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Geometry of Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

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Abstract: We study the geometry of lightlike submanifolds of golden semi-Riemannian manifolds. We also study invariant lightlike submanifolds of golden semi-Riemannian manifolds and obtain some conditions for an invariant lightlike submanifold to be a locally product manifold of a golden semi-Riemannian manifold.

Keywords: Golden semi-Riemannian manifolds, lightlike submanifolds, invariant lightlike submanifolds, product manifolds.

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1. Introduction

A $(1,1)-$tensor field f of constant rank on a smooth manifold is called a polynomial structure of degree d [1], if f satisfies the equation

$$f^{d}+a_{d}f^{d-1}+......+a_{2}f+a_{i}I=0,$$

where $a_1, a_2,.....a_d$ are real numbers and I is the identity tensor of type $(1,1)$. The golden ratio, also known as divine ratio or golden proportion, is a real positive root of the equation $x^2-x+1=0$. In fact, when a line segment AB is divided by a point C (which belongs to the segment AB) in the ratio $\frac{AB}{AC}=\frac{AC}{CB}$ (the ratio of whole segment by the major subsegment and the ratio of major subsegment by the minor subsegment) then on taking

$\frac{AC}{CB} = x$, the ratio becomes a real positive root of the equation $x^2 - x - 1 = 0$. This golden ratio has many significant applications in geometry, in special theory of relativity, in atomic physics, in mathematical probability, in architecture, in art & music and many more (see[2-4] and many reference there in). Due to significant applications of golden ratio, Crasmareanu and Hretcanu in[4] defined a golden structure on M as: if the polynomial $X^2 - X - 1$ is the minimal polynomial of the structure f, satisfying $f^2 - f - 1 = 0$ then the polynomial structure f is called a golden structure on M and (M, f) is called a golden manifold. Particularly, for $f^2 + 1 = 0$ or $f^2 - 1 = 0$, we have an almost complex structure or an almost product structure, respectively.

The geometry of semi-Riemannian submanifolds have many similarities with their Riemannian case but the geometry of lightlike submanifolds is different since their normal vector bundle intersect with the tangent bundle making it more difficult and interesting to study. The theory of lightlike submanifolds has growing importance in mathematical physics and relativity, particularly have interaction with some results on Killing horizon, electromagnetic, and radition fields and asymptotically flat spacetimes (see[5,8] and many references therein). Therefore the study of lightlike submanifolds of semi-Riemannian manifolds is very active area of study. Due to significant applications of the geometry of lightlike submanifolds in mathematical physics and very limited information available motivated us to work on this subject mater.

Poyraz and Yasar[6] introduced lightlike hypersurfaces of a golden semi-Riemannian manifold and investigated several properties of lightlike hypersurfaces of a golden semi-Riemannian manifold. They obtained some results for screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold and also studied screen conformal screen semi-invariant lightlike hypersurfaces. Recently, Poyraz and Yasar[7] also studied the concept of lightlike submanifolds of golden semi-Riemannian manifolds. In present paper, we studied the geometry of lightlike submanifolds and invariant lightlike submanifolds of golden semi-Riemannian manifolds.

2. Lightlike Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\overline{M}$, g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $\mathrm{Rad}(TM): x \in M \to \mathrm{Rad}T_xM$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an $r-$lightlike submanifold and Rad(TM) is called the radical distribution on M (for detail see[8]). Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of Rad(TM) in TM, that is, $TM = \mathrm{Rad}(TM) \perp S(TM)$. Let $S(TM^{\perp})$ be a complementary vector subbundle to Rad(TM) in $TM^{\perp}$ which is also non-degenerate with respect to $\bar{g}$. Let $\mathrm{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$ then $\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp})$, where $\mathrm{ltr}(TM)$ is complementary to Rad(TM) in $S(TM^{\perp})^{\perp}$ and is an arbitrary lightlike transversal vector bundle of M. Thus we have

$$T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM)) \oplus ltr(TM) \perp S(TM) \perp S(TM^{\perp}),$$

(for detail see[5]). Let u be a local coordinate neighborhood of M then local quasi-orthonormal field of frames on $\bar{M}$ along M is $\{\xi_1, \ldots, \xi_r, X_{r+1}, \ldots X_m, N_1, \ldots N_r, W_{r+1}, \ldots W_n\}$, where $\{\xi_i\}_{i=1}^{r}$ and $\{N_i\}_{i=1}^{r}$ are lightlike basis of $\Gamma(Rad(TM)|_u)$ and $\Gamma(ltr(TM)|_u)$, respectively and $\{X_\alpha\}_{\alpha=r+1}^{m}$ and $\{W_a\}_{a=r+1}^{n}$ are orthonormal basis of $\Gamma(S(TM)|_u)$ and $\Gamma(S(TM^{\perp})|_u)$, respectively. These local quasi-orthonormal field of frames on $\bar{M}$ satisfy

$$\bar{g}(N_i, \xi_j) = \delta_j^i, \ \bar{g}(N_i, N_j) = \bar{g}(N_i, X_\alpha) = \bar{g}(N_i, W_\alpha) = 0 .$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla_X^t U, \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^t U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is a linear operator on M and known as the shape operator. Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.1) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \tag{2.2}$$

where we put

$$h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)),$$

$$D_X^l U = L(\nabla_X^{\perp} U), \ D_X^s U = S(\nabla_X^{\perp} U).$$

Since h^l and h^s are $\Gamma(ltr(TM))$ – valued and $\Gamma(S(TM^{\perp}))$ – valued therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M, respectively. For any $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W), \tag{2.3}$$

since $TM = S(TM) \perp TRad(TM)$ then we can induce some new geometric objects on the screen distribution $S(TM)$ on M. Let Q be the projection morphism of TM on $S(TM)$, then we have

$$\nabla_X QY = \nabla_X^* QY + h^*(X, QY), \quad \nabla_X \xi = -A_\xi^* X + \nabla_X^{*t} \xi \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla_X^* QY, -A_\xi^* X\}$ and $\{h^*(X, QY), \nabla_X^{*t}\xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $\Gamma(Rad(TM))-$valued and $\Gamma(S(TM))$-valued bilinear forms and they are called as the second fundamental forms of distributions $S(TM)$ and Rad(TM), respectively. Using (2.2) and (2.4), we obtain

$$(2.5)\qquad \begin{cases} \bar{g}\left(h^l(X, QY), \xi\right) = g\left(A_\xi^* X, QY\right), \\ \bar{g}\left(h^*(X, QY), N\right) = g\left(A_N X, QY\right), \end{cases},$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

From the geometry of Riemannian submanifolds and non-degenerate submanifolds, it is known that the induced connection ∇ on a non degenerate submanifold is a metric connection. Unfortunately, this is not true for a lightlike submanifold. Indeed, considering $\bar{\nabla}$ a metric connection, we have

$$(2.6)\qquad (\nabla_X g)(Y, Z) = \bar{g}\left(h^l(X, Y), Z\right) + \bar{g}\left(h^l(X, Z), Y\right),$$

for any $X, Y, Z \in \Gamma(TM)$.

3. Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Let $\bar{M}$ be a differentiable manifold and $\bar{P}$ be a tensor field of the type $(1,1)$ on $\bar{M}$. If $\bar{P}$ satisfies

$$(3.1)\qquad \bar{P}^2 = \bar{P} + 1,$$

then $\bar{P}$ is called a golden structure on $\bar{M}$. Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and $\bar{P}$ is a golden structure on $\bar{M}$. If the metric $\bar{g}$ is $\bar{P}-$compatible, i.e

(3.2) $$\bar{g}\left(\bar{P}X, Y\right) = \bar{g}\left(X, \bar{P}Y\right),$$

then $\left(\bar{M}, \bar{g}, \bar{P}\right)$ is called a golden semi-Riemannian manifold. On using (3.1) and (3.2), we have

(3.3) $$\bar{g}\left(\bar{P}X, \bar{P}Y\right) = \bar{g}\left(\bar{P}X, Y\right) + \bar{g}\left(X, Y\right),$$

for any $X, Y \in \Gamma\left(T\bar{M}\right)$.

Theorem[4] 3.1: *A Golden structure $\bar{P}$ on the manifold $\bar{M}$ has theproperty*

$$\bar{P}^n = f_n\bar{P} + f_{n-1}I,$$

for every integer $n > 0$, where $\{f_n\}_n$ is the Fibonacci sequence.

If $\tilde{P}$ is an almost product structure on $\bar{M}$ then

$$\bar{P} = \frac{1}{2}\left(I + \sqrt{5}\tilde{P}\right),$$

is a golden structure on $\bar{M}$. Conversely, if $\bar{P}$ is a golden structure on $\bar{M}$ then

$$\tilde{P} = \frac{1}{\sqrt{5}}\left(2\bar{P} - I\right),$$

is an almost product structure on $\bar{M}$.

Let (M, g) be an $m-$dimensional lightlike submanifold of $(m+n)-$dimensional golden semi-Riemannian manifold $\left(\bar{M}, \bar{g}, \bar{P}\right)$. Assume that $\{\xi_i\}_{i=1}^{r}$ and $\{N_i\}_{i=1}^{r}$ be lightlike bases of $\Gamma\left(Rad(TM)|_u\right)$ and $\Gamma\left(ltr(TM)|_u\right)$, respectively and $\{X_\alpha\}_{\alpha=r+1}^{m}$ and $\{W_\alpha\}_{\alpha=r+1}^{n}$ be orthonormal bases of $\Gamma\left(S(TM)|_u\right)$ and $\Gamma\left(S\left(TM^\perp\right)|_u\right)$, respectively. We suppose that the indices verify that $i, j, k, ... \in \{1, ..., r\}$, $\alpha, \beta, \gamma, ... \in \{r+1, ..., m\}$ and $a, b, c \in \{r+1, ., n\}$. Then for any $X \in \Gamma(TM)$, $\bar{\phi}(X)$, $\bar{\phi}\left(N_i\right)$ and $\bar{\phi}\left(W_\alpha\right)$ can be decomposed in to tangential and transversal components as below

$$\bar{P}X = PX + \sum_{\alpha} u_{\alpha}(X) N_{\alpha} + \sum_{\alpha} \omega_{\alpha}(X) W_{\alpha},$$

$$\bar{P}N_{\alpha} = U_{\alpha} + \sum_{\beta} A_{\alpha\beta} N_{\beta} + \sum_{\alpha} B_{\alpha\alpha} W_{\alpha},$$

$$\bar{P}W_{\alpha} = \phi_{\alpha} + \sum_{\beta} C_{\alpha\beta} N_{\beta} + \sum_{b} B_{\alpha b} W_{b},$$

where P is $(1,1)$ tensor field on M, U_{α} and ϕ_{α} are tangent vector fields on M, u_{α} and ω_{α} are 1-forms on M. Let N_i, W_{α} be a basis of $\Gamma\left(tr(TM)|_u\right)$ on a coordinate neighborhood u of M, where $N_i \in \Gamma\left(ltr(TM)|_u\right)$ and $W_{\alpha} \in \Gamma\left(S(TM^{\perp})|_u\right)$ then (2.2) becomes

(3.4) $$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^l(X,Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^s(X,Y) W_{\alpha},$$

for any $r-$lightlike submanifold, where $\{h_i^l\}$ and $\{h_{\alpha}^s\}$ are the local lightlike second fundamental forms and the local screen second fundamental forms of M on u, respectively. Using the basis $\{N_i, W_{\alpha}\}$ of $\Gamma\left(tr(TM)|_u\right)$, (2.3) becomes

(3.5) $$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{i=1}^{r} \rho_{ij}(X) N_i + \sum_{\alpha=r+1}^{n} \sigma_{i\alpha}(X) W_{\alpha},$$

$$\bar{\nabla}_X W_{\alpha} = -A_{W_{\alpha}} X + \sum_{i=1}^{r} \gamma_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \mu_{\alpha\beta}(X) W_{\beta},$$

for any $r-$lightlike submanifold, where

(3.6) $$\rho_{ij}(X) = \bar{g}\left(\nabla_X^l N_i, \xi_j\right),\ \varepsilon_{\alpha}\sigma_{i\alpha}(X) = \bar{g}\left(D^s(X, N_i), W_{\alpha}\right),$$
$$\gamma_{i\alpha}(X) = \bar{g}\left(D^l(X, W_{\alpha}), \xi_i\right),\ \varepsilon_{\beta}\mu_{\alpha\beta}(X) = \bar{g}\left(\nabla_X^s W_{\alpha}, W_{\beta}\right)$$

and ε_{α} is the signature of W_{α}. Similarly (2.4) becomes

(3.7) $$\nabla_X QY = \nabla_X^* QY + \sum_{i=1}^{r} h_i^*(X, QY)\xi_i,$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X + \sum_{j=1}^{r} \mu_{ij}(X)\xi_j$$

where $h^*_i(X, QY) = \bar{g}\left(h^*(X, QY), N_i\right)$ and $\mu_{ij}(X) = g\left(\nabla^{*t}_X \xi_i, N_j\right)$. Using (3.4) -(3.7), we obtain $\mu_{ij} = -\rho_{ji}(X)$. Therefore (3.7) becomes

$$\nabla_X \xi_i = -A^*_{\xi_i} X + \sum_{j=1}^{r} \rho_{ji}(X)\xi_j .$$

Now, locally define the differential $1-$forms as $\eta_i(X) = \bar{g}\left(X, N_i\right)$, for all $X \in \Gamma\left(TM|_u\right)$ and $i \in \{1,......,r\}$. Then on u, (2.6) becomes

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \left\{h^l_i(X, Y)\eta_i(Z) + h^l_i(X, Z)\eta_i(Y)\right\}.$$

Theorem 3.2: *Let M be a lightlike submanifold of agolden semi-Riemannian manifold $\left(\bar{M}, \bar{g}, \bar{P}\right)$, equipped with $(1,1)$ tensor field $\bar{P}$ such that $\bar{g}$ and $\bar{P}$ satisfies condition (3.1) and (3.2). Then (M, g, P) structure induces on the submanifold M satisfies the following equalities:*

$$P^2 X = PX + X - \sum_\alpha u_\alpha(X)U_\alpha - \sum_\alpha \omega_\alpha(X)\phi_\alpha ,$$

$$u_\beta(X) = u_\beta(PX) + \sum_\alpha u_\alpha(X)A_{\alpha\beta} + \sum_\alpha \omega_\alpha(X)C_{\alpha\beta} ,$$

$$\omega_\alpha(X) = \omega_\alpha(PX) + \sum_\alpha u_\alpha(X)B_{\alpha\alpha} + \sum_b \omega_b(X)D_{b\alpha} ,$$

$$PU_\alpha = U_\alpha - \sum_\beta A_{\alpha\beta}U_\beta - \sum_\alpha B_{\alpha\alpha}\phi_\alpha ,$$

$$u_\beta(U_\alpha) = N_\alpha + A_{\alpha\beta} - \sum_\gamma A_{\alpha\gamma}A_{\gamma\beta} - \sum_\alpha B_{\alpha\alpha}C_{\alpha\beta} ,$$

$$\omega_\alpha(U_\alpha) = B_{\alpha\alpha} - \sum_\beta A_{\alpha\beta}B_{\beta\alpha} - \sum_b B_{\alpha b}D_{b\alpha} ,$$

$$P\phi_\alpha = \phi_\alpha - \sum_\alpha C_{\alpha\alpha}U_\alpha - \sum_b D_{ab}\phi_b ,$$

$$u_\alpha(\phi_\alpha) = C_{a\alpha} - \sum_\beta C_{a\beta} A_{\beta\alpha} - \sum_b D_{ab} C_{b\alpha} \,,$$

$$\omega_b(\phi_\alpha) = W_a + D_{ab} - \sum_\alpha C_{a\alpha} B_{\alpha b} - \sum_c D_{ac} D_{cb}$$

and the induced metric g on submanifold M is as follows

$$g(PX, Y) = g(X, PY) + \sum_\alpha u_\alpha(Y)\eta_\alpha(X) - \sum_\alpha u_\alpha(X)\eta_\alpha(Y),$$

$$g(PX, N_\alpha) = g(X, U_\alpha) + \sum_\beta A_{\alpha\beta}\eta_\beta(X),$$

$$g(PX, \xi_\alpha) = g(X, P\xi_\alpha) + \sum_\alpha u_\alpha(\xi_\alpha)\eta_\alpha(X) - \sum_\alpha u_\alpha(X),$$

$$g(PX, W_\alpha) = g(X, \phi_\alpha) + \sum_\beta C_{a\beta}\eta_\beta X - \sum_a \omega_a(X),$$

$$g(PX, PY) = g(PX, Y) + g(X, Y) + \sum_\alpha u_\alpha(X)\eta_\alpha(Y)$$

$$-\sum_\alpha u_\alpha(Y)\eta_\alpha(PX) - \sum_\alpha u_\alpha(X)\eta_\alpha(PY) - \sum_\alpha \omega_\alpha(Y)\omega_\alpha(X)$$

Theorem 3.3: *Let M be m-dimensional lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{P})$, equipped with $(1,1)$ tensor field $\bar{P}$. If the structure $\bar{P}$ is parallel with respect to connection then the (M, g, P) structure induces on the submanifold M satisfies the following equalities:*

(3.8)
$$(\nabla_X P)Y = \sum_\alpha u_\alpha(Y) A_{N_\alpha} X + \sum_a \omega_a(Y) A_{W_a} X$$
$$+\sum_\alpha h^l_\alpha(X, Y) U_\alpha + \sum_a h^s_a(X, Y)\phi_a \,,$$

$$(\nabla_X u_\alpha)Y = \sum_\beta h^l_\beta(X, Y) A_{\beta\alpha} + \sum_a h^s_a(X, Y) C_{a\alpha} - h^l_\alpha(X, PY)$$
$$-\sum_\beta u_\beta(Y)\rho_{\beta\alpha}(X) - \sum_a \omega_a(Y)\gamma_{a\alpha}(X),$$

(3.9)

$$\left(\nabla_X \omega_a\right)Y = \sum_\alpha h^l_\alpha\left(X, Y\right)B_{\alpha a} + \sum_b h^s_b\left(X, Y\right)D_{ba} - h^s_a\left(X, PY\right)$$

$$-\sum_\alpha u_\alpha\left(Y\right)\sigma_{\alpha a}\left(X\right) - \sum_b \omega_b\left(Y\right)\mu_{ba}\left(X\right),$$

(3.10)
$$\nabla_X U_\alpha = \sum_\beta A_{N_\beta}A_{\alpha\beta} + \sum_a B_{\alpha a}A_{W_a}X - PA_{N_\alpha} + \sum_\beta \rho_{\alpha\beta}\left(X\right)U_\beta$$

$$+\sum_a \sigma_{\alpha a}\left(X\right)\phi_\alpha ,$$

$$h^l_\gamma\left(X, U_\gamma\right) = -X\left(A_{\alpha\gamma}\right) - \sum_\beta A_{\alpha\beta}\rho_{\beta\gamma} - \sum_a B_{\alpha a}\gamma_{a\gamma}\left(X\right)$$

$$-u_\gamma\left(A_{N_\alpha}X\right) + \sum_\beta \rho_{\alpha\beta}\left(X\right)A_{\beta\gamma} + \sum_a \sigma_{\alpha a}\left(X\right)C_{a\gamma} ,$$

$$h^s_b\left(X, U_\alpha\right) = -\sigma_{\beta b}\left(X\right) - X\left(B_{\alpha b}\right) - \sum_a B_{\alpha a}\mu_{ab}\left(X\right)$$

$$-\omega_b\left(A_{N_\alpha}X\right) + \sum_\beta \rho_{\alpha\beta}\left(X\right)B_{\beta b} + \sum_a \sigma_{\alpha a}\left(X\right)D_{ab} ,$$

$$\nabla_X \phi_a = \sum_\beta C_{a\beta}A_{N_\beta}X + \sum_b D_{ab}A_{W_b}X - PA_{W_a}X$$

$$+\sum_\beta \gamma_{a\beta}\left(X\right)U_\beta + \sum_b \mu_{ab}\left(X\right)\phi_b ,$$

$$h^l_\alpha\left(X, \phi_\alpha\right) = -X\left(C_{a\alpha}\right) - \sum_\beta C_{a\beta}\rho_{\beta\alpha}\left(X\right) - \sum_b D_{ab}\gamma_{b\alpha}\left(X\right)$$

$$-u_\alpha\left(A_{W_a}X\right) + \sum_\beta \gamma_{a\beta}\left(X\right)A_{\beta\alpha} + \sum_b \mu_{ab}\left(X\right)C_{b\alpha} ,$$

$$h^s_c\left(X, \phi_c\right) = -X\left(D_{ac}\right) - \sum_\beta C_{a\beta}\sigma_{\beta c}\left(X\right) - \sum_b D_{ab}\mu_{bc}\left(X\right)$$

$$-\omega_c\left(A_{W_c}X\right) + \sum_\beta \gamma_{a\beta}\left(X\right)B_{\beta c} + \sum_b \mu_{ab}\left(X\right)D_{bc} .$$

Theorem 3.4: *Let M be a lightlike submanifold of a semi-Riemannian golden manifold $\left(\bar{M}, \bar{g}, \bar{P}\right)$, equipped with $\left(1,1\right)$ tensor field $\bar{P}$. If normal connections vanishes identically then u_α and ω_α are exact forms.*

***Proof*:** We know that

$$2du_\alpha(X,Y)=X\left(u_\alpha(Y)\right)-Y\left(u_\alpha(X)\right)-u_\alpha\left([X,Y]\right),$$

for any $X, Y \in \Gamma(TM)$ and

$$(\nabla_X u_\alpha)(Y)=X\left(u_\alpha(Y)\right)-u_\alpha(\nabla_X Y).$$

Thus we get

$$X\left(u_\alpha(Y)\right)=\left(\nabla_X u_\alpha\right)(Y)+u_\alpha(\nabla_X Y).$$

On interchanging X by Y in above relation, we get

$$Y\left(u_\alpha(X)\right)=\left(\nabla_Y u_\alpha\right)(X)+u_\alpha(\nabla_Y X).$$

From last three equations, we obtain

$$2du_\alpha(X,Y)=\left(\nabla_X u_\alpha\right)(Y)-\left(\nabla_Y u_\alpha\right)(X)+u_\alpha\left(\nabla_X Y-\nabla_Y X-[X,Y]\right).$$

This implies

$$2du_\alpha(X,Y)=\left(\nabla_X u_\alpha\right)(Y)-\left(\nabla_Y u_\alpha\right)(X).$$

From (3.9) we have

$$\begin{aligned}2du_\alpha(X,Y)=&\sum_\beta u_\beta(X)\rho_{\beta\alpha}(Y)+\sum_\alpha \omega_\alpha(X)\gamma_{\alpha\alpha}(Y)\\&-\sum_\beta u_\beta(Y)\rho_{\beta\alpha}(X)-\sum_\alpha \omega_\alpha(Y)\gamma_{\alpha\alpha}(X).\end{aligned}$$

On using the given assertion, we get $2du_\alpha(X,Y)=0$. Similarly using (3.10) we can prove $2d\omega_\alpha(X,Y)=0$. Hence result follows.

Theorem 3.5: *Let M be a lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{P})$, equipped with $(1,1)$ tensor field $\bar{P}$. If the structure $\bar{P}$ is parallel with respect to connection such that*

$\bar{\nabla}_X V \in \Gamma(tr(TM))$ *for any* $V \in \Gamma(tr(TM))$ *then the* (M, g, P) *structure induces on the submanifold* M *is parallel if and only if* M *is totally geodesic.*

***Proof*:** From (3.8) and (3.9) our assertion follows.

4. Invariant Lightlike Submanifold

Definition 4.1: *A lightlike submanifold* M *of a semi-Riemannian manifold* $\bar{M}$ *is said to be an invariant lightlike submanifold of* $\bar{M}$ *if* $\bar{P}(TM) \subset TM$ *and* $\bar{P}(ltr(TM)) \subset ltr(TM), \bar{P}(S(TM^{\perp})) \subset S(TM^{\perp})$.

If M is an invariant lightlike submanifold of a semi Riemannian manifold $\bar{M}$ then the vector fields U_α and ϕ_α are zero vector fields and one forms u_α and ω_α also vanishes. Thus from theTheorem 3.2, the structure (P, g) induced on the invariant lightlike submanifold M from the golden structure $(\bar{P}, \bar{g})$ on a golden semi-Riemannian manifold $(\bar{M}, \bar{P}, \bar{g})$ satisfies following equalities

$$P^2 X = PX + X ,$$

$$g(PX, Y) = g(X, PY),$$

$$g(PX, N_\alpha) = \sum_\beta A_{\alpha\beta} \eta_\beta X ,$$

$$g(PX, PY) = g(PX, Y) + g(X, Y).$$

Hence from the above equalities,thefollowing theorem follows immediately.

Theorem 4.2: *Let* M *be a lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \bar{P})$. *Then the necessary and sufficient condition for lightlike submanifold* M *to be an invariant is that the induced structure* (P, g) *on the submanifold* M *is also a golden semi-Riemannian structure.*

Theorem 4.3: *Let* M *be an invariant lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \bar{P})$. *If the golden structure* $\bar{P}$ *is parallel* *with respect to the Levi-Civita connection* $\bar{\nabla}$ *of* $\bar{M}$ *then the induced structure* P *on the submanifold* M *is also parallel with respect to the induced connection* ∇ *.*

Proof: The proof follows from the expression (3.8) of the Theorem 3.3. The Nijenhuis torsion tensor field of P has the form

$$N_P(X, Y) = [PX, PY] + P^2[X, Y] - P[PX, Y] - P[X, PY].$$

Theorem 4.4: *Let* M *be an invariant lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \bar{P})$ *then the expression for Nijenhius tensor of the induced golden structure* P *on* M *is given by*

$$N_P(X, Y) = (\nabla_{PX}P)Y - (\nabla_{PY}P)X + (\nabla_X P)PY - (\nabla_Y P)PX$$
$$+ (\nabla_Y P)X - (\nabla_X P)Y,$$

for any $X, Y \in \Gamma(TM)$.

Theorem 4.5: *Let* M *be an invariant lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \bar{P})$*. If the golden structure* $\bar{P}$ *is parallel with respect to the Levi-Civita connection* $\bar{\nabla}$ *on* $\bar{M}$ *then the Nijenhius tensor of the induced golden structure* P *on* M *vanishes.*

Proof: The assertion follows directly from the Theorem 4.3 and the Theorem 4.4.

Theorem 4.6: *Let* M *be an invariant lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \bar{P})$*. Then* M *is totally geodesic lightlike submanifold of* $\bar{M}$ *.*

Proof: The proof of the theorem follows immediately from (3.9) and (3.10).

It is known that $TM = Rad(TM) \perp S(TM)$, hence if the distributions Rad(TM) and $S(TM)$ are parallel distributions on M then by the decomposition Theorem of de Rham[9], the lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is locally a product manifold of the type $L_\xi \times M^s$, where L_ξ is a null curve tangent to the radical distribution Rad(TM) and M^S is a leaf of the screen distribution $S(TM)$.

Theorem 4.7: *Let M be an invariant lightlike submanifold of a golden semi-Riemannian manifold $\left(\bar{M}, \bar{g}, \bar{P}\right)$ such that the golden structure $\bar{P}$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$. Then M is locally a product manifoldof the type $L_{\xi}\times M^{s}$, where L_{ξ} is a null curve tangent to the radical distribution Rad(TM) and M^{S} is a leaf of the screen distribution $S(TM)$.*

Proof: It is known that the screen distribution $S(TM)$ is a parallel distribution on M if and only if $\nabla_X Y \in \Gamma\left(S(TM)\right)$, that is, $g\left(\nabla_X Y, N\right)=0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma\left(S(TM)\right)$. Then using (2.2) and (3.3) we have

$$\bar{g}\left(\nabla_X Y, N\right)=\bar{g}\left(\bar{\nabla}_X Y, N\right)=\bar{g}\left(\bar{P}\bar{\nabla}_X Y, \bar{P}N\right)-\bar{g}\left(\bar{P}\bar{\nabla}_X Y, N\right). \tag{4.1}$$

Since the golden structure $\bar{P}$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, that is, $\left(\bar{\nabla}_X \bar{P}\right)Y=0$, this implies that $\bar{\nabla}_X \bar{P}Y=\bar{P}\bar{\nabla}_X Y$, therefore from (4.1), we derive

$$\begin{aligned}\bar{g}\left(\nabla_X Y, N\right)&=\bar{g}\left(\bar{\nabla}_X \bar{P}Y, \bar{P}N\right)-\bar{g}\left(\bar{\nabla}_X \bar{P}Y, N\right)\\ &=-\bar{g}\left(\bar{P}Y, \bar{\nabla}_X \bar{P}N\right)+\bar{g}\left(\bar{P}Y, \bar{\nabla}_X N\right)\\ &=-\bar{g}\left(PY, \bar{\nabla}_X N\right)+\bar{g}\left(PY, \bar{\nabla}_X N\right)=0,\end{aligned} \tag{4.2}$$

as the lightlike submanifold M is invariant therefore $\bar{P}Y=PY$ and $\bar{P}N=N$.

Next, the radical distribution Rad(TM) is a parallel distribution on M if and only if $\nabla_X \xi \in \Gamma\left(Rad(TM)\right)$ that is, $g\left(\nabla_X \xi, Y\right)=0$ for any $X \in \Gamma(TM)$, $\xi \in \Gamma\left(Rad(TM)\right)$ and $Y \in \Gamma\left(S(TM)\right)$. Analogous to above steps, we can similarly derive that

$$g\left(\nabla_X \xi, Y\right)=0. \tag{4.3}$$

Hence using the decomposition Theorem of de Rham[12], the lightlike submanifold M of a golden semi-Riemannian manifold $\bar{M}$ is locally a product manifold of the type $L_{\xi}\times M^{s}$.

A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is said to be totally geodesic[10] if and only if $h^l(X,Y)=0$ and $h^s(X,Y)=0$, for any $X, Y \in \Gamma(TM)$.

Corollary 4.8: *Let M be a totally geodesic lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{P})$ such that the screen distribution $S(TM)$ is parallel distribution on M. Then M is locally a product manifold of the type $L_\xi \times M^s$.*

Proof: Let $X \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$and $Y \in \Gamma(S(TM))$, then using (2.2), we get

$$g(\nabla_X \xi, Y) = -\bar{g}(\xi, \bar{\nabla}_X Y) = -\bar{g}(\xi, h^l(X,Y)). \tag{4.4}$$

Since M is totally geodesic therefore the radical distribution Rad(TM) becomes a parallel distribution on M. Hence M is locally a product manifold of the type $L_\xi \times M^s$. A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be screen locally conformal[11] submanifold if on any coordinate neighbourhood, the local screen second fundamental forms h_i^* of the screen distribution are conformally related to the corresponding local lightlike second fundamental forms h_i^l of M by $h_i^*(X, QY) = \psi_i h_i^l(X, QY)$, where ψ_i are non-vanishing smooth functions om a coordinate neighbourhood in M.

Corollary 4.9: *Let M be a totally geodesic screen conformal lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{P})$. Then M is locally a product manifold of the type $L_\xi \times M^s$.*

A lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is said to be irrotational[12] if and only if $\bar{\nabla}_X \xi \in \Gamma(TM)$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Hence, if M is an irrotational lightlike submanifold then using (2.2), $h^l(X,\xi) = h^s(X,\xi)$.

Theorem 4.10: *Let M be a totally geodesic or irrotational lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{P})$. Suppose* *there exists a transversal vector bundle of* M *which is parallel with respect to the Levi-Civita connection* $\bar{\nabla}$*, that is,* $\bar{\nabla}_X V \in \Gamma(tr(TM))$*, for any* $V \in \Gamma(tr(TM))$*. Then* M *is locally a product manifoldof the type* $L_\xi \times M^s$*.*

Proof: Assume that $\bar{\nabla}_X V \in \Gamma(tr(TM))$,fo r any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(TM)$ then using (2.1), we obtain

(4.5) $$A_V X = 0.$$

Let $Y \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$ then using (2.3) and (4.5), we have

(4.6) $$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\bar{g}(Y, \bar{\nabla}_X N) = g(Y, A_N X) = 0,$$

hence the screen distribution $S(TM)$ is a parallel distribution on M. Further, as M is totally geodesic or irrotational then using (4.4), the radical distribution Rad(TM) becomes a parallel distribution on M. Thus M is locally a product manifoldof the type $L_\xi \times M^s$.

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