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On Finsler Space with Special (α,β)- Metric

H. S. Shukla and S. K. Mishra
Department of Mathematics and Statistics,
D. D. U. Gorakhpur University, Gorakhpur
Email: profhsshuklagkp@rediffmail.com; kumar.sandip62@yahoo.com

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Abstract: We considered a special (α,β)-metric given by

$$L = A_1\left(\alpha + \frac{\beta^2}{\alpha}\right) + A_2\left(\frac{\alpha^2}{\alpha - \beta}\right),$$

where A_1 and A_2 are constants, α is Riemannian metric given by $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ and β is one form given by $\beta = b_i(x)y^i$ and obtained the Berwald connection and the condition under which a Finsler space with above metric is a Berwald space. Further main scalar of two dimensional Finsler space and equations of geodesics of the Finsler space with this metric have been also determined.

Key words: (α,β)-metric, Berwald connection, Geodesics

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1. Preliminaries

M. Matsumoto[1] while studying a Finsler space with (α,β)-metric of Douglas type, introduced a special (α,β)-metric given by

$$L = \alpha + \frac{\beta^2}{\alpha}. \tag{1.1}$$

While measuring the slope of mountain with respect to time measure, he[2] introduced a metric given by $L = \dfrac{\alpha^2}{\alpha - \beta}$. The metric is called Matsumoto metric. In this paper we have considered a special (α,β)-metric, which is the linear combination of metric (1.1) and Matsumoto metric, and is given by

$$L = A_1\left(\alpha + \frac{\beta^2}{\alpha}\right) + A_2\left(\frac{\alpha^2}{\alpha - \beta}\right), \tag{1.2}$$

if $A_1 = 0$, the above metric is homothetic to Matsumoto metric and if $A_2 = 0$, it is homothetic to the metric (1.1). We shall denote the n-dimensional Finsler space with metric (1.2) as F^n and associated Riemannian space by R^n.

In the following Riemannian metric α is not supposed to be positive definite and we shall restrict our discussion to a domain of (x, y), where β does not vanish. The covariant differentiation with respect to the Levi - Civita connection $\{\gamma^i_{jk}(x)\}$ of R^n is denoted by semi-colon. Let us list the symbols[4] here for the use

$$b^i = a^{ir} b_r,\ b^2 = a^{rs} b_r b_s,\ 2r_{ij} = b_{i;j} + b_{j;i},\ 2s_{ij} = b_{i;j} - b_{j;i}, \tag{1.3}$$
$$r^i_j = a^{ir} r_{rj},\ s^i_j = a^{ir} s_{rj},\ r_i = b_r r^r_i,\ s_i = b_r s^r_i.$$

The Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ of F^n plays a leading role in the present paper. We shall denote by B^i_{jk} the difference tensor[4] of G^i_{jk} from γ^i_{jk}:

$$G^i_{jk}(x, y) = \gamma^i_{jk}(x) + B^i_{jk}(x, y). \tag{1.4}$$

Transvecting (1.4) by y^k and y^j successively, we get

$$G^i_j = \gamma^i_{0j} + B^i_j. \qquad 2G^i = \gamma^i_{00} + 2B^i, \tag{1.5}$$

where $\quad B^i_j = \dot{\partial}_j B^i, \quad B^i_{jk} = \dot{\partial}_k B^i_j.$

It is noted that the Cartan connection also has the non-linear connection $\{G^i_j\}$ common to $B\Gamma . B^i(x, y)$ is called the difference vector in the present paper and for an (α, β)-metric it is given by[2]

$$B^i = \frac{E}{\alpha} y^i + \frac{\alpha L_2}{L_1} s^i_0 - \frac{\alpha L_{11}}{L_1} C^* \left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta} b^i\right), \tag{1.6}$$

where

(1.7) $$E=\frac{\beta L_2}{L}C^*,\qquad C^*=\frac{\alpha\beta(r_{00}L_1-2\alpha s_0 L_2)}{2(\beta^2 L_1+\alpha\gamma^2 L_{11})},\qquad \gamma^2=b^2\alpha^2-\beta^2,$$

and subscripts 1 and 2 denote partial derivatives with respect to α and β respectively.

2. Finsler space with metric (1.2)

From (1.2) we have

(2.1) $$L=\frac{H_3}{\alpha(\alpha-\beta)},\quad L_1=\frac{A_4}{\alpha^2(\alpha-\beta)^2},\quad L_2=\frac{B_3}{\alpha(\alpha-\beta)^2},$$

$$L_{11}=\frac{2\beta^2 C_3}{\alpha^3(\alpha-\beta)^3},\quad L_{22}=\frac{2C_4}{\alpha(\alpha-\beta)^3},\quad L_{12}=\frac{-2BC_3}{\alpha^2(\alpha-\beta)^3},$$

where

(2.2) $$H_3=A_1(\alpha^2+\beta^2)(\alpha-\beta)+A_2\alpha^3,$$

$$A_4=A_1(\alpha-\beta)^3(\alpha+\beta)+A_2\alpha^3(\alpha-2\beta),$$

$$B_3=2A_1\beta(\alpha-\beta)^2+A_2\alpha^3,\; C_3=A_1(\alpha-\beta)^3+A_2\alpha^3.$$

Since $B\Gamma$ is L-metrical, i.e., $L_{|i}=L_1\,\alpha_{|i}+L_2\beta_{|i}=0,$ therefore from (2.1) we have $A_4\alpha_{|i}+\alpha B_3\beta_{|i}=0$ and so

(2.3) $$\alpha_{|i}=-\frac{\alpha B_3}{A_4}\beta_{|i}\;.$$

It is observed that $\beta_{|i}=b_{s|i}\,y^s+(b_{s;i}-b_r\,B^r_{si})y^s$, which implies

(2.4) $$\beta_{|i}\,y^i=r_{00}-2b_rB^r\,.$$

For the scalar b^2 we have $b^2_{;i}y^i=(\partial_i b^2)y^i=2b^r(r_{ri}+s_{ri})y^i$ which shows that

(2.5) $$b^2_{;i}\,y^i=2(r_0+s_0).$$

Next the quadratic form $\gamma^2=b^2\alpha^2-\beta^2=(b^2a_{ij}-b_ib_j)y^i\,y^j$, plays a role in the following. From above equation it is easy to show

(2.6) $$A_4\,\gamma^2_{|i}\,y^i=2A_4(r_0+s_0)\alpha^2-2B_3b^2\alpha+A_4\beta(r_{00}-2b_rB^r).$$

The following lemma has been shown:

Lemma (2.1)[6]. *If* $\alpha^2 \equiv 0 \pmod{\beta}$, *that is,* $a_{ij}(x)y^iy^j$ *contains* $b_i(x)\, y^i$ *as a factor, then the dimension is equal to two and* b^2 *vanishes. In this case we have* $\delta = d_i\,(x) y^i$ *satisfying* $\alpha^2 = \beta\delta$, *and* $d_i\, b^i = 2$.

In the following we consider that $\alpha^2 \not\equiv 0 (mod\, \beta)$.

3. The Berwald space with metric (1.2)

The equation $L_{|i} = \partial_i L - G^k_j (\partial_k\, L) = 0$ is written in the form

$$(3.1) \qquad A_4\, B^k_{ji} y^j y_k = \alpha^2 B_3\,(b_{j;\,i} - B^k_{ji} b_k) y^j\,,$$

where we have put $y_k = a_{ki}\, y^i$. Putting the values of A_4 and B_3 from (2.2) and rearranging, the equation (3.1) may be written in the form

$$(3.2) \qquad P_4\, B^k_{ji} y^j y_k + Q_5\,(b_{j;i} - B^k_{ji} b_k) y^j$$
$$- 2\alpha [R_3\, B^k_{ji} y^j y_k + S_4\,(b_{j;i} - B^k_{ji} b_k) y^j] = 0,$$

where P_4 , S_4 are homogeneous function of degree 4, R_3 is homogeneous function of degree 3 and Q_5 is homogeneous function of degree 5 in α and β, and these are given as

$$(3.3) \qquad P_4 = (A_1 + A_2)\alpha^4 - A_1\,\beta^4\,, \qquad Q_5 = -2A_1\alpha^2\beta(\alpha^2 + \beta^2)$$
$$R_3 = (A_1 + A_2)\,\alpha^2\beta - A_1\,\beta^3, \qquad S_4 = -\alpha^2(A_1\beta^2 - A_2\alpha^2).$$

Assume that the Finsler space with metric (1.2) is a Berwald space, i.e, $G^i_{jk} = G^i_{jk}(x)$. Then we have $B^i_{jk} = B^i_{jk}(x)$, so that $B^i_{jk} y^j y_k (= B^i_{jk} a_{kh} y^j y^h)$ is a quadratic from in y. Since P_4 , S_4 , R_3 , Q_5 and $b_{j;i}\, y^i$ are rational functions of y^i whereas α is an irrational of y^i , we have

$$(3.4) \qquad P_4 B^k_{ji} y^j y_k + Q_5 (b_{j;i} - B^k_{ji} b_k) y^j = 0,$$

and

$$(3.5) \qquad R_3 B^k_{ji} y^j y_k + S_4 (b_{j;i} - B^k_{ji} b_k) y^j = 0,$$

If $\alpha^2 \neq 0 \pmod{\beta}$, then $P_4\, S_4 - Q_5\, R_3 \neq 0$. Hence

$$B^k_{ji} a_{kh} + B^k_{hi} a_{kj} = 0, \qquad b_{j\,;\,i} - B^k_{ji} b_k = 0.$$

The former yields $B^k_{ji}=0$ immediately. Consequently the latter gives $b_{j\,;\,i}=0.$

Conversely if $b_{j\,;\,i}=0,$ then (1.3) leads to $r_{ij}=s_{ij}=0.$ Hence (1.6) and (1.7) lead to $B^i=0, B^k_{ji}=0$ and so $G^i_{jk}=\gamma^i_{jk}(x).$ Hence F^n is a Berwald space. Therefore we have

Theorem (3.1). *The Finsler space with the metric (1.2) is Berwald space if and only if the vector* b_i *is covariantly constant* $(b_{i;j}=0)$ *and the Berwald connection is given by* $B\Gamma=(\gamma^i_{jk},\gamma^i_{0j},0)$.

4. Main scalar of two dimensional Finsler space with (α,β)-metric (1.2)

The main scalar I of two dimensional Finsler space F^2 with metric $L(\alpha,\beta)$ is given by[5]:

(4.1) $$\varepsilon I^2=\left(\frac{L}{\alpha}\right)^4\left\{\frac{\gamma^2(T_2)^2}{4T^3}\right\},$$

where ε is signature of the space, $\gamma^2=b^2\alpha^2-\beta^2,$

(4.2) $$T=p(p+p_0b^2+p_{-1}\beta)+\{(p_0p_{-2}-(p_{-1})^2)\}\gamma^2,$$

(4.3) $$p=L\,L_1\,\alpha^{-1},\quad p_0=L\,L_{22}+L_2^2,\quad p_{-1}=(L\,L_{12}+L_1\,L_2)\alpha^{-1},$$
$$p_{-2}=L\;\alpha^{-2}(L_{11}-L_1\alpha^{-1})+L_1^2\alpha^{-2}$$

and $T_2=\dfrac{\partial T}{\partial\beta}$. if g denote the determinant of matrix (g_{ij}) and *a* denote that of matrix (a_{ij}) then in n-dimensional Finsler space with (α,β)-metric, we have[5]:

(4.4) $$g=(p^{n-2}T)a.$$

Putting the values of $L, L_1, L_2, L_{11}, L_{22},$ *and* L_{12} from (2.1) in (4.3) we get

(4.5) $$p=\frac{H_3A_4}{\alpha^4(\alpha-\beta)^3},\; p_0=\frac{2H_3C_3}{\alpha^2(\alpha-\beta)^3}+\frac{B_3^2}{\alpha^2(\alpha-\beta)^4},$$
$$P_{-1}=\frac{A_4B_3-2\beta H_3C_3}{\alpha^4(\alpha-\beta)^4},\qquad p_{-2}=\frac{2\beta C_3H_3+A_4^2}{\alpha^6(\alpha-\beta)^4}-\frac{H_3A_4}{\alpha^6(\alpha-\beta)^3}$$

where H_3, A_4, B_3 *and* C_3 are given in equation (2.2)

For a two dimensional Finsler space with (α, β)-metric, we have[5]:

$$\frac{g}{a} = T = \left(\frac{L}{\alpha}\right)^3 \left(L_1 + \frac{L_{22}}{\alpha}\gamma^2\right). \tag{4.6}$$

Substituting the values of L, L_1, L_{22} from (2.1) in (4.6), we have

$$T = \frac{H_3^3[(\alpha-\beta)A_4 + 2\gamma^2 C_3]}{\alpha^8(\alpha-\beta)^6}, \tag{4.7}$$

where H_3, A_4 and C_3 are given in (2.2). From (4.6) it follows that

$$T_2 = \frac{\partial T}{\partial \beta} = 3\left(\frac{L^2}{\alpha^3}\right) L_2 \left(L_1 \frac{L_{22}}{\alpha}\gamma^2\right) + \left(\frac{L}{\alpha}\right)^3 \left(L_{12} + \frac{\gamma^2 L_{222} - 2\beta L_{22}}{\alpha}\right), \tag{4.8}$$

where $L_{222} = \dfrac{\partial^3 L}{\partial \beta^3} = \dfrac{6A_2\alpha^2}{(\alpha-\beta)^4}$. Substituting the values of L, L_2, L_{12} and L_{222} in (4.8) we get

$$T_2 = 3\left(\frac{H_3^2 B_3}{\alpha^8(\alpha-\beta)^7}\right)[(\alpha-\beta)A_4 + 2\gamma^2 C_3] - \frac{6H_3^3}{\alpha^8(\alpha-\beta)^7}[\beta(\alpha-\beta)C_3 - \alpha^2\gamma^2 A_2], \tag{4.9}$$

where H_3, A_4, B_3 and C_3 are given in equation (2.2). Putting the values of L in (4.1) we get the main scalar of two dimensional space with metric (1.2) as

$$\varepsilon I^2 = \frac{H_3^4}{\alpha^8(\alpha-\beta)^4}\left\{\frac{\gamma^2(T_2)^2}{4T^3}\right\}, \tag{4.10}$$

where H_3, T and T_2 are given by (2.2), (4.7) and (4.9) respectively. Thus we have the following

Theorem (4.1). *The main scalar of two dimensional Finsler space with* (α, β)*-metric (1.2) is given by (4.10).*

5. Equations of geodesic of a Finsler space with (α, β)-metric (1.2)

By the arc length s (Finslerian parameter) the equations of geodesic of F^n are written in the well- known form[5]

$$(5.1)\qquad \frac{d^2x^i}{ds^2}+2G^i\left(x,\frac{dx}{ds}\right)=0,$$

where functions $G^i(x,y)$ are given by

$$2G^i=g^{ir}(y^j\partial_r\partial_jF-\partial_iF),\qquad F=\frac{L^2}{2}.$$

Using the parameter τ, (5.1) may be written as

$$(5.2)\qquad \frac{d^2x^i}{d\tau^2}+2G^i\left(x,\frac{dx}{d\tau}\right)=-\frac{\tau''}{\tau'}\frac{dx^i}{d\tau},$$

where $\tau'=\dfrac{d\tau}{ds}$.

But for an (α,β)-metric L the equation of geodesic is given by[5]

$$(5.3)\qquad \frac{d^2x^i}{d\tau^2}+\gamma^i_{00}+\frac{2L_2}{L_1}s^i_0+\frac{2L\,L_{11}E}{L_1\,L_2\beta^2}p^i=0,$$

where $p^i=b^i-\dfrac{\beta}{\alpha^2}y^i$ and E is given by equation (1.7). Substituting the values of L_1, L_2, L_{11} from (2.1) in the expression of C* and E given by (1.7) we get

$$(5.4)\qquad C^*=\frac{\alpha(\alpha-\beta)[r_{00}A_4-2s_0\alpha^2B_3]}{2\beta[(\alpha-\beta)A_4+2\gamma^2C_3]},$$

$$(5.5)\qquad E=\frac{\alpha B_3[r_{00}A_4-2s_0\alpha^2B_3]}{2H_3[(\alpha-\beta)A_4+2\gamma^2C_3]}.$$

Putting the values of L, L_1, L_2, L_{11} and E in (5.3) we get equations of geodesic for a Finsler space with metric (1.2) as

$$\frac{d^2x^i}{d\tau^2} + \gamma^i_{00} + \frac{2\alpha B_3}{A_4} s^i_0 + \frac{2C_3[r_{00}A_4 - 2s_0\alpha^2 B_3]}{A_4(\alpha-\beta)[(\alpha-\beta)A_4 + 2\gamma^2 C_3]} p^i = 0. \tag{5.6}$$

Theorem (5.1). *In terms of arc length* τ *in the associated Riemannian space* $\mathrm{R}^n = (\mathrm{M}^n, \alpha)$ *the equations of the geodesic of Finsler space* F^n *with* (α, β)*-metric (1.2) are written as (5.6) where* $\gamma^i_{jk}(x)$ *are the Christoffel sumbols of* R^n *and* A_4, B_3, C_3 *are given by (2.2).*

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