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On L_P-Approximation for Modified Baskakov Durrmeyer Type Operators

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Abstract: In the present paper, we study a certain integral modification of the well known Baskakov operators with the weight function of Beta basis function. We establish asymptotic Voronovskaja type asymptotic formula and error estimation in Lp-approximation for these operators. The linear approximating method, namely steklov mean is used to prove the main result.

Key words and phrases: Baskakov operators, Lp-approximation, Direct results, Asymptotic formula, Error estimation.

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1. Introduction

For $x \in [0,\infty)$ and $\alpha > 0$, we consider the following type of Baskakov-Durrmeyer type operators

$$(1.1)\quad B_{n,\alpha}\left(f(t),x\right) = \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^{\infty} b_{n,k,\alpha}(t)\, f(t)\,dt$$

$$+\left(1+\alpha x\right)^{-n/\alpha}\, f(0),$$

$$= \int_0^{\infty} W_{n,\alpha}(x,t)\, f(t)\,dt,$$

where $$p_{n,k,\alpha}(x)=\frac{\Gamma\left(\frac{n}{\alpha}+k\right)}{\Gamma(k+1)\Gamma\left(\frac{n}{\alpha}\right)}\cdot\frac{(\alpha x)^k}{(1+\alpha x)^{\frac{n}{\alpha}+k}},$$

$$b_{n,k,\alpha}(t)=\frac{\alpha\Gamma\left(\frac{n}{\alpha}+k+1\right)}{\Gamma(k)\Gamma\left(\frac{n}{\alpha}+1\right)}\cdot\frac{(\alpha t)^{k-1}}{(1+\alpha t)^{\frac{n}{\alpha}+k+1}}$$

and $$W_{n,\alpha}(x,t)=\sum_{k=1}^{\infty}p_{n,k,\alpha}(x)\,b_{n,k,\alpha}(t)+(1+\alpha x)^{-n/\alpha}\,\delta t,$$

δt being the Dirac's Delta function.

The operators (1.1) were first introduced by Gupta[1]. As a special case, for $\alpha=1$, these operators reduce to the operators studied by Z. Finta[2]. The order of approximation of the operators (1.1) is at best $o(n^{-1})$ howsoever smooth the function may be. We will solicit help from the technique of linear combination of linear positive operators to improve the order of approximation of the operators (1.1).

The approximation process is defined as follows:

The linear combination $B_{n,\alpha}(f,k,x)$ of $B_{d_j n,\alpha}(f(t):x),\ j=0,1,2.......,k$

is defined as:

(1.2) $$B_{n,\alpha}(f,k,x)=\frac{1}{\Delta}\begin{vmatrix} B_{d_0,n}(f,x) & d_0^{-1} & d_0^{-2} & .... & d_0^{-k} \\ B_{d_1,n}(f,x) & d_1^{-1} & d_1^{-2} & .... & d_1^{-k} \\ B_{d_2,n}(f,x) & d_2^{-1} & d_2^{-2} & .... & d_2^{-k} \\ .... & .... & .... & .... & .... \\ B_{d_k,n}(f,x) & d_k^{-1} & d_k^{-2} & .... & d_k^{-k} \end{vmatrix},$$

where $d_0, d_1, d_2,........, d_k$ are $k+1$ arbitrary but fixed distinct positive integers and $\mathbf{\Delta}$ is the Vandermonde determinant obtained by replacing the operator's column of the above determinant with the enteries 1. On simplification (1.2) reduces to

(1.3) $$B_{n,\alpha}\left(f,k,x\right) = \sum_{j=o}^{k} C(j,k)\, B_{d_j n,\alpha}(f,x),$$

where $$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i},\ k\neq 0, \qquad C(0,0)=1.$$

Let $m\in\mathbb{N}$ (the set of positive integers) and $0<a<b<\infty$. For $f\in L_p[a,b]$, $1\le p<\infty$, the m^th^ order integral modulus of smoothness of f is defined as $\omega_m(f,\gamma,p,[a,b])=Sup\left\|\Delta_\delta^m f(t)\right\|_{L_p[a,b-m\delta]}$, where $\Delta_\delta^m f(t)$ is the m^th^ order forward difference of the function f with step length δ and $0<\gamma\le\dfrac{b-a}{m}$.

The spaces AC[a,b] and BV[a,b] are defined as the classes of absolutely continuous functions and functions of bounded variation over [a,b] respectively. The seminorm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on [a,b].

Throughout the paper, we assume that $0<a_1<a_2<a_3<b_3<b_2<b_1<\infty$, $I_i=[a_i,b_i]$, $i=1,2,3$ and C denotes a positive constant, not necessarily the same at all occurrences.

For $1\le p<\infty$, let

$$L_p^{2k+2}(I_i)=\left\{f\in L_p[0,\infty): f^{2k+1}\in AC(I_i)\, and\ f^{2k+2}\in L_p(I_i)\right\},$$

for $f\in L_p[a,b]$, $1\le p<\infty$, the Hardy Littlewood Majorant of f is defined by

$$h_f(x)=\sup_{\xi\neq x}\frac{1}{\xi-x}\int_x^{\xi} f(t)dt, \quad (a\le\xi\le b).$$

2. Preliminary Results

In order to prove the main results, we shall require the following Lemmas.

The following lemma gives L_p bound for Hardy Littlewood Majorant h_f in terms of f.

Lemma 2.1: *For* $1 < p < \infty$ *and* h_f , $f \in L_p[a,b]$, *we have*

$$\|h_f\|_{L_p[a,b]} \le 2^{1/p} \frac{p}{p-1} \|f\|_{L_p[a,b]}.$$

Steklov mean: Let $f \in L_p[a,b]$, $1 \le p < \infty$. Then for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m[th] order corresponding to f is defined as follows:

$$f_{n,m}(t) = \eta^{-m} \left(\int_{-\eta/2}^{\eta/2} \right)^m \left\{ f(t) + (-1)^{m-1} \Delta_u^m f(t) \right\} \prod_{i=1}^{m} dt_i ,$$

where $t \in I_1$ and $u = \sum_{i=1}^{m} t_i$.

Lemma 2.2: *For the function* $f_{n,m}(t)$ *defined above, we have*

(a) $f_{\eta,m}(t)$ *has derivatives up to the order m over* I_1 , $f_{\eta,m}^{(m-1)}$ *is absolutely continuous on* I_1 , $f_{\eta,m}^{(m)}$ *exists almost everywhere and belongs to* $L_P(I_1)$,

(b) $\| f_{\eta,m}^{(r)} \|_{LP(I_2)} \le C_1 \eta^{-r} \omega_r (f, \eta, I_1), r = 1, 2, 3, ...m,$

(c) $\| f - f_{\eta,m} \|_{LP(I_2)} \le C_2 \omega_r (f, \eta, I_1),$

(d) $\| f_{\eta,m} \|_{LP(I_2)} \le C_3 \| f \|_{LP(I_1)} \le C_4 \| f \|_{\gamma},$

(e) $\| f_{\eta,m}^{(m)} \|_{LP(I_2)} \le C_5 \| f \|_{\gamma} ,$

where $C_i's$ *are certain constants that depend on* i *but are independent of* f *and* η.

The next lemma gives a bound for the intermediate derivatives in terms of highest derivative and the function in L_p *norm,* $(1 \le p < \infty)$.

Lemma2.3[3]: *Let* $1 \le p < \infty$, $f \in L_p[a,b]$, $f^k \in AC[a,b]$ *and* $f^{k+1} \in L_p[a,b]$. *Then*

$$\left\| f^{j} \right\|_{L_p[a,b]} \le C_j \left\{ \left\| f^{k+1} \right\|_{L_p[a,b]} + \| f \|_{L_p[a,b]} \right\} \qquad j = 1, 2, 3....k$$

where $C_i's$ *are certain constants depending only on j, k, p, a and b.*

Lemma 2.4[4]: *For m*ϵ $\mathbb{N}$ *(the set of positive integers) and n sufficiently large, there holds*

$$B_{n,\alpha}[(t-x)^m, k, x] = \begin{cases} 0, & m = 1, 2, 3, ..., k, k+1 \\ n^{-(k+1)}\{Q(m,k,x) + o(1)\}, & for\, m = k+2, k+3, ..., 2k+2 \\ o(n^{-(k+1)}), & m = 2k+3, 2k+4, ... \end{cases}$$

where $Q(m,k,x)$ *are certain polynomials in* x *of degree* m *and* $x \in [0,\infty)$ *is arbitrary but fixed.*

Lemma 2.5 : *For* $f \in BV(I_1)$ *, the following inequality holds*

$$\left\| B_{n,\alpha}\left(\phi(t) \int_x^t (t-w)^{2k+1} \, df(w); x \right) \right\|_{L_1(I_2)} \le C n^{-(k+1)} \| f \|_{BV(I_1)},$$

where $\phi(t)$ *is the characteristic function of* I_1*.*

Proof : For each n there exists a non-negative integer $r = r(n)$ such that

$$r\; n^{-1/2} \le \max\{b_1 - a_2, b_2 - a_1\} \le (r+1)\, n^{-1/2}.$$ Then

$$K := \left\| B_{n,\alpha}\left(\int_x^t (t-w)^{2k+1} \, df(w) \phi(t); x \right) \right\|_{L_1(I_2)}$$

$$\le \sum_{i=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+l\,n^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(t) W_{n,\alpha}(t,x) |t-x|^{2k+1} \left[\int_x^{x+(l+1)n^{-1/2}} \phi(W) |df(W)| \right] dt \right.$$

$$\left. + \int_{x-(l+1)n^{-1/2}}^{x-l\,n^{-1/2}} \phi(t) W_{n,\alpha}(t,x) |t-x|^{2k+1} \left[\int_{x-(l+1)n^{-1/2}}^{x} \phi(W) |df(W)| \right] dt \right\} dx.$$

Let $\phi_{x,c,d}(w)$ denote the characteristic function of w in the interval $[x-cn^{-1/2}, x+d\,n^{-1/2}]$, where c and d are non-negative integers. Then we have

$$K \le \sum_{l=1}^{r}\left(n^2 I^{-4}\int_{a_2}^{b_2}\left\{ \int_{x+l n^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(t)W_{n,\alpha}(t,x)|t-x|^{2k+5}\left[\int_{a_1}^{b_1}\phi_{x,l+1,0}(w)|\,df(w)|\right]dt\right.\right.$$

$$\left.\left.+\int_{x-(l+1)n^{-1/2}}^{x-l n^{-1/2}} \phi(t)W_{n,\alpha}(t,x)|t-x|^{2k+5}\left[\int_{a_1}^{b_1}\phi_{x,l+1,0}(w)|\,df(w)|\right]dt\right\}dx\right)$$

$$+\int_{a_2}^{b_2}\int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} \phi(t)W_{n,\alpha}(t,x)|t-x|^{2k+1}\left[\int_{a_1}^{b_1}\phi_{x,1,1}(w)|\,df(w)|\right]dt\,dx.$$

Using Fubini's theorem, we get

$$K \le Cn^{-(2k+1)/2}\left\{\sum_{l=1}^{r} l^{-4}\left[\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,0,l+1}(w)\,dx\right)|\,df(w)|\right.\right.$$

$$\left.\left.+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,l+1,0}(w)\,dx\right)|\,df(w)|\right] + \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,1,1}(w)\,dx\right)|\,df(w)|\right\}$$

$$\le\ Cn^{-(k+1)}\,\|f\|_{BV(I_1)}.$$

In order to prove our main result, we first discuss the approximation in the smooth subspace $L_p^{(2k+2)}(I_1)$ *or* $L_p[0,\infty)$.

Lemma 2.6: *For* $m\in\mathbb{N}\cup\{0\}$, *if the mth order moment be defined as*

$$U_{n,m,\alpha}(x)=\sum_{k=0}^{\infty} p_{n,k,\alpha}(x)\left(\frac{k}{n}-x\right)^m.$$

Then $U_{n,0,\alpha}(x)=1,\quad U_{n,1,\alpha}(x)=0$ *and*

$$nU_{n,m+1,\alpha}(x)=x(1+\alpha x)\left[U'_{n,m,\alpha}(x)+mU_{n,m-1,\alpha}(x)\right],\quad m\ge 1.$$

Further, we have the following consequences of $U_{n,m,\alpha}(x)$*:*

(i) $U_{n,m,\alpha}(x)$ *is a polynomial in x of degree m, m ≠ 1;*

(ii) for every $x \in [0,\infty)$, $U_{n,m,\alpha}(x) = o\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, *where [β] denotes the integral part of β.*

Lemma 2.7 : *Let the function* $T_{n,m,\alpha}(x)$, $m \in \mathbb{N} \bigcup \{0\}$, *be defined as*

$$T_{n,m,\alpha}(x) = B_{n,\alpha}((t-x)^m, x)$$

$$= \sum_{k=0}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^m \, dt + (1+\alpha x)^{-n/\alpha}(-x)^m.$$

Then $T_{n,0,\alpha}(x) = 1$, $T_{n,1,\alpha}(x) = 0$, $T_{n,2,\alpha}(x) = \dfrac{2x(1+\alpha x)}{n-\alpha}$ *and*

$$(n-\alpha m)T_{n,m+1,\alpha}(x) = x(1+\alpha x)[T'_{n,m,\alpha}(x) + 2mT_{n,m-1,\alpha}(x)] + m(1+2\alpha x)T_{n,m,\alpha}(x)$$

$$n > \alpha\, m.$$

Further, we have the following consequences of $T_{n,m,\alpha}(x)$*:*

(i) $T_{n,m,\alpha}(x)$ *is a polynomial in x of degree m, m ≠ 1;*

(ii) for every $x \in [0,\infty)$, $T_{n,m,\alpha}(x) = o\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, *where [β] denotes the integral part of β.*

Lemma 2.8: *The function* $\mu_{n,m,\alpha}(x), m \in N^0$ *, can be defined as*

$$\mu_{n,m,\alpha}(x) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^m \, dt.$$

Then $\mu_{n,0,\alpha}(x) = 1$, $\mu_{n,1,\alpha}(x) = \dfrac{\alpha(1+\alpha x)}{n-\alpha}$, $n > \alpha$ *and*

$$\mu_{n,2,\alpha}(x) = \alpha^2 \left[\frac{2\alpha(n+\alpha)x^2 + (n+2\alpha)(x+2)}{(n-\alpha)(n-2\alpha)}\right], \quad n > 2\alpha$$

Consequently for each $x \in [0,\infty)$, $\mu_{n,m,\alpha}(x) = o\left(n^{-\left[\frac{m+1}{2}\right]}\right)$.

3. Direct Results

In this section, we establish some direct results viz-Voronovskaja type asymptotic formula (Lemma 3.1) and error estimation formula (Theorem 3.2) in terms of higher order integral modulus of smoothness.

Lemma 3.1 : (i) *Let* $1<p<\infty$ *and* $f\in L_p[0,\infty)$, $f^{(2k+2)}\in L_p(I_1)$, *then for sufficiently large n, the following inequality holds*

$$(3.1)\quad \left\|B_{n,\alpha}(f,k,x)-f\right\|_{L_p(I_2)}\le C_1 n^{-(k+1)}\left\{\left\|f^{(2k+2)}\right\|_{L_p(I_1)}+\left\|f\right\|_{L_p[0,\infty)}\right\}.$$

(ii)For $p=1$

Let $f\in L_1[0,\infty)$. *If* $f^{(2k+1)}\in BV(I_1)$ *with* $f^{(2k)}\in AC(I_1)$, *then for all n sufficiently large, the following inequality holds*

$$(3.2)\quad \left\|B_{n,\alpha}(f,k,x)-f\right\|_{L_1(I_2)}\le C_2 n^{-(k+1)}\left\{\left\|f^{(2k+1)}\right\|_{BV(I_1)}+\left\|f^{(2k+1)}\right\|_{L_1(I_1)}+\left\|f\right\|_{L_1[0,\infty)}\right\},$$

where $C_1=C_1(k,p)$ *and* $C_2=C_2(k)$.

Proof: Let $p>1$. With the given assumptions on f, for $x\in I_2$ and $t\in I_1$, we can write

$$f(t)=\sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x)+\frac{1}{(2k+1)!}\int_x^t (t-w)^{(2k+1)}f^{(2k+2)}(w)\,dw.$$

If $\phi(t)$ is the characteristic function of I_1, then

$$f(t)=\sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x)+\frac{1}{(2k+1)!}\int_x^t (t-w)^{(2k+1)}\phi(t)f^{(2k+2)}(w)\,dw$$
$$+F(t,x)(1-\phi(t)),$$

where $F(t,x)=f(t)-\sum_{j=0}^{2k+1}\frac{(t-x)^j}{j!}f^{(j)}(x)$, $\forall t\in I_1$ *and* $x\in I_2$.

In view of $B_{n,\alpha}(1,k,x)=1$, we obtain

$$
\begin{aligned}
B_{n,\alpha}(f,k,x)-f(x) &= \sum_{j=1}^{2k+1}\frac{f^{(j)}(x)}{j!}B_{n,\alpha}((t-x)^j,k,x) \\
&\quad +\frac{1}{(2k+1)!}B_{n,\alpha}\left(\phi(t)\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\,dw,k,x\right) \\
&\quad + B_{n,\alpha}(F(t,x)(1-\phi(t)),k,x) \\
&= \sum{}_1+\sum{}_2+\sum{}_3.
\end{aligned}
$$

It follows from Lemmas 2.3 and 2.4 that

$$
\|\Sigma_1\|_{L_p(I_2)} \le Cn^{-(k+1)}\left(\|f\|_{L_p(I_2)}+\left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right).
$$

To estimate Σ_2, let h_f be the Hardy Littlewood majorant of $f^{(2k+2)}$ on I_1. Then using Holder's inequality and Lemma 2.7, we get

$$
\begin{aligned}
J_1 &:= \left|B_{n,\alpha}\left(\phi(t)\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\,dw;x\right)\right| \\
&\le B_{n,\alpha}\left(\phi(t)|t-x|^{2k+1}\int_x^t \left|f^{(2k+2)}(w)\right|.|\,dw\,|;x\right) \\
&\le B_{n,\alpha}\left(\phi(t)(t-x)^{2k+2}\left|h_f(t)\right|;x\right), \\
&\le \left\{B_{n,\alpha}\left(\phi(t)|t-x|^{(2k+2)q};x\right)^{1/q}\right\}\left\{B_{n,\alpha}\left(\phi(t)\left|h_f(t)\right|^p;x\right)^{1/p}\right\} \\
&\le Cn^{-(k+1)}\left\{\int_{a_1}^{b_1} W_{n,\alpha}(t,x)\left|h_f(t)\right|^p dt\right\}^{1/p}.
\end{aligned}
$$

Hence by Fubini's theorem, Lemmas 2.1 and 2.8, we have

$$
\|J_1\|_{L_p(I_2)}^p \le Cn^{-(k+1)p}\int_{a_2}^{b_2}\int_{a_1}^{b_1} W_{n,\alpha}(t,x)\left|h_f(t)\right|^p dt\,dx,
$$

$$\leq Cn^{-(k+1)p} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[W_{n,\alpha}(t,x)dx \right] \left| h_f\ (t) \right|^p dt$$

$$\leq Cn^{-(k+1)p} \frac{n}{n-1} \int_{a_1}^{b_1} \left| h_f\ (t) \right|^p dt$$

$$\leq Cn^{-(k+1)p} \left\| h_f \right\|_{L_p(I_1)}^p ,$$

since n is sufficiently large,

$$\leq Cn^{-(k+1)p} \left\| f^{(2k+2)} \right\|_{L_p(I_2)}^p ,$$

consequently $\left\| J_1 \right\|_{L_p(I_2)} \leq Cn^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_{p(I_1)}}$.

Thus, we have $\left\| \Sigma_2 \right\| \leq Cn^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}$.

For $x \in I_2$ and $t \in [0,\infty)/I_1$ there exist a $\delta > 0$ such that $|t-x| \geq \delta$ which leads us to what follows

$$\left| B_{n,\alpha}\left(F\left(t,x\right)\left(1-\phi(t)\right);x\right) \right| \leq \delta^{-(2k+2)} B_{n,\alpha}\left(\left| f(t) \right| (t-x)^{2k+2}, x \right)$$
$$+ \sum_{j=0}^{2k+1} \frac{f^{(j)}(x)}{j!} B_{n,\alpha}((t-x)^{2k+j+2}, x)$$
$$:= J_2 + J_3 .$$

It follows from Holder’s inequality and Lemma 2.7 that

$$\left| J_2 \right| \leq Cn^{(-k+1)} \left\{ B_{n,\alpha}\left(\left| f\left(t\right) \right|^p ; x \right) \right\}^{\frac{1}{p}} .$$

Again, applying Fubini's theorem, we get

$$\left\| J_2 \right\|_{L_p(I_2)} \leq Cn^{-(k+1)} \left\| f \right\|_{L_p[0,\infty)} .$$

Moreover, using Lemmas 2.3 and 2.7, we have

$$\left\| J_3 \right\|_{L_p(I_2)} \leq Cn^{-(k+1)} \left(\left\| f \right\|_{L_p(I_2)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right),$$

Hence $$\|\Sigma_3\|_{L_p(I_2)} \le Cn^{-(k+1)}\left(\|f\|_{L_p[0,\infty)} + \|f^{(2k+2)}\|_{L_p(I_2)}\right).$$

On combining the estimates of Σ_1, Σ_2 and Σ_3, (3.1) follows

(ii) Now let $p=1$. With the given assumptions on f for almost all $x \in I_2$ and for $t \in I_1$, we can write by Taylor's Theorem,

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\,dw.$$

Hence, if $\phi(t)$ is the characteristic function of t in I_1. Then

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!}\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\,dw$$
$$+F(t,x)(1-\phi(t)).$$

Where $F(t,x) = f(t) - \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j!} f^{(j)}(x)$, for almost all $x \in I_2$ and $t \in [0,\infty)$.

Operating by $B_{n,\alpha}$ on both sides of the above equation we get

$$B_{n,\alpha}(f(t),x) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} B_{n,\alpha}((t-x)^j, x)$$
$$+\frac{1}{(2k+1)!} B_{n,\alpha}\left(\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\phi(t)dw, x\right)$$
$$+B_{n,\alpha}\left(F(t,x)(1-\phi(t),k,x)\right)$$

$$:= \Sigma_1 + \Sigma_1 + \Sigma_3.$$

Applying Lemmas 2.3 and 2.4, we have

$$\|\Sigma_1\|_{L_1(I_2)} \le Cn^{-(k+1)}\left(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)}\right).$$

Next, using Lemma 2.5, we obtain

$$\|\Sigma_2\|_{L_1(I_2)} \le Cn^{-(k+1)}\left(\|f^{(2k+1)}\|_{BV(I_1)}\right).$$

For all $x \in I_2$ and $t \in [0,\infty)/I_1$ we can choose a $\delta > 0$ such that $|t-x| \geq \delta$. Then

$$\left\| B_{n,\alpha}\left(F(t,x)(1-\phi(t));x\right)\right\|_{L_1(I_2)} \leq \int_{a_2}^{b_2}\int_0^\infty W_{n,\alpha}(x,t)\left| f(t)\right|.(1-\phi(t))\,dt\,dx$$

$$+\sum_{j=0}^{2k+1}\frac{1}{j!}\int_{a_2}^{b_2}\int_0^\infty W_{n,\alpha}(x,t)\left| f^{j}(x)\right|.|t-x|^{j}(1-\phi(t))\,dt\,dx$$

$$:= J_1 + J_2.$$

For sufficiently large t, we can find positive constants M and C' such that

$$\frac{(t-x)^{2k+2}}{t^{2k+2}+1} > C' \quad \forall x \in I_2 \text{ and } t \geq M.$$

Applying Fubini's theorem, we get

$$J_1 = \left(\int_0^M\int_{a_2}^{b_2} + \int_M^\infty\int_{a_2}^{b_2}\right) W_{n,\alpha}(x,t)\left| f(t)\right| dx\,dt := J_3 + J_4.$$

Now using Lemma 2.8, we have

$$J_3 \leq \delta^{-(2k+2)}\int_0^M\int_{a_2}^{b_2} W_{n,\alpha}(x,t)\left| f(t)\right||t-x|^{2k+2}\,dt\,dx$$

$$\leq Cn^{-(k+1)}\int_0^M\left| f(t)\right| dt.$$

$$J_4 = \frac{1}{C'}\int_M^\infty\int_{a_2}^{b_2} W_{n,\alpha}(x,t)\frac{(t-x)^{2k+2}}{t^{2k+1}+1}\left| f(t)\right| dx\,dt$$

$$\leq Cn^{-(k+1)}\int_M^\infty\left| f(t)\right| dt, \quad \text{since t is sufficiently large.}$$

On combining the estimates of J_3 and J_4 we get,

$$J_1 \leq Cn^{-(k+1)}\| f\|_{L_1[0,\infty)}.$$

Further using Lemmas 2.3 and 2.7 we obtain,

$$J_2 \le Cn^{-(k+1)}\left(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)}\right).$$

Hence

$$\|B_{n,\alpha}(F(t,x)(1-\phi(t);x)\|_{L_1(I_2)} \le Cn^{-(k+1)}\left(\|f\|_{L_1[0,\infty)} + \|f^{2k+1}\|_{L_1(I_2)}\right).$$

Consequently, $\|\Sigma_3\|_{L_1(I_2)} \le Cn^{-(k+1)}\left(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)}\right).$

Finally combining the estimates of Σ_1 , Σ_2 and Σ_3, we get (3.2).

Theorem 3.1: *Let* $f \in L_p[0,\infty), 1 \le p < \infty.$ *For all* n *sufficiently large, we have*

$$\|B_{n,\alpha}(f,k,x) - f\|_{L_p(I_2)} \le M\left(\omega_{2k+2}\left(f, n^{-1/2}, p, I_1\right) + n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right),$$

where M is a constant depending on k *and* p *but independent of* f *and* n.

Proof : Let $f_{\eta,2k+2}(t)$ be the Steklov mean of $(2k+2)^{th}$ order corresponding to $f(t)$ over I_1, where $\eta > 0$ is sufficiently small and $f_{\eta,2k+2}(t)$ is defined as zero outside I_1. Then we have

$$\begin{aligned}\left\|B_{n,\alpha}(f,k,x) - f(x)_{L_p(I_2)}\right\| &= \|B_{n,\alpha}(f - f_{\eta,2k+2},k,x)\|_{L_p(I_2)} \\ &\quad + \|B_{n,\alpha}(f_{\eta,2k+2},k,x) - f_{\eta,2k+2}\|_{L_p(I_2)} + \|f_{\eta,2k+2} - f\|_{L_p(I_2)} \\ &= \Sigma_1 + \Sigma_1 + \Sigma_3, \quad \text{say.}\end{aligned}$$

Let $\phi(t)$ be the characteristic function of I_1, we get

$$\begin{aligned}B_{n,\alpha}(f - f_{\eta,2k+2}(t):x) &= B_{n,\alpha}(\phi(t)(f - f_{\eta,2k+2})(t):x) \\ &\quad + B_{n,\alpha}(1-\phi(t)(f - f_{\eta,2k+2})(t):x) \\ &= J_1 + J_2, \quad \text{say}.\end{aligned}$$

Clearly, the following inequality holds for $p=1$. For $p > 1$, it follows from Holders inequality

$$\int_{a_2}^{b_2} |J_1|^p \, dx = \int_{a_2}^{b_2}\int_{a_3}^{b_3} W_{n,\alpha}(x,t)\left|\left(f - f_{\eta,2k+2}\right)(t)\right|^p dt\, dx.$$

Using Fubini's theorem and Lemma 2.8, we get

$$\|J_1\|_{L_{p(I_2)}} \leq \left\| f - f_{\eta,2k+2} \right\|_{L_{p(I_3)}}.$$

Proceeding in a similar manner, for all $p \geq 1$

$$\|J_2\|_{L_{p(I_2)}} \leq C n^{-(k+1)} \left\| f - f_{\eta,2k+2} \right\|_{L_{p[0,\infty)}}.$$

Consequently by the property (C) of Steklov mean, we get

$$\Sigma_1 \leq C\left(\omega_{2k+2}(f,\eta,p,I_1) + \eta^{-(k+1)} \|f\|_{L_p[0,\infty)}\right).$$

Since $\left\| f_{\eta,2k+2}^{(2k+1)} \right\|_{BV(I_1)} = \left\| f_{\eta,2k+2}^{(2k+1)} \right\|_{L_1(I_1)}$, by Lemma 2.4 for all $p \geq 1$ it follows that

$$\Sigma_2 \leq C n^{-(k+1)}\left(\left\| f_{\eta,2k+2}^{(2k+2)} \right\|_{L_{p}(I_1)} + \left\| f_{\eta,2k+2} \right\|_{L_p[0,\infty)} \right)$$

$$\leq C n^{-(k+1)}\left(\eta^{-(k+2)} \omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)} \right),$$

in view of properties (b) and (d) of Steklov means.

Finally by property (c) of Steklov mean

$$\Sigma_3 \leq C\omega_{2k+2}(f,\eta,p,I_1).$$

Choosing $\eta = n^{-1/2}$ and combining the estimates of Σ_1, Σ_2 and Σ_3 the desired result follows.

References

1. V. Gupta, Approximation for modified Baskakov Durrmeyer type operators, *Rocky Mountain J. Math.*, **39 (3)** (2009) 825-841.

2. Z. Finta, On converse approximation theorems, *J. Math. Anal. Appl*., **312(1)**(2005) 159-180.
3. S. Goldberg and A. Meir, Minimum moduli of ordinary differential operators, *Proc. London Math. Soc.*, **23** (1971)1-15.
4. P. N. Agrawal and A. J. Mohammad, Linear combination of a new sequence of linear positive operators, *Revista de la U.M.A.*, **42(2)** (2001) 57-65.
5. P. N. Agrawal and A. J. Mohammad, On Lp-approximation by a linear combination of a new sequence of linear positive operators, *Turk. J. Math*., **27** (2003) 389-405.
6. P. N. Agrawal and K. H. Thamar, Approximation of unbounded function by a new sequence of linear positive operators, *J. Math. Anal. Appl*., **225** (1998) 660-672.
7. V. Gupta, A note on modified Baskakov type operators, *Approx Theory and its Appl*., **10 (3)** (1994) 74-78.
8. V. Gupta and M. K Gupta, Rate of convergence for certain families of summation integral type operators, *J. Math. Anal. Appl*., **296** (2004) 608-618.