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Some Properties of Various Tensors of Semi-Symmetric Non-Metric Connection in β – Kenmotsu Manifolds

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Abstract: Semi- symmetric connection in a Riemannian manifold was studied systematically by Yano[1] and semi- symmetric non- metric connection is initiated by M. R. Chafle and N. S. Agashe[2]. Further it is studied by De and S. C. Biswas[3], U. C. De and D. Kamilya[4], R. N. Singh and Pondey[5] and others. In the present paper we study some properties of the various tensors of semi symmetric non- metric connection in $\beta -$ Kenmotsu manifold.

Keywords: Curvature tensor, Ricci tensor, scalar tensor, concircular tensor, conharmonic curvature tensor, $W_4 -$ curvature tensor, semi-symmetric non-metric connection, $\beta -$ Kenmotsu.

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1. Preliminaries

Let M be a $n-$ dimensional almost contact metric manifold equipped with almost contact metric structure (φ,ξ,η,g) ; where φ is $(1,1)$ tensor field, ξ is a structural vector field, η is a 1- form and g is compatible Riemannian metric[6, 7, 8] such that

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \varphi o \xi = 0,\ \eta o \varphi = 0,$$

(1.1) $$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

(1.2) $g(X,\xi)=\eta(X),\ g(\varphi X,Y)=-g(X,\varphi Y)$,

for every $X,Y\in TM$.

A manifold M is called $\beta-$Kenmotsu manifold if

(1.3) $(\nabla_X\varphi)Y=\beta\{g(\varphi X,Y)\xi-\eta(Y)\varphi X\}$,

Where ∇ is Levi- Civita connection of Riemannian metric g and β is smooth function on M.

From equations (1.1) to (1.4) we have

(1.4) $\nabla_X\xi=\beta(X-\eta(X)\xi)$,

(1.5) $(\nabla_X\eta)Y=\beta g(\varphi X,\varphi Y)$,

M, an almost contact metric manifold is said to be $\eta-$Einstein if its Ricci-tensor S is of the form

(1.6) $S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)$,

where a and b are smooth functions on M.

A linear connection $\overline{\nabla}$ in an almost contact metric manifold M is said to be

(a) Semi–symmetric connection[1] if its torsion tensor $T(X,Y)=\overline{\nabla}_XY-\overline{\nabla}_YX-[X,Y]$ satisfies

(1.7) $T(X,Y)=\eta(Y)X-\eta(X)Y$,

(b) Non– metric connection[2] if

(1.8) $\overline{\nabla}g\neq 0$.

A semi- symmetric non- metric connection $\overline{\nabla}$ in an almost contact metric manifold M can be defined as[2]

(1.9) $\overline{\nabla}_XY=\nabla_XY+\eta(Y)X$,

also

(1.10) $\overline{R}(X,Y)Z=R(X,Y)Z+A(X,Z)Y-A(Y,Z)X$,

where $\overline{R}$ is the curvature tensor of the semi- symmetric non- metric connection $\overline{\nabla}$, R is the curvature tensor of the Levi-Civita connection ∇ and A is a tensor field of type $(0,2)$ satisfies

(1.11) $A(X,Y)=(\overline{\nabla}_X\eta)Y=(\nabla_X\eta)Y-\eta(X)\eta(Y)$.

From equation (1.11), we have

(1.12) $\overline{S}(X,Y)=S(X,Y)-(n-1)A(X,Y)$,

(1.13) $\overline{\rho}=\rho-(n-1)traceA,$

where $\overline{S}$ is the Ricci tensor having scalar curvature $\overline{\rho}$ of the semi-symmetric non- metric connection $\overline{\nabla}$ and S is Ricci tensor having scalar curvature ρ of the Levi- Civita connection ∇.

Let $\{e_1,e_2,........,e_n=\xi\}$ be a local orthonormal basis of vector fields in a $n-$dimensional almost contact manifold M then $\{\varphi e_1,\varphi e_2,......\varphi e_{n-1},\xi\}$ is also a local orthonormal basis. It is easy to show that

(1.14) $\sum_{i=1}^{n}g(e_i,e_i)=\sum_{i=1}^{n-1}g(\varphi e_i,\varphi e_i)+g(\xi,\xi)=n.$

The following are also easy to verify that

(1.15) $(\overline{\nabla}_X\varphi)Y=\overline{\nabla}_X(\varphi Y)-\varphi(\overline{\nabla}_XY)$,

(1.16) $(\overline{\nabla}_X\varphi)Y=\overline{\nabla}_X(\eta(Y))-\eta(\overline{\nabla}_XY)$,

2. Curvature Tensor of Semi- symmetric Non- metric Connection in $\beta-$Kenmotsu Manifolds

Proposition: *Let M be a almost contact manifold with the semi-symmetric non- metric connection, then*

(2.1) $(\overline{\nabla}_X\eta)Y=\beta g(\varphi X,\varphi Y)-\eta(X)\eta(Y)$,

(2.2)
$$\overline{R}(X,Y)Z=R(X,Y)Z-\beta\{g(Y,Z)X-g(X,Z)Y\}+(\beta+1)\eta(Z)\{\eta(Y)X-\eta(X)Y\},$$

(2.3)
$$'\overline{R}(X,Y,Z,W)={}'R(X,Y,Z,W)-\beta\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}+(\beta+1)\eta(Z)\{\eta(Y)g(X,W)-\eta(X)g(Y,W)\},$$

(2.4) $\left(\overline{\nabla}_X\varphi\right)Y = \beta\left\{g\left(\varphi X, Y\right)\xi - \eta\left(Y\right)\varphi X\right\} - \eta\left(Y\right)\varphi X$,

(2.5) $\overline{\nabla}_X\xi = X + \beta\left\{X - \eta\left(X\right)\xi\right\}$,

where $'\overline{R}\left(X,Y,Z,W\right) = g(\overline{R}\left(X,Y\right)Z,W)$

and $'R\left(X,Y,Z,W\right) = g(R\left(X,Y\right)Z,W)$

Proof: From equations (1.6), (1.10) and (1.17) we have (2.1). Using equations (1.11), (1.12) and (2.1) we get (2.2). Equation (2.3) follows from equation (2.2). Using (1.4), (1.10) and (1.16) we get (2.4) and finally equation (2.5) follows from (1.1), (1.5) and (1.10).

Theorem: *In a $\beta-$Kenmotsu manifold with the semi- symmetric non-metric connection, we have*

(2.6) $\overline{R}\left(X,Y\right)Z + \overline{R}\left(Y,Z\right)X + \overline{R}\left(Z,X\right)Y = 0$.

Proof: Above equation (2.6) follows from (2.2) and using first Bianchi identity with respect to Levi- Civita connection ∇.

Theorem: *In a $n-$dimensional $\beta-$Kenmotsu manifold the Ricci tensor $\overline{S}$ and scalar curvature $\overline{\rho}$ with respect to semi- symmetric non- metric connection $\overline{\nabla}$ are given by*

(2.7) $\overline{S}\left(X,Y\right) = S\left(X,Y\right) - \left(n-1\right)\left\{\beta g\left(X,Y\right) - \left(\beta+1\right)\eta\left(X\right)\eta\left(Y\right)\right\}$,

(2.8) $\overline{\rho} = \rho - \left(n-1\right)\left\{\left(n-1\right)\beta - 1\right\}$.

Proof: From equation (1.12), (1.13) and (2.1) we find (2.7). Further suppose $\{e_1, e_2, ...e_n = \xi\}$ is a local orthonormal basis of vector fields then it is known that

(2.9) $\rho^2 = \sum_{i=1}^{n}\overline{S}\left(e_i, e_i\right)$.

Equation (2.8) follows from (1.3), (1.15), (2.7) and (2.9).

Proposition: *If Ricci tensor $\overline{S}$ of the semi- symmetric non- metric connection $\overline{\nabla}$ in $\beta-$ Kenmotsu manifold vanishes, then the manifold is an $\eta-$Einstein manifold.*

3. Conformal Curvature Tensor of Semi- symmetric Non-metric Connection in $\beta-$Kenmotsu Manifolds

Let $\overline{C}$ is a conformal curvature tensor of semi- symmetric non-metric connection $\overline{\nabla}$, then

(3.1) $$\overline{C}(X,Y)Z=\overline{R}(X,Y)Z-\frac{1}{n-2}\{\overline{S}(Y,Z)X-\overline{S}(X,Z)Y+g(Y,Z)\overline{Q}X-g(X,Z)\overline{Q}Y\}+\frac{\overline{\rho}}{(n-1)(n-2)}\{g(Y,Z)X-g(X,Z)Y\},$$

where $\overline{Q}$ and $\overline{S}$ are Ricci operator and Ricci tensor of semi- symmetric non- metric connection $\overline{\nabla}$ respectively.

From equation (3.1) we have,

(3.2) $$'\overline{C}(X,Y,Z,W)='\overline{R}(X,Y,Z,W)-\frac{1}{n-2}\{\overline{S}(Y,Z)g(X,W)-\overline{S}(X,Z)g(Y,W)+g(Y,Z)\overline{S}(X,W)-g(X,Z)\overline{S}(Y,W)\}+\frac{\overline{\rho}}{(n-1)(n-2)}\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\},$$

where $'\overline{C}(X,Y,Z,W)=g(\overline{C}(X,Y)Z,W)$

Now from equations (2.3), (2.7), (2.8) and (3.2) we have

(3.3) $$\overline{C}(X,Y,Z,W)='C(X,Y,Z,W)+\frac{(1+\beta)}{(n-2)}[\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}+\eta(Z)\{\eta(X)g(Y,W)-\eta(Y)g(X,W)\}+(n-1)\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}\eta(W)],$$

where $'C(X,Y,Z,W)=g(C(X,Y)Z,W)$; C is the conformal curvature tensor with respect to Levi-Civita connection ∇.

Theorem: *The conformal curvature tensors $\overline{C}$ of semi- symmetric non-metric connection $\overline{\nabla}$ and C of Levi- Civita connection ∇ in a $\beta-$Kenmotsu manifold are related by*

(3.4) $$\overline{C}(X,Y)Z = C(X,Y)Z + \frac{(1+\beta)}{(n-2)}[\{g(Y,Z)X - g(X,Z)Y\} + \eta(Z)\{\eta(X)Y - \eta(Y)X\} + (n-1)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi]$$

Proof: From equation (3.3) we have the result (3.4)

4. Conharmonic Curvature Tensor of Semi- symmetric non-metric Connection in $\beta-$ Kenmotsu Manifolds

Let $\overline{L}$ is a conharmonic curvature tensor of semi- symmetric non- metric connection $\overline{\nabla}$, then

(4.1) $$\overline{L}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-2}\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y + g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\}.$$

which gives

(4.2) $$'\overline{L}(X,Y,Z,W) = {}'\overline{R}(X,Y,Z,W) - \frac{1}{n-2}\{\overline{S}(Y,Z)g(X,W) - \overline{S}(X,Z)g(Y,W) + g(Y,Z)\overline{S}(X,W) - g(X,Z)\overline{S}(Y,W)\},$$

where $'\overline{L}(X,Y,Z,W) = g(\overline{L}(X,Y)Z,W)$

Theorem: *The conharmonic curvature tensors* $\overline{L}$ *of semi- symmetric non- metric connection* $\overline{\nabla}$ *and* L *of Levi- Civita connection* ∇ *in a* $\beta-$ *Kenmotsu manifold are related by*

(4.3) $$\overline{L}(X,Y)Z = L(X,Y)Z - \frac{n\beta}{(n-2)}\{g(X,Z)Y - g(Y,Z)X\} + \frac{(1+\beta)}{(n-2)}[\eta(Z)\{\eta(Y)X - \eta(X)Y\} + (n-1)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi]$$

Proof: Using equation (2.3) and (2.7) in (4.2), we get

$$'\bar{L}(X,Y,Z,W) = 'L(X,Y,Z,W) - \frac{n\beta}{(n-2)}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}$$
$$-\frac{(1+\beta)}{(n-2)}[\eta(Z)\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\}$$
$$-(n-1)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W)],$$

where $'L(X,Y,Z,W) = g(L(X,Y)Z,W)$

which gives equation (4.3). This proves the theorem.

5. Concircular Curvature Tensor of Semi- symmetric Non-metric Connection in $\beta-$Kenmotsu Manifolds

Let $\bar{V}$ is the concircular curvature tensor of semi- symmetric non- metric connection $\bar{\nabla}$ and V is of the Levi- Civita connection ∇ in a $\beta-$Kenmotsu manifold are related by

(5.1)
$$\bar{V}(X,Y)Z = V(X,Y)Z + (1+\beta)\eta(Z)\{\eta(Y)X - \eta(X)Y\}$$
$$+\frac{(1+\beta)}{n}\{g(X,Z)Y - g(Y,Z)X\}.$$

For, concircular curvature tensor $\bar{V}$ of semi- symmetric non- metric connection $\bar{\nabla}$ on a Riemannian manifold of dimension n is defined by

(5.2)
$$\bar{V}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\rho}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}$$

From equations (2.2), (2.8) and (5.2) we get the result.

6. W_4-Curvature Tensor of Semi- symmetric Non-metric Connection in $\beta-$Kenmotsu Manifolds

$\bar{W}_4-$Curvature tensor on semi- symmetric non- metric connection $\bar{\nabla}$ [3] is defined as

(6.1)
$$\bar{W}_4(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}\{g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y\}$$

which implies

$$'\overline{W}_4(X,Y,Z,W) = '\overline{R}(X,Y,Z,W) - \frac{1}{n-1}\{g(Y,Z)\overline{S}(X,W) - g(X,Y)\overline{S}(Z,W)\}, \tag{6.2}$$

where $'\overline{W}_4(X,Y,Z,W) = g(\overline{W}_4(X,Y)Z,W)$

From equations (2.3), (2.7) and (6.2), we have

$$\begin{aligned}'\overline{W}_4(X,Y,Z,W) &= 'W_4(X,Y,Z,W) + \beta[\{g(X,Y)g(Y,W) - g(Y,Z)g(X,W)\}] \\ &+ (\beta+1)[\eta(Z)\{\eta(Y)g(X,W) - \eta(X)g(Y,W)\} \\ &+ \{\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\}\eta(W)],\end{aligned} \tag{6.3}$$

where $'W_4(X,Y,Z,W) = g(W_4(X,Y)Z,W)$

Theorem: *The* $\overline{W}_4$ *– curvature tensors of semi- symmetric non- metric connection* $\overline{\nabla}$ *and* W_4 *of the Levi- Civita connection* ∇ *in* β *– Kenmotsu manifold are related by*

$$\overline{W}_4(X,Y)Z = W_4(X,Y)Z + \beta[\{g(X,Y)Y - g(Y,Z)X\} + \eta(Y)g(X,Z) - \eta(Z)g(X,Y)\}] \tag{6.4}$$

Proof: From equation (6.3) we get the result (6.4).

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