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Some Geometric Properties of Generalized β−Change of Finsler Metric with an h–Vector

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Abstract: In the present paper, we have considered the generalized $\beta-$change of Finsler metric with an h-vector given by $\overline{L}(x,y)=f(L(x,y),\beta^{1)},\beta^{2)},\ldots,\beta^{m)})$, where $\beta^{1)},\beta^{2)},\beta^{3)},\ldots,\beta^{m)}$ all are linearly independent one-form defined as $\beta^{r)}=b_i^{r)}(x,y)\,y^i$ and $b_i^{r)}(x,y)$ is an h-vector in (M^n,L). We have obtained the v-curvature of Finsler space characterized by generalized $\beta-$change of Finsler metric and derive some results on S-4 like Finsler space for the change.

Keywords: Generalized $\beta-$change, $\beta-$change, h-vector, Finsler space.

1. Introduction

In 1971, Matsumoto[1] introduced the transformation of Finsler metric

$$\overline{L}(x,y)=L(x,y)+\beta(x,y),$$

where $\beta(x,y)=b_i(x)\,y^i$ is a one-form. In 1984, Shibata[2] introduced the transformation of Finsler metric

$$\overline{L}(x,y)=f(L,\beta),$$

where $\beta=b_i(x)\,y^i$, $b_i(x)$ are components of a covariant vector in (M^n,L) and f is positively homogeneous function of degree one in L and β.

In[3], we studied generalized $\beta-$ change defining as

$$L(x,y) \rightarrow \bar{L}(x,y) = f(L, \beta^{1)}, \beta^{2)}, \ldots, \beta^{m)}),$$

where f is positively homogeneous function of degree one in $L, \beta^{1)}, \beta^{2)}, \ldots, \beta^{m)}$ where $\beta^{1)}, \beta^{2)}, \ldots, \beta^{m)}$ are linearly independent one-form.

H. Izumi[4] while studying the conformal transformation of Finsler spaces, introduced the concept of h-vector b_i, which is v-covariant constant with respect to the Cartan connection and satisfies $LC_{ij}^{h} b_h = \rho h_{ij}$, where ρ is a non-zero scalar function, C_{ij}^{h} are components of Cartan tensor and h_{ij} are components of angular metric tensor. Thus if b_i is $h-$vector then

$$\text{(1.1)} \qquad \begin{cases} (a) & b_i|_k = 0, \\ (b) & LC_{ij}^{h} b_h = \rho h_{ij}. \end{cases}$$

This gives

$$\text{(1.2)} \qquad L\dot{\partial}_j b_i = \rho h_{ij}.$$

Since $\rho \neq 0$ and $h_{ij} \neq 0$, the $h-$vector b_i depends not only on positional coordinates but also on directional arguments. Izumi[5] showed that ρ is independent of directional arguments.

A Finsler space $F^n = (M^n, L)$ is said to be S-4 like Finsler space if there exists a symmetric and indicatory tensor K_{ij} such that the v-curvature tensor has the form[5]

$$\text{(1.3)} \qquad L^2 S_{hijk} = (h_{hj}K_{ik} + h_{ik}K_{hj} - j\,|\,k),$$

where $-j\,|\,k$ means interchange of j and k and subtract the quantities within the bracket.

In the present paper, we have found out the relation between the v-curvature tensor of the original Finsler space and the other which is Finsler space given by generalized $\beta-$ change $\bar{L} = f(L, \beta^{1)}, \beta^{2)}, \ldots, \beta^{m)})$.

2. Preliminaries

Let M^n be an n-dimensional smooth manifold and $F^n=(M^n,L)$ be an n-dimensional Finsler space equipped with the fundamental function L on M^n. The metric tensor $g_{ij}(x,y)$ and Cartan's C-tensor $C_{ijk}(x,y)$ are given by

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\,\partial y^j}, \qquad C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k},$$

respectively and we can introduce the Cartan's connection $C\Gamma=(F^i_{jk},-N^i_j,C^i_{jk})$ in F^n. Here, we have considered the following transformation of Finsler metric:

$$\bar{L}(x,y)=f(L,\beta^{1)},\beta^{2)},\ldots,\beta^{m)}), \tag{2.1}$$

and such transformation is called generalized $\beta-$conformal change of Finsler metric, where f is positively homogeneous function of degree one in $L,\beta^{1)},\beta^{2)},\ldots,\beta^{m)}$ and $\beta^{1)}$, $\beta^{2)},\ldots,\beta^{m)}$ all are linearly independent one-form and defined as $\beta^{r)}=b^{r)}_i(x,y)\,y^i$ and $b^{r)}_i(x,y)$ is an $h-$vector in (M^n,L). Homogeneity of f gives

$$Lf_0+f_r\beta^{r)}=f, \tag{2.2}$$

where the subscripts 0 and r denote the partial derivative with respect to L and $\beta^{r)}$ respectively. If we write $\overline{F^n}=(M^n,\bar{L})$, then the Finsler space $\overline{F^n}$ is said to be obtained from F^n by generalized $\beta-$conformal change with $h-$vector. The quantities corresponding to $\bar{F}^n$ are denoted by putting bar on those quantities.

To find the relation between fundamental quantities of (M^n,L) and $(M^n,\bar{L})$, we use the following results:

$$\dot{\partial}_i\beta^{r)}=b^{r)}_i, \quad \dot{\partial}_i L=l_i, \quad \dot{\partial}_i l_i=L^{-1}h_{ij}, \tag{2.3}$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by

$$h_{ij} = g_{ij} - l_i\, l_j = L\dot{\partial}_i\, \dot{\partial}_j L\,.$$

The v-curvature tensor S_{hijk} of (M^n, L) is given by[5],

$$S_{hijk} = C^{a)}_{hk} C_{aij} - C^{a)}_{hj} C_{aik}\,.$$

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Differentiating equation (2.2) with respect to L and $\beta^{s)}$, respectively, we get

(3.1) $$L f_{00} + f_{0r} \beta^{r)} = 0\,,$$

and

(3.2) $$L f_{0s} + f_{rs} \beta^{r)} = 0\,.$$

The successive differentiation of (2.1) with respect to y^i and y^j give

(3.3) $$\overline{l_i} = f_0\, l_i + f_r\, b_i^{r)}\,,$$

(3.4) $$\overline{h_{ij}} = \frac{f f_0}{L} h_{ij} + f f_{00}\, l_i\, l_j + f f_{0r}\, (b_i^{r)} l_j + b_j^{r)} l_i) + f f_{rs}\, b_i^{r)}\, b_j^{s)}\,.$$

Using equations (3.1) and (3.2) in equation (3.4), we have

(3.5) $$\overline{h_{ij}} = \frac{f f_0}{L} h_{ij} + f f_0 \left(b_i^{r)} - \frac{\beta^{r)}}{L} l_i \right)\left(b_j^{r)} - \frac{\beta^{s)}}{L} l_j \right).$$

If we put $m_i^{r)} = b_i^{r)} - \frac{\beta^{r)}}{L} l_i$, equation (3.5) may be written as

(3.6) $$\overline{h_{ij}} = \frac{f f_0}{L} h_{ij} + f f_{rs}\, m_i^{r)} m_j^{s)}$$

From equations (3.3) and (3.6), we get the following relation between metric tensors of (M^n, L) and $(M^n, \bar{L})$.

(3.7) $$\bar{g}_{ij} = \frac{ff_0}{L} g_{ij} + \left(f_0^2 - \frac{ff_0}{L} \right) l_i\, l_j + ff_{rs}\, m_i^{r)} m_j^{s)} + f_0\, f_r\, (b_i^{r)} l_j + b_j^{r)} l_i) + f_r\, f_s\, b_i^{r)} b_j^{s)} .$$

The inverse metric tensor of $\bar{F}^n$ is derived as[6]

(3.8) $$\bar{g}^{ij} = \frac{L}{ff_0} g^{hk} + \frac{L^3}{f^3 f_0\, (f_0 + Lf_{rs'} b^{rs'})} \Big\{ (f_0\, f_r + ff_{0r})(\frac{f\beta^{r)}}{L} + f_s\, b^{rs})\, l^h\, l^k$$
$$- \frac{f^2}{L} f_{rs} b^{r)h}\, b^{s)k} - \frac{f}{L}(f_0\, f_r + ff_{0r})(l^h\, b^{r)k} + l^k b^{r)h}) \Big\},$$

where we put $b^{r)i} = g^{ij} b_j^{r)}$, $l^i = g^{ij} l_j$ and $b^{rs} = b_i^{r)}\, b^{s)i} - \dfrac{\beta^{r)} \beta^{s)}}{L^2}$.

Now,

(3.9) $$\begin{cases} (a)\ \dot{\partial}_j\, m_i^{r)} = -\dfrac{1}{L} m_i^{r)}\, l_j - \dfrac{\beta^{r)}}{L^2} h_{ij}, \\ (b)\ \dot{\partial}_i\, f = f_0\, l_i + f_r\, b_i^{r)}, \\ (c)\ \dot{\partial}_i\, f_{rs} = f_{rs0}\, l_i + f_{rst}\, b_i^{r)}. \end{cases}$$

Differentiating equation (3.6) with respect to y^i and using equations (3.7), (2.3), (3.2), (3.1), (3.3) and (3.9), we get

(3.10) $$2\bar{C}_{ijk} = \frac{2\, ff_0}{L} C_{ijk} + \frac{(f_0\, f_r + ff_{0r})}{L} \left(h_{ij} m_k^{r)} + h_{jk} m_i^{r)} + h_{ki} m_j^{r)} \right)$$
$$+ \left(f_{rs}\, f_t + f_{st}\, f + f_{tr}\, f_{sr} + ff_{rst} \right) m_i^{r)} m_j^{s)} m_k^{t)} .$$

It is to be noted that

(3.11) $$m_i^{r)} l^i = 0,\ m_i^{r)}\, b^{s)i} = m^{rs} = m_i^{r)} m^{s)i},\ h_{ij} l^j = 0,\ h_{ij} m^{r)j} = h_{ij} b^{r)j} = m_i^{r)},$$

where $m^{r)i} = g^{ij} m_j^{r)} = b^{r)i} - \dfrac{\beta^{r)}}{L} l^i$.

To find $\bar{C}^h_{ij} = \bar{g}^{hk}\bar{C}_{ijk}$, we use equations (3.8), (3.10) and (3.11), we get

(3.12) $$\bar{C}^h_{ij} = C^h_{ij} + \frac{1}{2ff_0}(f_0 f_r + ff_{0r})\left(h_{ij}m^{r)h} + h^h_j m^{r)}_i + h^h_i m^{r)}_j\right)$$

$$+\frac{L}{2ff_0}(f_{rs}f_t + f_{st}f_r + f_{tr}f_s + ff_{rst})m^{r)}_i m^{s)}_j m^{t)h}$$

$$\left. -\frac{L}{f\left(f_0 + Lf_{rs}m^{rs}\right)}C^a_{.ij}n^h_{r)} - \frac{Lm^{rs}\left(f_0 f_r + ff_{0r}\right)}{2f^2 f_0\left(f_0 + Lf_{rs}m^{rs}\right)}h_{ij}n^h_r \right),$$

where $C^a_{.jk} = C_{ijk}b^{r)i}$ and $n^h_{r)} = ff_{rs}b^{s)h} + (f_0 f_r + ff_{0r})l^h$.

We have the following corresponding to the vector with components $n^{r)i}$ and $m^{r)i}$

(3.13) $$C_{ijk}m^{r)i} = C^a_{.jk},\ C_{ijk}n^i_{r)} = ff_{rs}C^a_{.jk},\ m^{r)}_i n^i_{s)} = ff_{rs}m^{rs}.$$

To find the v-curvature tensor of $(M^n, \bar{L})$ with respect to Cartan's connection, we use the following:

(3.14) $$C^h_{ij}h_{hk} = C_{ijk},\ h^k_j h^i_k = h^i_j,\ h_{ij}n^i_{r)} = ff_{rs}m^{s)}_j.$$

The v-curvature tensor $\bar{S}_{hijk}$ of $(M^n, \bar{L})$ is defined as

(3.15) $$\bar{S}_{hijk} = \bar{C}^{r'}_{ij}\bar{C}_{r'hk} - \bar{C}^{r'}_{ik}\bar{C}_{r'hj}.$$

From equations (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15), we get the following relation between v-curvature tensor of the original Finsler space and the other which is Finsler space given by generalized $\beta-$change with $h-$vector

(3.16) $$\bar{S}_{hijk} = \frac{ff_0}{L}S_{hijk} + [HC^{r)}_{.ik}C^{s)}_{.hj} + I(C^{r)}_{.ij}m^{s)}_h m^{t)}_k + C^{r)}_{.hk}m^{s)}_i m^{t)}_j) + Jh_{ij}h_{hk}$$

$$+K(C^{r)}_{.ij}h_{hk} + C^{r)}_{.hk}h_{ij}) + N(h_{ij}m^{r)}_h m^{s)}_k + h_{hk}m^{r)}_i m^{s)}_j) - (j/k)],$$

where

$$H = \frac{ff_0 f_{rs}}{f_0 + Lf_{rs'}m^{rs'}},$$

$$I = \frac{1}{(f_0 + Lf_{rs'}m^{rs'})}\left\{ \frac{f_0(f_{rs}f_t + f_{st}f_r + f_{tr}f_s + ff_{rst})}{2} - f_{rs}(f_0 f_t + ff_{0t}) \right\},$$

$$J = \frac{m^{rs}(f_0 f_r + ff_{0r})(f_0 f_s + ff_{0s})}{4Lf(f_0 + Lf_{rs'}m^{rs'})},$$

$$K = \frac{f_0(f_0 f_r + ff_{0r})}{2L(f_0 + Lf_{rs'}m^{rs'})},$$

$$N = \left\{ \frac{(f_0 f_r + ff_{0r})(f_0 f_s + ff_{0s})}{2Lf(f_0 + Lf_{rs'}m^{rs'})} + \frac{m^{r't}(f_0 f_{r'} + ff_{0r'})(f_{rs}f_t + f_{st}f_r + f_{tr}f_s + ff_{rst})}{4f(f_0 + Lf_{rs'}m^{rs'})} \right\}.$$

Theorem 3.1: *The v-curvature tensor $\bar{S}_{hijk}$ of Finsler space $\bar{F}^n$ characterized by generalized $\beta-$change of Finsler metric is given by (3.16).*

Using (3.6), equation (3.16) gives us

$$\text{(3.17)} \qquad \bar{L}^2\bar{S}_{hijk} = \frac{f^3 f_0}{L}S_{hijk} + A_{ijkh} + \bar{h}_{hk}M_{ij} + \bar{h}_{ij}M_{hk} - \bar{h}_{hj}M_{ik} - \bar{h}_{ik}M_{hj},$$

where

$$A_{ijkh} = f^2 H(C^{r)}_{.ik}C^{s)}_{.hj} - C^{r)}_{.ij}C^{s)}_{.hk}) + f^2 I(C^{r)}_{.ij}m^{s)}_h m^{t)}_k + C^{r)}_{.hk}m^{s)}_i m^{t)}_j$$
$$-C^{r)}_{.ik}m^{s)}_h m^{t)}_j - C^{r)}_{.hj}m^{s)}_i m^{t)}_k) - \frac{KLf_{rs}f^2}{f_0}(C^{r)}_{.ij}m^{s)}_h m^{t)}_k + C^{r)}_{.hk}m^{s)}_i m^{t)}_j$$
$$-C^{r)}_{.ik}m^{s)}_h m^{t)}_j - C^{r)}_{.hj}m^{s)}_i m^{t)}_k)$$

and

$$M_{ij} = \frac{Lf}{e^\sigma f_0}\left(\frac{J}{2}h_{ij} + KC^{r)}_{.ij} + Nm^{r)}_i m^{s)}_j \right) - \frac{L^2 fJf_{rs}}{2f_0^2}m^{r)}_i m^{s)}_j.$$

Thus, we have

Theorem 3.2: *If the v-curvature tensor S_{hijk} and A_{ijkh} of Finsler space F^n vanishes, then the Finsler space $\bar{F}^n$ is S-4 like Finsler space.*

Further, if $F^n = (M^n, L)$ is S-4 like space, i.e. $L^2 S_{hijk} = (h_{hj}K_{ik} + h_{ik}K_{hj} - j/k)$, then equation (3.17) gives us

(3.18) $$\bar{L}^2\bar{S}_{hijk} = [\bar{h}_{hj}H_{ik} + \bar{h}_{ik}H_{jh} - j/K] + A_{ijkh} - B_{ijkh},$$

where $H_{ij} = \frac{f^2}{L^2}K_{ij} - M_{ij}$ and $B_{ijkh} = \frac{f^3}{L^2} f_{rs}[m_h^{r)}m_k^{s)}K_{ij} + m_i^{r)}m_j^{s)}K_{hk} - j/k]$.

Thus, we have

Theorem 3.3: *If F^n is S-4 like Finsler space then generalized $\beta -$ changed Finsler space with h-vector is S-4 like Finsler space provided A_{ijkh} and B_{ijkh} vanish.*

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