ISSN 0974 - 9373

Vol. 24 No. 3 (2020) **Journal of International Academy of Physical Sciences** pp. 227- 246

Modeling the Removal of two Gaseous Pollutants and a Mixture of Particulate Matters formed by these Gases in a Rain System

Shyam Sundar

Department of Mathematics
Pranveer Singh Institute of Technology, Kanpur-209305, India
Email: ssmishra15@gmail.com

S. N. Mishra

Department of Mathematics
Brahmanand College, The Mall, Kanpur-208004, India
Email: snmishra2006@gmail.com

Ram Naresh

Department of Mathematics, School of Basic & Applied Sciences
Harcourt Butler Technical University, Kanpur-208002, India
Email: ramntripathi@yahoo.com

(Received August 05, 2020)

Abstract: In this paper, a nonlinear mathematical model is proposed and analyzed to study the removal of two gaseous pollutants and the mixture of particulate matters formed separately by these two gases from the atmosphere of a city by rain. It is assumed that particulate matters are formed in the atmosphere by chemical transformations of two gaseous pollutants emitting in the atmosphere in presence of rain. The atmosphere, during rain, is assumed to consist of six interacting phases namely, the raindrops phase, the two phases of gaseous pollutants, their absorbed phases and the phase of the mixture of particulate matters. It is also assumed that gaseous pollutants are removed from the atmosphere by the process of absorption while the particulate matters are removed only by the process of impaction with different removal rates. By analyzing the model, it is shown that under appropriate conditions, these gaseous pollutants and particulate matters are removed from the atmosphere and their equilibrium levels, remaining in the atmosphere, would depend mainly upon the rates of emission of pollutants, growth rate of raindrops and the rate of raindrops falling on the ground, etc. It is found that if the rates of transformations of gaseous pollutants into particulate matters and the rainfall are very large, then these pollutants may be removed completely from the atmosphere. The numerical simulations performed support the analytical findings.

Keywords: Nonlinear model, gaseous pollutants, particulate matters, precipitation, stability, numerical simulation.

1. Introduction

The most challenging problem that our society faces today is the removal of gaseous pollutants and particulate matters present in the atmosphere of a city caused by anthropogenic sources such as industrial and household emissions, vehicular exhausts, etc. These gaseous pollutants and particulate matters affect human health, plant species, monuments, etc. and therefore need to be removed from the atmosphere. In the atmosphere, particulate matters may be formed due to chemical transformation of gases or by deposition of gases on the surface of the existing dust particles[1-3]. For example, sulfuric acid vapor (H_2SO_4) reacts with ammonia (NH_3) and forms ammonium bi-sulfate (NH_4HSO_4),

$$\underbrace{SO_3 + H_2O}_{\text{Sulfuric acid vapour}} + NH_3(\mathrm{g}) \rightarrow NH_4HSO_4(\mathrm{p})$$

Now this ammonium bi-sulfate, in presence of excess ammonia, gives ammonium sulfate ($(NH_4)_2SO_4$)

$$NH_4HSO_4(\mathrm{p}) + NH_3(\mathrm{g}) \rightarrow (NH_4)_2SO_4(\mathrm{p}).$$

Also, SO_2 is oxidized to H_2SO_4 and may undergo hetero-molecular nucleation with H_2O to form new H_2SO_4/H_2O particles.

Some mathematical models have been proposed to study the phenomenon of removal of pollutants in a rain system by considering uniform distribution of rain droplets in the atmosphere[4-7]. Hales[7] studied a linearized model for predicting the washout of gaseous pollutants from industrial plumes by taking into account the reversibility of adsorption process assuming the rain to be composed of spherical, non-interacting drops of static size distribution.

Some experimental studies also affirm that the pollutants are removed from the atmosphere by precipitation[8-22]. In particular, the removal of SO_2 by rain in an industrial area of Sheffield, U. K. has been studied by Davies[8] and significant reduction in its concentration after rain was found. In a study by Sharma et al.[11], the concentrations of suspended particulate matters were measured in Kanpur city, U.P., India and considerable decrease in their concentrations during and after monsoon was observed. Similar results have been reported in[12-14]. Further, Guo et al.[21] have studied the washout effects of rainfall on atmospheric particulate pollution in two cities of China and found considerable decrease in the concentration level of particulate matter $PM_{2.5}$.

It is pointed out here that the process of removal of gaseous pollutants and particulate matters from the atmosphere by rain is a nonlinear process and as such it involves bilinear interactions of various phases[23-31]. For example, Pandis and Seinfeld[24] studied the interaction between equilibration processes and wet or dry deposition. They have shown that the equilibration processes between two atmospheric phases, with different removal rates, can affect significantly the amount deposited on the ground. Shukla et al.[26] presented a nonlinear mathematical model for removal of a single gaseous air pollutant by rain to study the effect of precipitation on the equilibrium levels of pollutants in the atmosphere. Shukla et al.[27] have also proposed and analyzed a nonlinear mathematical model for the removal of a gaseous pollutant and two particulate matters assuming that one particulate matter is formed from gaseous pollutants. They have shown that, under appropriate conditions, both the gaseous pollutants and particulate matters can be washed out from the atmosphere. Shukla et al.[28] further developed a mathematical model to study the effect of traffic pollutants on human population with some control measures such as spray of liquid drops used for abatement of traffic pollutants. They have shown that the concentration of traffic pollutants decreases significantly when some liquid species are sprayed in the atmosphere.

Since various chemical reactions occur in the atmosphere, certain gaseous species may also get transformed into particulate matters due to these chemical conversions. This aspect has not been explored in the above studies. Using this concept, in the present study, we have assumed that two gaseous pollutants emitted in the atmosphere are converted into two different particulate matters forming a combined phase of mixture of these particulates. Thus, a six dimensional nonlinear mathematical model is proposed to study the removal by rain of these gaseous pollutants and mixture of particulate matters formed by these gaseous pollutants from the atmosphere of a city.

2. Mathematical Model and Description

Consider the atmosphere of a polluted city where the phenomenon of removal of two gaseous pollutants and the mixture of particulate matters formed separately by these two gases takes place in presence of water in the rain system. During rain, both gaseous pollutants and the particulate matters interact with raindrops and get removed from the atmosphere by the process of absorption in the case of gases and by impaction in the case of mixture of particulate matters. The other natural removal phenomenon such as gravitational settling also occurs. Thus, the following six interacting phases are taken into account in the atmosphere,

1. The raindrops phase, which occurs due to rain fall.
2. The separate phases of two gaseous pollutants which occur due to emission of gaseous pollutants into the atmosphere by various sources such as industries, power plants, vehicles, etc.
3. The phase of the mixture of particulate matters, which is formed due to transformation of gaseous pollutants to particulate matters in the atmosphere.
4. The separate absorbed phases of gaseous pollutants due to absorption of these in raindrops.

Let C_r be the number density of raindrops in the atmosphere occurring with a constant growth rate q, C_1, C_2 be the cumulative concentrations of two different gaseous pollutants respectively and C_p be the cumulative concentration of the mixture of particulate matters formed by C_1 and C_2 in the atmosphere. Further, C_{1a} and C_{2a} be the concentrations of respective gaseous pollutants absorbed in the raindrops. It is assumed that the number density of raindrops may deplete naturally and also due to interactions with gaseous pollutants (which may be discharged heated). This depletion is assumed to be directly proportional to the number density of raindrops as well as the cumulative concentrations of gaseous pollutants (i.e. $r_1 C_1 C_r$ and $r_2 C_2 C_r$) where r_1 and r_2 are depletion rate coefficients of raindrops due to interactions with C_1 and C_2 respectively, r_0 is the natural depletion rate coefficient of raindrops. Further, Q_1 and Q_2 be the constant emission rates of two different gaseous pollutants with their natural depletion rates $\delta_1 C_1$ and

$\delta_2 C_2$ respectively where constants δ_1 and δ_2 are natural depletion rate coefficients of C_1 and C_2. It is assumed that the absorption of these gaseous pollutants by raindrops is in direct proportion to the cumulative concentrations of these pollutants and the number density of raindrops (i.e. $\alpha_1 C_1 C_r$, $\alpha_2 C_2 C_r$ respectively) where constants α_1 and α_2 are the removal rate coefficients of these gaseous pollutants due to interaction with raindrops. As pointed out above, the two gaseous pollutants are converted to particulate matters due to chemical transformation with a rate $\gamma_1 C_1$ and $\gamma_2 C_2$ to form a combined phase of the mixture of the particulate matters so formed where γ_1 and γ_2 are the conversion rate coefficients of C_1 and C_2 respectively into particulate form. The constants δ_p and α_p are the natural depletion rate coefficient and the removal rate coefficient of the combined phase of particulate matters formed by these gaseous pollutants. The gaseous pollutants in the absorbed phase may also be removed naturally with rates $k_1 C_{1a}$ and $k_2 C_{2a}$ where k_1, k_2 are removal rate coefficients. It is also assumed that the removal of gaseous pollutants in the absorbed phase is directly proportional to their concentrations in the respective absorbed phases and the number density of raindrops (i.e. $\nu_1 C_r C_{1a}$ and $\nu_2 C_r C_{2a}$ respectively) where ν_1, ν_2 are their removal rate coefficients.

In view of the above considerations, the dynamics of the system is assumed to be governed by the following nonlinear differential equations,

(2.1) $$\frac{dC_r}{dt} = q - r_0 C_r - r_1 C_1 C_r - r_2 C_2 C_r,$$

(2.2) $$\frac{dC_1}{dt} = Q_1 - \delta_1 C_1 - \alpha_1 C_1 C_r,$$

(2.3) $$\frac{dC_2}{dt} = Q_2 - \delta_2 C_2 - \alpha_2 C_2 C_r,$$

(2.4) $$\frac{dC_p}{dt} = \gamma_1 C_1 + \gamma_2 C_2 - \delta_p C_p - \alpha_p C_p C_r,$$

(2.5) $$\frac{dC_{1a}}{dt} = \alpha_1 C_1 C_r - k_1 C_{1a} - \nu_1 C_r C_{1a},$$

(2.6) $$\frac{dC_{2a}}{dt} = \alpha_2 C_2 C_r - k_2 C_{2a} - \nu_2 C_r C_{2a},$$

with $C_r(0) \geq 0,\ C_1(0) \geq 0,\ C_2(0) \geq 0,\ C_p(0) \geq 0,\ C_{1a}(0) \geq 0,\ C_{2a}(0) \geq 0$.

3. Equilibrium and Stability Analysis

Now we analyze the model system (2.1) – (2.6) using stability theory of differential equations which provides a strong mathematical tool to analyze the qualitative behavior of the solutions of a model system. The model has only one non-negative equilibrium point namely $E^*(C_r^*, C_1^*, C_2^*, C_p^*, C_{1a}^*, C_{2a}^*)$, where $C_r^*, C_1^*, C_2^*, C_p^*, C_{1a}^*$ and C_{2a}^* are the positive solutions of the following set of simultaneous equations obtained by setting right hand side of model system (2.1) - (2.6) to zero,

(3.1) $$C_r = \frac{q}{r_0 + r_1 C_1 + r_2 C_2},$$

(3.2) $$C_1 = \frac{Q_1}{\delta_1 + \alpha_1 C_r},$$

(3.3) $$C_2 = \frac{Q_2}{\delta_2 + \alpha_2 C_r},$$

(3.4) $$C_p = \frac{\gamma_1 C_1 + \gamma_2 C_2}{\delta_p + \alpha_p C_r},$$

(3.5) $$C_{1a} = \frac{\alpha_1 C_1 C_r}{k_1 + \nu_1 C_r},$$

(3.6) $$C_{2a} = \frac{\alpha_2 C_2 C_r}{k_2 + \nu_2 C_r}.$$

To show the existence of E^* from equations (3.1)-(3.3), we write

(3.7) $$F(C_r) = q - r_0 C_r - r_1 \frac{Q_1}{\delta_1 + \alpha_1 C_r} C_r - r_2 \frac{Q_2}{\delta_2 + \alpha_2 C_r} C_r = 0.$$

From above equation, we note that

(i) $F(0) = q > 0$,

(ii) $F\left(\frac{q}{r_0}\right) < 0$,

(iii) $F'(C_r) = -r_0 - \frac{r_1 \delta_1 Q_1}{(\delta_1 + \alpha_1 C_r)^2} - \frac{r_2 \delta_2 Q_2}{(\delta_2 + \alpha_2 C_r)^2} < 0$.

Thus, $F(C_r) = 0$ has exactly one root (say C_r^*) between 0 and $\frac{q}{r_0}$ without any condition. Using C_r^*, the values of $C_1^*, C_2^*, C_p^*, C_{1a}^*$ and C_{2a}^* can be found from (3.2)-(3.6) respectively.

It may also be noted from equations (3.2)-(3.6) that $C_1, C_2, C_p, C_{1a}, C_{2a} \to 0$ as $C_r \to \infty$ showing that all the pollutants would be removed completely from the atmosphere if the number density of raindrops is very high.

Now we check the characteristics of equilibrium values of some variables with respect to relevant parameter,

(i) Variation of C_r with q : We can write equation (3.7) as

$$r_0 C_r + r_1 \frac{Q_1}{\delta_1 + \alpha_1 C_r} C_r + r_2 \frac{Q_2}{\delta_2 + \alpha_2 C_r} C_r = q .$$

Differentiating this equation with respect to q ,we get

$$\frac{dC_r}{dq} = \frac{1}{r_0 + \frac{r_1 \delta_1 Q_1}{(\delta_1 + \alpha_1 C_r)^2} + \frac{r_2 \delta_2 Q_2}{(\delta_2 + \alpha_2 C_r)^2}} > 0 .$$

This implies that the number density of raindrops C_r increases in the atmosphere with increase in the growth rate of rain q .

(ii) Variation of C_1 with q : From equation (3.2), we get

$$\frac{dC_1}{dC_r} = -\frac{\alpha_1 Q_1}{(\delta_1 + \alpha_1 C_r)^2} < 0 \text{ and since } \frac{dC_r}{dq} > 0 ,$$

it follows that $\frac{dC_1}{dq}=\frac{dC_1}{dC_r}\frac{dC_r}{dq}<0$.

This shows that the cumulative concentration C_1 of gaseous pollutants emitted in the atmosphere decreases with increase in the growth rate of raindrops q.

(iii) Variation of C_2 with q: Similarly from equation (3.3), we get

$$\frac{dC_2}{dC_r}=-\frac{\alpha_2 Q_2}{(\delta_2+\alpha_2 C_r)^2}<0 \text{ and since } \frac{dC_r}{dq}>0,$$

it follows that $\frac{dC_2}{dq}=\frac{dC_2}{dC_r}\frac{dC_r}{dq}<0$.

Therefore, the cumulative concentration C_2 of gaseous pollutants also decreases with increase in the growth rate of raindrops q in the atmosphere.

It can be easily shown that $\frac{dC_p}{dq}<0$ which indicates that as the growth rate of raindrops q increases (i.e. as the rain intensity increases), the cumulative concentration of mixture of particulate pollutants decreases in the atmosphere. Now to analyze the stability behavior of the equilibrium point, we state the following theorems for its local and nonlinear stability.

Theorem 3.1: *The equilibrium E^* is locally asymptotically stable without any condition.*

Proof: To establish the local stability behavior of E^*, we compute the eigenvalues of the Jacobian matrix evaluated at E^*. The three eigenvalues of the Jacobian matrix are obtained as, $-(k_1+\nu_1 C_r^*)$, $-(k_2+\nu_2 C_r^*)$ $-(k_1+\nu_1 C_r^*)$ and $-(\delta_p+\alpha_p C_r^*)$ which are negative. The remaining eigenvalues are given by the following characteristic equation,

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0,$$

where $a_1=\frac{q}{C_r^*}+\delta_1+\alpha_1 C_r^*+\alpha_2 C_r^*$,

$$a_2=(r_0+r_2C_2^*)(\delta_1+\alpha_1 C_r^*)+(\delta_1+\alpha_1 C_r^*)(\delta_2+\alpha_2 C_r^*)+\delta_1 r_1 C_1^*$$

$$+(r_0 + r_1 C_1^*)(\delta_2 + \alpha_2 C_r^*) + \delta_2 r_2 C_2^*,$$

$$a_3 = [r_0(\delta_2 + \alpha_2 C_r^*) + \delta_2 r_2 C_2^*] + r_2 C_2^*)[(\delta_1 + \alpha_1 C_r^*) + \delta_1 r_1 C_1^*(\delta_2 + \alpha_2 C_r^*)]$$

It is easy to check that the conditions of Routh-Hurwitz criterion for the above characteristics equation i.e. $a_1 > 0, a_2 > 0, a_3 > 0$ and $a_1 a_2 > a_3$ are satisfied. Hence, all the eigenvalues of the Jacobian matrix corresponding to E^* are either negative or have negative real parts. Hence, E^* is locally asymptotically stable.

To establish the nonlinear stability of E^*, we need the bounds of dependent variables involved. For this we find the region of attraction for which we state the following lemma without proof.

Lemma 3.1: *The set*

$$\Omega = \left\{ (C_r, C_1, C_2, C_p, C_{1a}, C_{2a}) : 0 \le C_r \le \frac{q}{r_0}, 0 \le C_1 + C_{1a} \le \frac{Q_1}{\delta_m}, \right.$$
$$\left. 0 \le C_2 + C_{2a} \le \frac{Q_2}{\delta_n}, 0 \le C_p \le \frac{1}{\delta_p}\left(\frac{\gamma_1 Q_1}{\delta_m} + \frac{\gamma_2 Q_2}{\delta_n} \right) \right\}$$

attracts all solutions initiating in the interior of the positive octant, where $\delta_m = \min\{\delta_1, k_1\}, \delta_n = \min\{\delta_2, k_2\}$.

Theorem 3.2: *Let the following inequalities hold in* Ω,

(3.8) $$r_1 \alpha_1 Q_1 C_r^* < \frac{1}{15}(\delta_1 + \alpha_1 C_r^*) r_0 \delta_1,$$

(3.9) $$r_2 \alpha_2 Q_2 C_r^* < \frac{1}{15}(\delta_2 + \alpha_2 C_r^*) r_0 \delta_2,$$

then E^* *is nonlinearly asymptotically stable.*

Proof: Consider the following positive definite function about E^*,

(3.10) $$U = \frac{1}{2}\Big[m_0(C_r - C_r^*)^2 + m_1(C_1 - C_1^*)^2 + m_2(C_2 - C_2^*)^2$$
$$+ m_3(C_p - C_p^*)^2 + m_4(C_{1a} - C_{1a}^*)^2 + m_5(C_{2a} - C_{2a}^*)^2 \Big],$$

where $m_i, i=0-5$ are positive constants to be chosen appropriately.
Differentiating U with respect to 't' along the solutions of the system (2.1) - (2.6), we get after some simplifications,

$$
\begin{aligned}
(3.11)\quad \frac{dU}{dt} = & -m_0(r_1C_1+r_2C_2)(C_r-C_r^*)^2 - m_1\alpha_1C_r(C_1-C_1^*)^2 \\
& -m_2\alpha_2C_r(C_2-C_2^*)^2 - m_3\alpha_pC_r(C_p-C_p^*)^2 - m_0r_0(C_r-C_r^*)^2 \\
& -m_1\delta_1(C_1-C_1^*)^2 - m_2\delta_2(C_2-C_2^*)^2 - m_3\delta_p(C_p-C_p^*)^2 \\
& -m_4(k_1+\nu_1C_r^*)(C_{1a}-C_{1a}^*)^2 - m_5(k_2+\nu_2C_r^*)(C_{2a}-C_{2a}^*)^2 \\
& -(m_0r_1C_r^*+m_1\alpha_1C_1^*)(C_r-C_r^*)(C_1-C_1^*) - (m_0r_2C_r^*+m_2\alpha_2C_2^*) \\
& \times(C_r-C_r^*)(C_2-C_2^*) - m_3\alpha_pC_p^*(C_r-C_r^*)(C_p-C_p^*) + m_4(\alpha_1C_1-\nu_1C_{1a}) \\
& \times(C_r-C_r^*)(C_{1a}-C_{1a}^*) + m_5(\alpha_5C_5-\nu_5C_{2a})(C_r-C_r^*)(C_{2a}-C_{2a}^*) \\
& +m_3\gamma_1(C_1-C_1^*)(C_p-C_p^*) + m_4\alpha_1C_r^*(C_1-C_1^*)(C_{1a}-C_{1a}^*) \\
& +m_3\gamma_2(C_2-C_2^*)(C_p-C_p^*) + m_5\alpha_2C_r^*(C_2-C_2^*)(C_{2a}-C_{2a}^*).
\end{aligned}
$$

Now for $\frac{dU}{dt}$ to be negative definite, the following sufficient conditions must be satisfied

$$(3.12)\qquad (m_0r_1C_r^*+m_1\alpha_1C_1^*)^2 < \frac{4}{15}m_0m_1r_0\delta_1,$$

$$(3.13)\qquad (m_0r_2C_r^*+m_2\alpha_2C_2^*)^2 < \frac{4}{15}m_0m_2r_0\delta_2,$$

$$(3.14)\qquad m_3(\alpha_pC_p^*)^2 < \frac{4}{15}m_0r_0\delta_p,$$

$$(3.15)\qquad m_4(\alpha_1C_1-\nu_1C_{1a})^2 < \frac{2}{5}m_0r_0(k_1+\nu_1C_r^*),$$

$$(3.16)\qquad m_5(\alpha_2C_2-\nu_2C_{2a})^2 < \frac{2}{5}m_0r_0(k_2+\nu_2C_r^*),$$

$$(3.17)\qquad m_3\gamma_1^{\,2} < \frac{4}{9}m_1\delta_1\delta_p,$$

(3.18) $$m_3\gamma_2^{\;2} < \frac{4}{9} m_2\delta_2\delta_p ,$$

(3.19) $$m_4(\alpha_1 C_r^*)^2 < \frac{2}{3} m_1(k_1 + \nu_1 C_r^*)\delta_1 ,$$

(3.20) $$m_5(\alpha_2 C_r^*)^2 < \frac{2}{3} m_2(k_2 + \nu_2 C_r^*)\delta_2 .$$

After maximizing the L.H.S. of above conditions, the stability conditions can be obtained appropriately.

Now choosing $m_0 = 1$, $m_1 = \dfrac{r_1 C_r^*}{\alpha_1 C_1^*}$, $m_2 = \dfrac{r_2 C_r^{\;*}}{\alpha_2 C_2^{\;*}}$,

$$m_3 < \frac{1}{3}\delta_p \min\left\{ \frac{4}{5}\frac{r_0}{(\alpha_p C_p^*)^2}, \frac{4}{3}\frac{\delta_1\delta_p r_1 C_r^*}{\gamma_1^{\;2}\alpha_1 C_1^*}, \frac{4}{3}\frac{\delta_2\delta_p r_2 C_r^*}{\gamma_2^{\;2}\alpha_2 C_2^*} \right\},$$

$$m_4 < (k_1 + \nu_1 C_r^*)\min\left\{ \frac{2}{5}\frac{r_0\delta_m^{\;2}}{(\alpha_1 + \nu_1)^2 Q_1^{\;2}}, \frac{2}{3}\frac{r_1\delta_1 C_r^*}{(\alpha_1 C_r^*)^2 \alpha_1 C_1^*} \right\},$$

$$m_5 < (k_2 + \nu_2 C_r^*)\min\left\{ \frac{2}{5}\frac{r_0\delta_n^{\;2}}{(\alpha_2 + \nu_2)^2 Q_2^{\;2}}, \frac{2}{3}\frac{r_2\delta_2 C_r^*}{(\alpha_2 C_r^*)^2 \alpha_2 C_2^*} \right\},$$

$\dfrac{dU}{dt}$ will be negative definite provided the conditions (3.8) and (3.9) are satisfied. This implies that U is a Lyapunov function and hence the theorem is proved.

Remark: We note here that if the parameters r_1, r_2, α_1 and α_2 tend to zero, the conditions (3.8) and (3.9) are automatically satisfied. This shows that the removal parameters have destabilizing effect on the system. From these conditions, we further note that if the rates of emission of both the gaseous pollutants i.e. Q_1 and Q_2 are small enough then the possibility of satisfying these is more plausible.

4. Numerical Simulation and Discussion

To study the model system (2.1)-(2.6) numerically, we consider the following set of parameter values so as to satisfy the stability conditions,

$$q = 5,\ r_0 = 0.4,\ r_1 = 0.02,\ r_2 = 0.04,\ Q_1 = 4,\ Q_2 = 2,$$
$$\delta_1 = 0.20,\ \delta_2 = 0.15,\ \delta_p = 0.35,\ \alpha_1 = 0.45, \alpha_2 = 0.30,\ \alpha_p = 0.75,$$

$$k_1 = 0.55,\ k_2 = 0.60,\ \nu_1 = 0.65, \nu_2 = 0.60,\ \gamma_1 = 0.16, \gamma_2 = 0.14.$$

The equilibrium values of different variables in E^* are calculated as follows,

$$C_r^* = 11.4334,\ C_1^* = 0.7483,\ C_2^* = 0.5586, C_p^* = 0.0022,\ C_{1a}^* = 0.4823,\ C_{2a}^* = 0.2568\ .$$

The eigenvalues corresponding to E^* are obtained as follows,

$$-36.1999, -31.6198, -29.2798, -0.3998, -21.7136, -14.4955.$$

Since all the eigenvalues are negative and hence E^* is locally asymptotically stable.

The nonlinear stability behavior of E^* with respect to above parameter values in $C_1 - C_2$ and $C_r - C_{1a}$ plane is shown in the Figs. (1)-(2) respectively. The variation of cumulative concentrations of both the gaseous pollutants (i.e. C_1 and C_2) and the mixture of particulate matters (C_p) with time 't' is shown in Figs. (3)-(5) for different values of growth rate of raindrops i.e. $q = 0, 10, 20$. From these figures, it can be seen that the cumulative concentrations of both the gaseous pollutants (i.e. C_1 and C_2) and the mixture of particulate matters (C_p) decrease as growth rate q of raindrops increases. This implies that the concentrations of gaseous pollutants and particulate matters can be washed out significantly if the intensity of rain increases. It is also noted that in the absence of raindrops in the atmosphere i.e. $q = 0$, the cumulative concentration of gaseous pollutants as well as particulate matters increase continuously. In Figs. (6)-(7), the variation of concentrations of absorbed phases of gaseous pollutants (i.e. C_{1a} and C_{2a}) with time 't' is shown for different values of growth rate of raindrops i.e. $q = 10, 15, 20$. It can be noted that the concentrations of gaseous pollutants in absorbed phases (i.e. C_{1a} and C_{2a}) decrease as growth rate of raindrops q increases. This is due to the fact that as the intensity of rain increases, both the gaseous pollutants will be washed out significantly leading to decline in the concentration of gaseous pollutants in the absorbed phases. In Figs. (8)-(9), the variation of concentrations C_{1a} and C_{2a} of the gaseous pollutants absorbed in raindrops with time '*t*' at $q = 5$ is shown at different levels of removal parameters ν_1 and ν_2 respectively. From these figures, it is observed that the concentrations of both the gaseous pollutants in absorbed phases decrease with increase in respective removal parameters. In Fig. (10), the variation of cumulative concentration C_p of the mixture of particulate matters with time '*t*' is shown at different levels of conversion parameters γ_1 and γ_2 of formation of particulate matters. From this figure, it is seen that the cumulative concentration of particulate matters (C_p) increases with increase in the values of of γ_1 and γ_2. It is also noted that if the conversion rate coefficients γ_1 and γ_2 of the two gaseous pollutants to particulate matters are zero then the equilibrium concentration of particulate matters (C_p) may tend to zero.

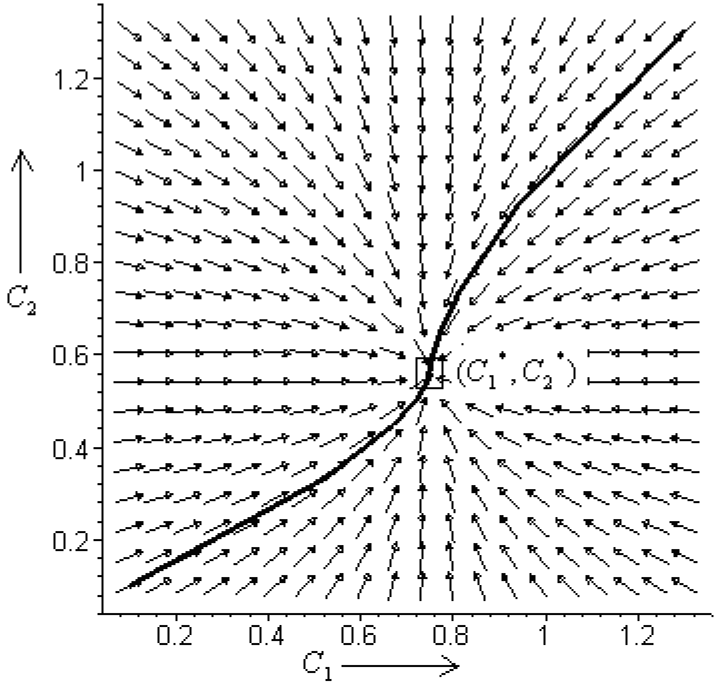

Figure 1. Nonlinear stability in $C_1 - C_2$ plane

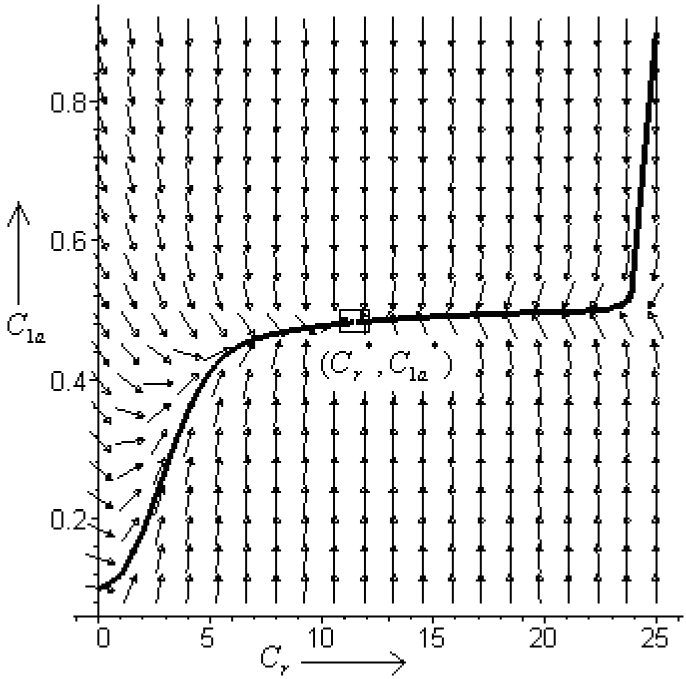

Figure 2. Nonlinear stability in $C_r - C_{1a}$ plane

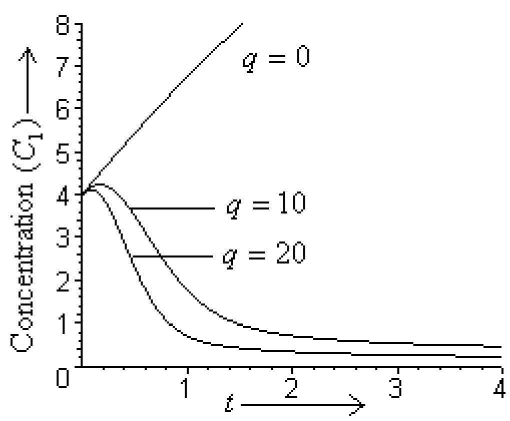

Figure 3. Variation of C_1 with time 't' for different values of growth rate of raindrops q

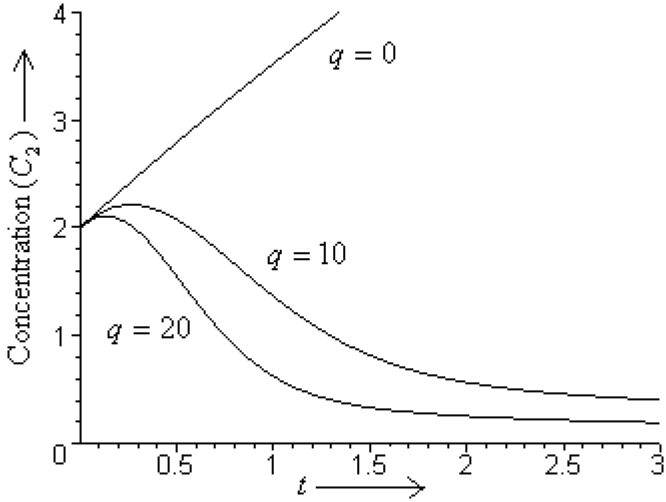

Figure 4. Variation of C_2 with time 't' for different values of growth rate of raindrops q

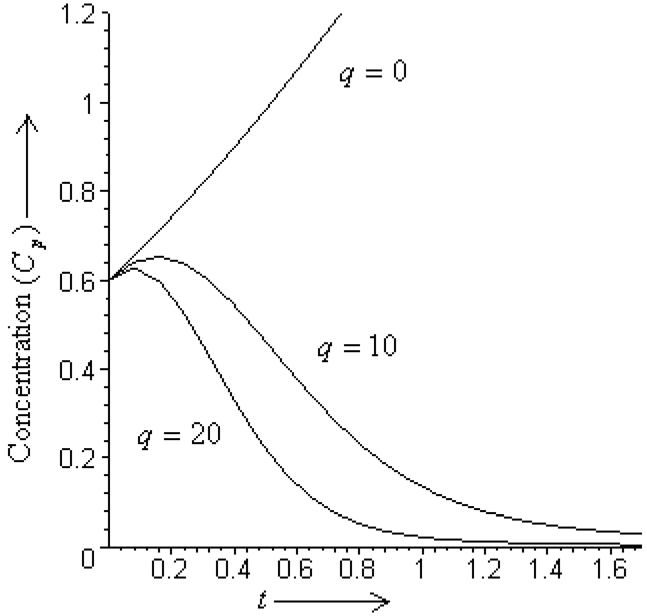

Figure 5. Variation of C_p with time 't' for different values of growth rate of raindrops q

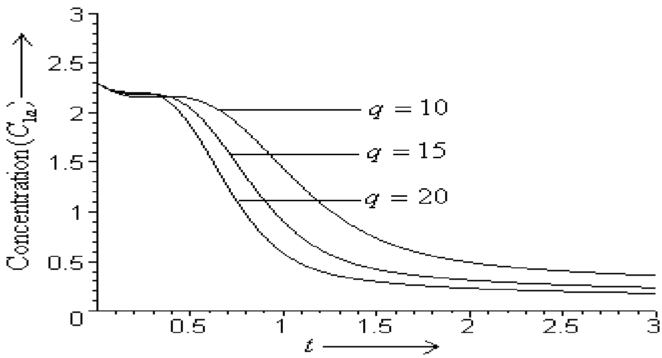

Figure 6. Variation of C_{1a} with time 't' for different values of growth rate of raindrops q

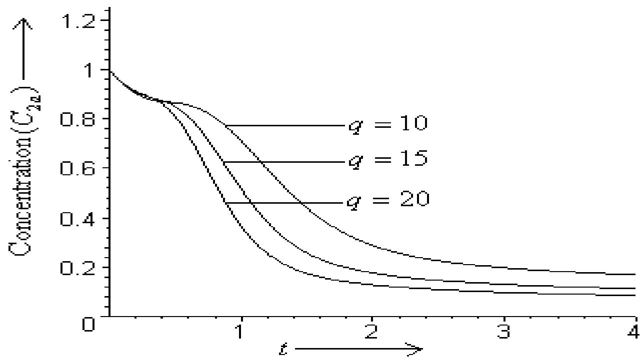

Figure 7. Variation of C_{2a} with time 't' for different values of growth rate of raindrops q

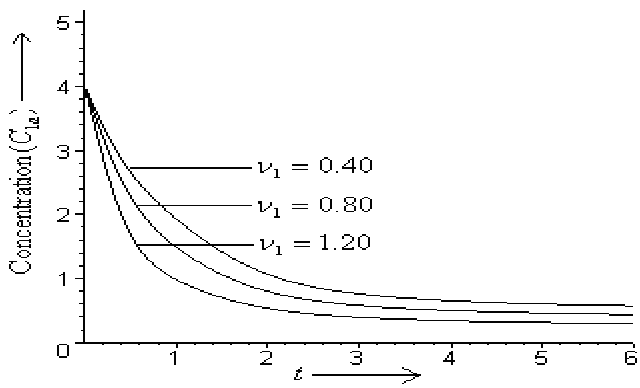

Figure 8. Variation of C_{1a} with time 't' for different values of removal rate ν_1

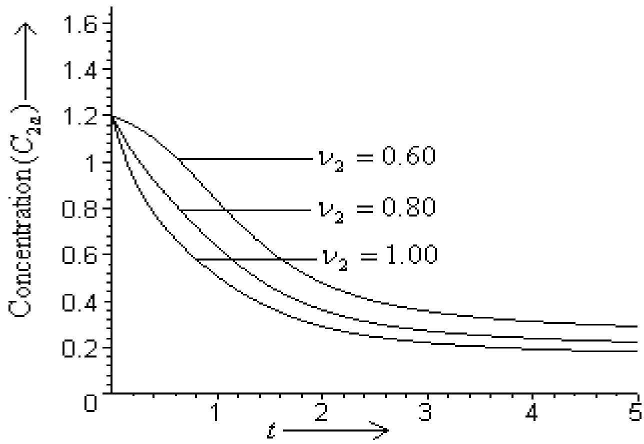

Figure 9. Variation of C_{2a} with time 't' for different values of removal rate ν_2

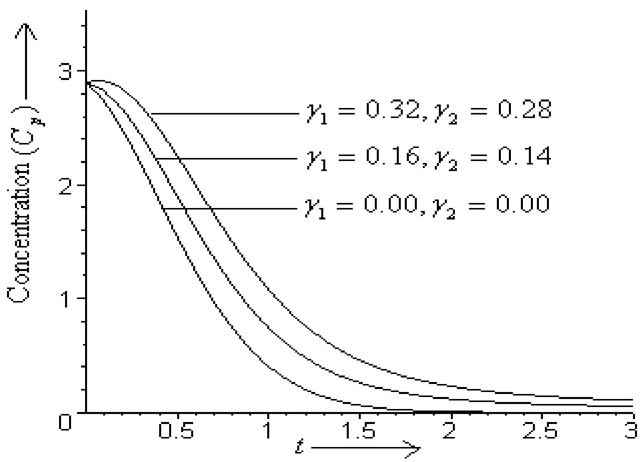

Figure 10. Variation of C_p with time 't' for different values of conversion rates γ_1 and γ_2

5. Conclusion

A nonlinear mathematical model is proposed for the removal of two gaseous pollutants and the mixture of particulate matters formed separately by these two gases from the atmosphere of a city by rain. The removal of gaseous pollutants is assumed to take place due to absorption by raindrops falling on the ground while the removal of mixture of particulate matters by the processes of impaction or entrapment by falling raindrops. The model has been analyzed qualitatively using stability theory of differential equations. It is shown that the gaseous pollutants and consequently formed mixture of particulate matters can be removed from the atmosphere under appropriate conditions and the rate of removal would depend upon the rates of emission of pollutants, the intensity of rain and other removal parameters. The equilibrium level of two gaseous pollutants and that of the mixture of particulate matters in the atmosphere has been found to be much smaller after rain than their corresponding value before rain. It has also been observed that if the intensity of rain is high enough (large growth rate of raindrops), the equilibrium concentration of all the pollutants reduces considerably in the atmosphere and may even lead to complete wash out of pollutants. The numerical simulations have also been carried out to support the analytical results quantitatively. Thus, a rain system provides an important removal mechanism for the gaseous pollutants and particulate matters formed by these gases in the atmosphere.

References

1. C. Andronache, W. L. Chameides, D. D. Davis, B. E. Anderson, R. F. Pueschel, A. R. Bandy, D. C. Thornton, R. W. Talbot, P. Kasibhatla and C. S. Kiang, Gas-to-particle Conversion of Tropospheric Sulfur as Estimated from Observations in the Western North Pacific During PEM-West B, *Journal of Geophysical Research*, **102(D23)** (1997), 28511-28538

2. C. Kesavachandran, B. S. Pangtey, V. Bihari, M. Fareed, M. K. Pathak, A. K. Srivastava et al., Particulate Matter Concentration in Ambient Air and Its Effects on Lung Functions Among Residents in the National Capital Region, India, *Environ Monitor. Assess*., **185** (2013), 1265–1272, pmid:22527464.

3. J. H. Seinfeld, *Atmospheric Chemistry and Physics of Air Pollution*, John Wiley and Sons, New York, 1986.

4. S. Kumar, An Eulerian Model for Scavenging of Pollutants by Raindrops, *Atmos. Environ.*, **19** (1985), 769-778.

5. F. L. T. Goncalves, A. M. Ramos, S. Freitas, M. A. Silva Dias and O. Massambani, In-Cloud and Below-Cloud Numerical Simulation of Scavenging Processes at Serra Do Mar Region, SE Brazil, *Atmos. Environ*., **36** (2002), 5245-55.

6. R. Naresh, Modeling the Dispersion of Air Pollutant from a Time Dependent Point Source: Effect of Precipitation Scavenging, *J. MACT,* **32** (1999), 77-91.

7. J. M. Hales, Fundamentals of the Theory of Gas Scavenging by Rain, *Atmos. Environ*., **6** (1972), 635-650.

8. T. D. Davies, Precipitation Scavenging of Sulfur Dioxide in an Industrial Area, *Atmos. Environ*., **10** (1976), 879 – 890.

9. R. J. Engelmann, The Calculation of Precipitation Scavenging, *Metrology and Atomic Energy*, (ed.: Slade D.H.) U.S.A.E.C. 68-60097 (1968), 208-218.

10. C. Johansson and P. Johansson, Particulate Matter in the Underground of Stockholm, *Atmos. Environ*., **37** (2003), 3-9.

11. V. P. Sharma, H. C. Arora and R. K. Gupta, Atmospheric Pollution Studies at Kanpur – Suspended Particulate Matter, *Atmos. Environ*., **17** (1983), 1307-1314.

12. J. Pandey, M. Agarwal, N. Khanam, D. Narayan and D. N. Rao, Air Pollutant Concentration in Varanasi, India, *Atmos. Environ*., **26B** (1992), 91-98.

13. A. Pillai, M. S. Naik, G. Momin, P. Rao, K. Ali, H. Rodhe and L. Granet, Studies of Wet Deposition and Dust Fall at Pune, India, *Water, Air and Soil Pollution,* **30** (2001), 475-480.

14. K. Ravindra, S. Mor, J. S. Kamyotra and C. P. Kaushik, Variation of Spatial Pattern of Air Pollutants Before and During Initial Rain of Monsoon, *Environ. Monitor. Assess.*, **87** (2003), 145-153.

15. R. Korber, Air Pollution on a Rainy Day: Understanding Weather & Air Quality, (2019), https://blog.breezometer.com/air-pollution-weather-rainy-day.

16. S. Kim, Ki-Ho Hong, H. Jun, Y-J. Park, M. Park and Y. Sunwoo, Effect of Precipitation on Air Pollutant Concentration in Seoul, Korea, *Asian J. Atmos. Environ.*, **8(4)** (2014), 202-211.

17. Y. Wu, J. Liu, J. Zhai, L. Cong, Y. Wang, W. Ma et al., Comparison of Dry and Wet Deposition of Particulate Matter in Near-Surface Waters During Summer, *PLoS ONE* **13(6)** (2018), e0199241. https://doi.org/10.1371/journal.pone.0199241.

18. A. Y. Tai, L. A. Chen, X. Wang, J. C. Chow and J. G. Watson, Atmospheric Deposition of Particles at a Sensitive Alpine Lake: Size-Segregated Daily and Annual Fluxes from Passive Sampling Techniques, *Sci. Total Environ.*, **579** (2017), 1736-1744. pmid:27932212.

19. L. J. Zhu, J. K. Liu, L. Cong, W. M. Ma, W. Ma and Z. M. Zhang, Spatiotemporal Characteristics of Particulate Matter and Dry Deposition Flux in the Cuihu Wetland of Beijing, *PLoS ONE*, **11** (2016), e0158616. pmid:27437688.

20. S. Y. Bae, C. H. Jung and Y. P. Kim, Derivation and Verification of an Aerosol Dynamics Expression for the Below-Cloud Scavenging Process Using the Moment Method, *J. Aerosol Sci.*, **41** (2010), 266–280.

21. L. C. Guo, Y. Zhang, H. Lin, et al., The Washout Effects of Rainfall on Atmospheric Particulate Pollution in Two Chinese Cities, *Environ. Pollut*. **215** (2016), 195-202. doi:10.1016/j.envpol.2016.05.003

22. N. Roldán-Henao, C. D. Hoyos, L. Herrera-Mejía, and A. Isaza, An Investigation of the Precipitation Net Effect on the Particulate Matter Concentration in a Narrow Valley: Role of Lower-Troposphere Stability, *J. Appl. Meteor. Climatol.*, **59 (**2020**)** 401-426, https://doi.org/10.1175/JAMC-D-18-0313.1.

23. R. Naresh, S. Sundar and J. B. Shukla, Modeling the Removal of Gaseous Pollutants and Particulate Matters from the Atmosphere of a City, *Nonlinear Analysis: RWA*, **8** (2007), 337-344.

24. S. N. Pandis and J. H. Seinfeld, On the Interaction Between Equilibration Processes and Wet or Dry Deposition, *Atmos. Environ*., **24**A (1990), 2313-2327.

25. J. B. Shukla, R. Nallaswamy, S. Verma and J. H. Seinfeld, Reversible Absorption of a Pollutant from an Area Source in a Stagnant Fog Layer, *Atmos. Environ*., **16** (1982) 1035-1037.

26. J. B. Shukla, M. Agarwal and R. Naresh, An Ecological Type Nonlinear Model for Removal Mechanism of Air Pollutants, *In: Precipitation Scavenging and Atmosphere Surface Exchange, (Ed: S. E. Schwartz and W.G.N. Slinn),* Hemisphere Publishing Corp., Richland, Washington, U.S.A. **3** (1992), 1255-1263.

27. J. B. Shukla, A. K. Misra , S. Sundar and R. Naresh, Effect of Rain on Removal of a Gaseous Pollutant and Two Different Particulate Matters from the Atmosphere of a City, *Math. Comp. Model.*, **48** (2008), 832-844.

28. J. B. Shukla, N. Swaroop, S. Sundar and R. Naresh, Modelling the Effect of Traffic Pollutants on Human Population with a Control Strategy, *Modeling Earth Systems and Environment*, (2020), Accepted, https://doi.org/10.1007/s40808-020-00825-7.

29. S. Sundar, R. Naresh, A. K. Misra and J. B. Shukla, A Nonlinear Mathematical Model to Study the Interactions of Hot Gases with Cloud Droplets and Raindrops, *Appl. Math. Model*., **33** (2009) 3015-3024.

30. S. Sundar, Rajan K. Sharma and R. Naresh, Modelling the Role of Cloud Density on the Removal of Gaseous Pollutants and Particulate Matters from the Atmosphere, *Appl. Appl. Math.: An Int. J.*, **8(2)** (2013), 416-435.

31. S. Sundar and R. Naresh, Modeling the Effect of Dust Pollutants on Plant Biomass and their Abatement from the Near Earth Atmosphere, *Modeling Earth Systems and Environment*, **3(1)** (2017), 42, DOI 10.1007/s40808-017-0302-3.