parallel tensor on an LP- Sasakian manifold is a constant multiple of the associated metric tensor. Further, it is shown that on an LP-Sasakian manifold there is no non-zero parallel 2-form.

2. Preliminaries

A differentiable manifold M of dimension n is called Lorentzian Para-Sasakian[3,4] if it admits a (1-1) tensor field ϕ, a contravariant vector field η, a covariant vector field ξ and a Lorentzian metric g which satisfy

(2.1) $\eta(\xi) = -1,$

(2.2) $\phi^2 X = X + \eta(X)\,\xi,$

(2.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\,\eta(Y),$

(2.4) $g(X, \xi) = \eta(X), \; \nabla_X \xi = \phi X,$

(2.5) $(\nabla_X \phi)(Y) = \eta(Y)\,X + g(X, Y)\xi + 2\eta(X)\,\eta(Y)\xi,$

where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g.

It can easily be seen that in an LP-Sasakian manifold M the following relations hold:

(2.6) $\phi\xi = 0, \; \eta(\phi X) = 0, \; \text{rank}(\phi) = n-1.$

Further, on an LP-Sasakian manifold with (Φ, ξ, η, g) structure, the following relations hold**:**

(2.7) $g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\,\eta(X) - g(X, Z)\,\eta(Y),$

(2.8) $R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$

(2.9) $R(\xi, X)\xi = X + \eta(X)\xi,$

(2.10) $R(X, Y)\,\xi = \eta(Y)\,X - \eta(X)\,Y,$

(2.11) $S(X, \xi) = (n-1)\,\eta(X),$

(2.12) $S(\phi X, \phi Y) = S(X, Y) + (n-1)\,\eta(X)\,\eta(Y),$

for any vector fields X, Y, Z, where R(X, Y)Z is the Riemannian curvature tensor and S is the Ricci tensor.

Definition: A tensor T of second order is said to be a second order parallel tensor if $\nabla T = 0$, where ∇ denotes the operator of covariant differentiation with respect to metric g.

Theorem 2.1: *On an LP-Sasakian manifold M, a second order symmetric parallel tensor is a constant multiple of associated metric tensor.*

Proof: Let α denotes a (0, 2)-symmetric tensor field on an LP-Sasakian manifold M such that $\nabla \alpha = 0$.Then it follows that

(2.13) $$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0,$$

for arbitrary vector fields W,X ,Y , Z on M.

Substituting $W = Y = Z = \xi$ in (2.13), we get

$$\alpha(\xi, R(\xi, X)\xi) = 0, \text{ Since } \alpha \text{ is symmetric.}$$

As the manifold is an LP-Sasakian manifold, using equation (2.9) in above equation, we have

$$\alpha(\xi, X + \eta(X)\xi) = 0,$$

which gives

(2.14) $$g(X, \xi)\alpha(\xi, \xi) + \alpha(\xi, X) = 0.$$

Differentiating (2.14) covariantly along Y, we get

(2.15) $$g(\nabla_Y X, \xi)\alpha(\xi, \xi) + g(X, \phi Y)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\xi, \phi Y) + \alpha(\phi Y, X) + \alpha(\nabla_Y X, \xi) = 0.$$

Putting $X = \nabla_Y X$ in (2.14), we get

(2.16) $$g(\nabla_Y X, \xi)\alpha(\xi, \xi) + \alpha(\xi, \nabla_Y X) = 0.$$

From (2.15) and (2.16), it follows that

(2.17) $$g(X, \phi Y)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\xi, \phi Y) + \alpha(\phi Y, X) = 0.$$

Replacing X by ϕY in (2.14) and using $\eta(\phi Y) = 0$, we get

$$g(\phi Y, \xi)\alpha(\xi, \xi) + \alpha(\xi, \phi Y) = 0,$$

which reduces to

(2.18) $$\alpha(\xi, \phi Y) = 0.$$

From equations (2.17) and (2.18), it follows that

(2.19) $$g(X, \phi Y)\alpha(\xi, \xi) + \alpha(\phi Y, X) = 0.$$

Replacing Y by ϕY in (2.19), we get

(2.20) $$g(X, Y)\alpha(\xi, \xi) + \eta(Y)\eta(X)\alpha(\xi, \xi) + \alpha(X, Y) + \eta(Y)\alpha(\xi, X) = 0.$$

Now from equations (2.14) and (2.4), we have

(2.21) $$\alpha(\xi, X) = -\eta(X)\alpha(\xi, \xi).$$

Thus from (2.20) and (2.21), we have

(2.22) $$\alpha(X, Y) = -g(X, Y)\alpha(\xi, \xi).$$

Differentiating (2.22) covariantly along any vector field on M, it can be easily seen that $\alpha(\xi, \xi)$ is constant. This completes the proof.

Corollary 1: *A Ricci symmetric LP-Sasakian manifold is an Einstein manifold.*

Theorem 2.2: *On an LP-Sasakian manifold, there is no non-zero parallel 2-form.*

Proof: Again putting $W = Y = \xi$ in equation (2.13), we get

$$\alpha(R(\xi, X)\xi, Z) + \alpha(\xi, R(\xi, X)Z) = 0$$

Using equations (2.8) and (2.9) in above equation, we get

(2.23) $\alpha(X, Z) = \eta(Z)\,\alpha(\xi, X) - \eta(X)\,\alpha(\xi, Z) - g(X, Z)\,\alpha(\xi, \xi).$

Now α being 2-form, is a (0, 2) skew-symmetric tensor and therefore $\alpha(\xi, \xi) = 0$. Hence equation (2.23) reduces to

(2.24) $\alpha(X, Z) = \eta(Z)\,\alpha(\xi, X) - \eta(X)\,\alpha(\xi, Z).$

Now suppose A be a (1, 1) tensor field which is metrically equivalent to α i.e., $\alpha(X, Y) = g(AX, Y)$. Then from (2.24), it follows that

$$g(AX, Z) = \eta(Z)\, g(A\xi, X) - \eta(X)\, g(A\xi, Z),$$

from which we have

(2.25) $AX = g(A\xi, X)\xi - \eta(X)A\xi.$

Since α is parallel, so A is parallel and hence, using $\nabla_X \xi = \phi X$, it follows that

$$\nabla_X(A\xi) = (\nabla_X A)\xi + A(\nabla_X \xi) = A(\phi X),$$

from which we obtain

(2.26) $\nabla_{\phi X}(A\xi) = AX + \eta(X)A\xi.$

Therefore, from equations (2.25) and (2.26), we have

(2.27) $\nabla_{\phi X}(A\xi) = g(A\xi, X)\xi$

From equation (2.27), we have

$$g(\nabla_{\phi X}(A\xi), A\xi) = g(A\xi, X)g(A\xi, \xi).$$

Since $g(A\xi, \xi) = \alpha(\xi, \xi) = 0$, therefore, the above equation reduces to

(2.28) $g(\nabla_{\phi X}(A\xi), A\xi) = 0.$

Replacing X by ΦX in above equation and using $\nabla_\xi \xi = 0$, we get

(2.29) $g(\nabla_X(A\xi), A\xi) = 0,$

for any tangent vector X and consequently $\|A\xi\|$ = constant on M.

From the above equation (2.29), we have

$$g(A(\nabla_X \xi), A\xi) = 0,$$

which reduces to

$$-\alpha(A\xi, \nabla_X\xi) = 0,$$

because α is metrically equivalent to A. The above equation can be written as

$$g(\nabla_X\xi, A^2\xi) = 0.$$

Replacing X by ΦX, we get

$$g(\nabla_{\phi X}\xi, A^2\xi) = 0.$$

Using equations (2.2) and (2.4) in above equation, we have

$$g(X, A^2\xi) = -g(\eta(X)\xi, A^2\xi),$$

which gives

$$A^2\xi = \|A\xi\|^2\xi. \tag{2.30}$$

Differentiating above equation covariantly along X, we get

$$\nabla_X(A^2\xi) = A^2(\nabla_X\xi) = A^2(\phi X)$$

$$= \|A\xi\|^2\nabla_X\xi$$

$$= \|A\xi\|^2(\phi X).$$

Hence $\qquad A^2(\phi X) = \|A\xi\|^2(\phi X).$

Replacing X by ΦX in above equation, we get

$$A^2(\phi^2 X) = \|A\xi\|^2(\phi^2 X),$$

which gives

$$A^2X + \eta(X)A\xi = \|A\xi\|^2X + \eta(X)\|A\xi\|^2\xi.$$

Using equation (2.30) in above equation, we get

$$A^2X = \|A\xi\|^2X.$$

Now, if $\|A\xi\| \neq 0$, then from above equation, we have

$$\left(\frac{A}{\|A\xi\|}\right)^2 X = X.$$

Let $\qquad F = \dfrac{A}{\|A\xi\|}$, then we have

$$F^2X = X. \tag{2.31}$$

Therefore F is an almost product structure on M. The fundamental 2-form is given by

$$g(FX, Y) = g\left(\frac{AX}{\|A\xi\|}, Y\right) = \frac{1}{\|A\xi\|}\, g(AX, Y)$$

$$= \lambda g(AX, Y)$$
$$= \lambda \alpha(X, Y),$$

where $\lambda = \dfrac{1}{\| A\xi \|}$ = constant. But from equation (2.25) we have

$$\alpha(X, Z) = \eta(Z)\, \alpha(\xi, X) - \eta(X)\, \alpha(\xi, Z),$$

which shows that α is degenerate, hence a contradiction. Therefore $\| A\xi \| = 0$ and so $\alpha = 0$. This completes the proof.

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