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Three Dimensional Conformally Flat Landsberg and Berwald Spaces

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Abstract: The purpose of the present paper is to find the condition under which a three dimensional conformally flat Landsberg space to be a Berwald space.

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1. Introduction

Let $(x, y) = (x^i, y^i)$ be a local coordinate system of the total space of the tangent bundle TM of a three dimensional differentiable manifold M.

Let us consider a Finsler space (M, L) which is equipped with the fundamental function L(x, y). Let g_{ij} be the fundamental tensor and C_{ijk} be the Cartan's C-tensor of the Finsler space (M, L) and the matrix (g^{ij}) be the inverse of the matrix (g_{ij}). Then

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k} \quad \text{and} \quad g^{ij} g_{jk} = \delta^i_k .$$

If in a Finsler space there exists a local coordinate system (x^i, y^i) in which the fundamental tensor g_{ij} can be written as a function of the variable y^i alone, we call the space, a locally Minkowski space and such a coordinate system (x^i, y^i) a rectilinear coordinate system. If a Finsler space (M, L) is conformal to a locally Minkowski space $(M, \bar{L})$, then (M, L) is called a conformally flat Finsler space.

2. Scalar Components and Conformal Changes in Moor's Frame

A. Moor[1] introduced a special orthonormal frame field (l^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element l^i, the second is the normalized torsion vector $m^i = C^i/C$, and third n^i is the unit vector orthogonal to them. Here

$C^i = g^{ij}C_{jkh}g^{kh}$ and $C^2 = g_{ij}C^iC^j$.

In a Moor's frame an arbitrary tensor field can be represented by scalar components along the unit vector l^i, m^i and n^i. For instance, let T^i_{jk} be a tensor of type (1, 2), then the scalar components $T_{\alpha\beta\gamma}$ are defined by

$$T_{\alpha\beta\gamma} = T^i_{jk} e_{\alpha)i} e^j_{\beta)} e^k_{\gamma)}, \qquad \alpha, \beta, \gamma = 1, 2, 3.$$

and the tensor T^i_{jk} may be expressed as

$$T^i_{jk} = T_{\alpha\beta\gamma}\, e^i_{\alpha)} e_{\beta)j} e_{\gamma)k} \qquad \alpha, \beta, \gamma = 1, 2, 3.$$

where $e^i_{1)} = l^i$, $e^i_{2)} = m^i$, $e^i_{3)} = n^i$, $e_{\alpha)i} = g_{ij}e^j_{\alpha)}$, $\alpha = 1, 2, 3$, and $g_{ij}e^i_{\alpha)}e^j_{\beta)} = \delta_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$. Therefore $g_{ij} = l_il_j + m_im_j + n_in_j$.

The C-tensor C_{ijk} satisfies

$$C_{ijk}l^i = C_{ijk}l^j = C_{ijk}l^k = 0,$$

So the expression of C_{ijk} in three dimensional Finsler space is written as[2]

$$\begin{aligned} \text{(2.1)} \qquad LC_{ijk} = {} & Hm_im_jm_k - J\left(m_im_jn_k + m_in_jm_k + n_im_jm_k\right) \\ & + I\left(m_in_jn_k + n_im_jn_k + n_in_jm_k\right) + J\,n_in_jn_k, \end{aligned}$$

where H, I, J are called main scalars[2], such that H + I =LC.

Now the h-covariant and v-covariant differentiations of the former fields with respect to Cartan's connection CΓ are given by[2].

$$\text{(2.2)} \qquad l_{i|j} = 0, \qquad m_{i|j} = n_ih_j, \qquad n_{i|j} = -m_ih_j,$$

(2.3) $\quad Ll_i\big|_j = m_i m_j + n_i n_j\,, \quad Lm_i\big|_j = -l_i m_j + n_i v_j\,,$

$Ln_i\big|_j = -l_i n_j - m_i v_j$, respectively, where h_i and v_i are components of vectors called the h-connection vector and v-connection vector respectively.

Let us consider a Finsler space (M, $\bar{L}$) which is conformal to a Minkowski space (M, L), i.e. $\bar{L}(x, y) = e^{\sigma(x)} L(y)$.

In this paper we shall use the symbol '–' on the top of the quantities to denote the quantities of the conformally flat Finsler space (M, $\bar{L}$).
We use the following notations

$$F = L^2/2, \quad \bar{F} = \bar{L}^2/2, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}, \quad \partial_i = \frac{\partial}{\partial x^i} \ .$$

So, we get[3]

(2.4) $$\bar{F} = e^{\sigma}F, \quad \bar{g}_{ij} = e^{2\sigma} g_{ij}\,, \quad \bar{g} = e^{4\sigma} g, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}\,,$$
$$\bar{l}_i = e^{\sigma} l_i, \quad \bar{m}_i = e^{\sigma} m_i, \quad \bar{n}_i = e^{\sigma} n_i, \ \bar{l}_i = e^{-\sigma} l_i, \quad \bar{m}_i = e^{-\sigma} m_i,$$
$$\bar{n}_i = e^{-\sigma} n_i, \ \bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \bar{C}^i_{jk} = C^i_{jk}, \ \bar{H} = H, \ \bar{I} = I, \ \bar{J} = J.$$

Now, we are concerned with the conformal change of Christoffel's symbols

$$\gamma_{ijk} = g_{jr}\gamma^r_{ik} = \frac{1}{2}(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik})\,.$$

From (2.4) we can easily obtain the following

(2.5) $$\bar{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - g_{jk}\sigma^i\,, \quad \left(\sigma_i = \frac{\partial \sigma}{\partial x^i}\,, \quad \sigma^i = g^{ij}\sigma_j\right).$$

Therefore, the conformal change of $2G^i = \ \gamma^i_{jk} y^j y^k = \gamma^i_{00}$ is given by

(2.6) $$2\bar{G}^i = 2G^i + 2\sigma_k y^k y^i - \sigma^i L^2.$$

If we write $\sigma_i = \sigma_1 l_i + \sigma_2 m_i + \sigma_3 n_i$ in three dimensional Finsler space (M,L), then

(2.7) $$2\bar{G}^i = 2G^i + L^2(\sigma_1 l^i - \sigma_2 m^i - \sigma_3 n^i)\,,$$

Differentiating equation (2.7) with respect to y^j and using equations (2.1), (2.3) and the fact that $G^i_j = \dfrac{\partial G^i}{\partial y^j}$, we get

(2.8) $$\overline{G}^i_j = G^i_j + Ll^i(\sigma_1 l_j + \sigma_2 m_j + \sigma_3 n_j) - Lm^i(\sigma_2 l_j - \sigma_4 m_j + \sigma_5 n_j) - Ln^i(\sigma_3 l_j + \sigma_5 m_j - \sigma_6 n_j).$$

where we have written

(2.8)′ $\sigma_4 = \sigma_1 + \sigma_2 H - \sigma_3 J$, $\sigma_5 = \sigma_2 J - \sigma_3 I$ and $\sigma_6 = \sigma_1 + \sigma_2 I + \sigma_3 J$.

On the other hand, the connection coefficients F^i_{jk} of Cartan's connection $C\Gamma$ are given by[4]

$$F_{ijk} = g_{jr} F^r_{ik} = \gamma_{ijk} - C_{ijr} G^r_k - C_{jkr} G^r_i - C_{ikr} G^r_j .$$

Therefore from (2.1), (2.5) and (2.8), we get

(2.9) $$\begin{aligned} \bar{F}^i_{jk} = F^i_{jk} &+ l^i\{\sigma_1 l_j l_k + \sigma_2 (l_j m_k + m_j l_k) + \sigma_3 \left(l_j n_k + n_j l_k\right) - \sigma_4 m_j m_k \\ &+ \sigma_5 \left(m_j n_k + n_j m_k\right) - \sigma_6 n_j n_k\} - m^i\{\sigma_2 l_j l_k - \sigma_4 \left(l_j m_k + m_j l_k\right) \\ &- \sigma_5 \left(l_j n_k + n_j l_k\right) - (\sigma_3 + \sigma_5 H + \sigma_6 J) \left(m_j n_k + n_j m_k\right) \\ &-(\sigma_2 - \sigma_4 H - \sigma_5 J)\, m_j m_k + (\sigma_2 - \sigma_4 I + 3\sigma_5 J + 2\sigma_6 I)\, n_j n_k\} \\ &- n^i[\sigma_3 l_j l_k + \sigma_5 \left(l_j m_k + m_j l_k\right) - \sigma_6 \left(l_j n_k + n_j l_k\right) \\ &-(\sigma_2 - \sigma_4 I + \sigma_6 J) \left(m_j n_k + n_j m_k\right) + \{\sigma_3 - 2\sigma_4 J + \sigma_6 J \\ &+ \sigma_5 (H - 2I)\}\, m_j m_k - (\sigma_3 + \sigma_5 I - \sigma_6 J)\, n_j n_k]. \end{aligned}$$

Now, we shall deal with the h-covariant derivative $S_{\perp i}$ of a conformally invariant scalar field S with respect to the conformally changed Cartan connection $C\bar{\Gamma}$: $S_{\perp i} = \partial_i S - \dot{\partial}_r S\, \overline{G}^r_i$, S is positively homogeneous of degree zero in y^i. Then from (2.8), we have

$$\begin{aligned} S_{\perp j} = \partial_j S - \dot{\partial}_r S\, \{ G^r_j + Ll^r(\sigma_1 l_j + \sigma_2 m_j + \sigma_3 n_j) - Lm^r(\sigma_2 l_j \\ - \sigma_4 m_j + \sigma_5 n_j) - Ln^r(\sigma_3 l_j + \sigma_5 m_j - \sigma_6 n_j), \end{aligned}$$

which gives immediately

$$(2.10) \qquad S_{\perp j} = S_{|j} + S;_2 (\sigma_2 l_j - \sigma_4 m_j + \sigma_5 n_j) + S;_3 (\sigma_3 l_j + \sigma_5 m_j - \sigma_6 n_j).$$

Since $S_{\perp i} = S;_1 \bar{l}_i + S;_2 \bar{m}_i + S;_3 \bar{n}_i$, from (2.2) and (2.10) we have the relations

$$(2.11) \qquad \begin{aligned} S;_1 &= S;_i \bar{l}^i = e^{-\sigma}(S,_1 + S;_2 \sigma_2 + S;_3 \sigma_3), \\ S;_2 &= S;_i \bar{m}^i = e^{-\sigma}(S,_2 - S;_2 \sigma_4 + S;_3 \sigma_5), \\ S;_3 &= S;_i \bar{l}^i = e^{-\sigma}(S,_3 - S;_2 \sigma_5 - S;_3 \sigma_6) \end{aligned}$$

For the conformal change of the adopted components h_α of h-connection vector h_i, from (2.2) and (2.4), we have $\bar{m}_{i\perp j} = e^{\sigma}(m_i \sigma_j + m_{i\perp j})$, which in view of (2.8) and (2.9) leads to

$$(2.12) \quad \begin{aligned} \bar{h}_i = h_i + (\sigma_2 v_2 + \sigma_3 v_3)\, l_i - (\sigma_4 v_2 - \sigma_5 v_3 + \sigma_3 + H\sigma_5 + J\sigma_6 \\ - J\sigma_4 - I\sigma_5)\, m_j + (v_2 \sigma_5 - v_3 \sigma_6 + 2J\sigma_5 + Is_6 - Is_4 + s_2)\, n_j . \end{aligned}$$

Thus the adopted components h_α α= 1, 2, 3 of h_i in (M, $\bar{L}$) are given by

$$(2.13) \qquad \begin{aligned} \mathrm{h}_1 &= \mathrm{e}^{-\sigma}(\mathrm{h}_1 + \sigma_2 \mathrm{v}_2 + \sigma_3 \mathrm{v}_3), \\ h_2 &= e^{-\sigma}\{h_2 - (\sigma_4 v_2 - \sigma_5 v_3 + \sigma_3 + H\sigma_5 + J\sigma_6 - J\sigma_4 - I\sigma_5)\}, \\ h_3 &= e^{-\sigma}(h_3 + v_2\sigma_5 - v_3\sigma_6 + 2J\sigma_5 + I\sigma_6 - I\sigma_4 + \sigma_2). \end{aligned}$$

3. Conformally flat Landsberg space

Berwald spaces are characterized by $C_{ijk|h} = 0$ and Landsberg spaces are characterized by $C_{ijk|0} = 0$ where the index '0' denotes the transvection by the supporting element y^i. If a Finsler space is a Berwald space, it is a Landsberg space.

it is shown[5,6,7] that Landsberg space becomes a Berwald space in many cases. We have discussed the same case with some condition in three dimensional Finsler space.

Definition (3.1)[1]: A Finsler space F^n is called conformally flat if F^n is conformal to a locally Minkowaski space.

Theorem (3.1)[1]**:** *A Finsler space F^3 with non zero C is a Berwald space if and only if the h-connection vector h_i vanishes and all the main scalars are h-covariant constant.*

Theorem (3.2)[1]**:** *A Finsler space F^3 with non zero C is a Landsberg space if and only if the h-connection vector h_i is orthogonal to the supporting element y^i, that is $h_1 = 0$ and the main scalars $H_{,1} = I_{,1} = J_{,1} = 0$.*

If the three dimensional Finsler space $\bar{F}^3 = (M, \bar{L})$ is conformal to a Finsler space (M, L), the main scalars $\bar{H}$, $\bar{I}$ and $\bar{J}$ of $(M, \bar{L})$ coincide with the main scalars H, I and J of (M, L). In particular we must notice that the main scalars H, I, J and h-connection vector h_i in our case are functions of the variable y^i alone.

Firstly, we suppose that the Finsler space $(M, \bar{L})$ be a Landsberg space. Then from Theorem (3.2) it follows that

$$\bar{H}_{,1} = 0, \quad \bar{I}_{,1} = 0, \quad \bar{J}_{,1} = 0 \text{ and } \bar{h}_1 = 0. \tag{3.1}$$

The scalar $\bar{H}_{,1}$ can be written in terms of Moor'a frame as

$$H_{,1} = \bar{H}_{,k}\bar{l}^k = \left(\frac{\partial\bar{H}}{\partial x^k} - \bar{G}^r_k\frac{\partial\bar{H}}{\partial y^r}\right)\bar{l}^k = \bar{G}^r_k\frac{\partial H}{\partial y^r}e^{-\sigma}l^k$$
$$= -Le^{-\sigma}\{l^r(\sigma_1 l_k + \sigma_2 m_k + \sigma_3 n_k) - m^r(\sigma_2 l_k - \sigma_4 m_k + \sigma_5 n_k)$$
$$- n^r(\sigma_3 l_k + \sigma_5 m_k - \sigma_6 n_k)\}\, l^k H|_r$$
$$= -Le^{-\sigma}(\sigma_1 H|_r\, l^r - \sigma_2 H|_r\, m^r - \sigma_3 H|_r\, n^r).$$

Therefore,

$$\bar{H}_{,1} = (\sigma_2 H;_2 + \sigma_3 H;_3)e^{-\sigma}. \tag{3.2}$$

Similarly, we have

$$\mathrm{I}_{,1} = (\sigma_2 \mathrm{I};_2 + \sigma_3 \mathrm{I};_3)\mathrm{e}^{-\sigma}, \tag{3.3}$$

$$\bar{\mathrm{J}}_{,1} = (\sigma_2 \mathrm{J};_2 + \sigma_3 \mathrm{J};_3)\mathrm{e}^{-\sigma}. \tag{3.4}$$

Therefore, from (3.1), (3.2), (3.3), (3.4) and (2.12), we get

$$\begin{aligned} &\sigma_2 H;_2 + \sigma_3 H;_3 = 0, \qquad \sigma_2 I;_2 + \sigma_3 I;_3 = 0, \\ &\sigma_2 J;_2 + \sigma_3 J;_3 = 0 \text{ and } h_1 + \sigma_2 v_2 + \sigma_3 v_3 = 0. \end{aligned} \tag{3.5}$$

Now, we prove that σ_2 and σ_3 never vanish simultaneously, for non homothetic transformation.

If possible suppose that σ_2 = 0, σ_3 = 0, then $\sigma_i = \sigma_1 l_i + \sigma_2 m_i + \sigma_3 n_i$, gives $\sigma_i = \sigma_1 l_i$. Differentiating this with respect to y^j, we get

$$0 = (\dot{\partial}_j \sigma_1) l_i + \sigma_1 \dot{\partial}_j l_i = (\dot{\partial}_j \sigma_1) l_i + \sigma_1 l_i \big|_j ,$$

which in view of (1.3) gives $0 = (\dot{\partial}_j \sigma_1) l_i + \frac{\sigma_1}{L}(m_i m_j + n_i n_j)$ or

$$\text{(3.6)} \qquad \frac{\sigma_1}{L}(m_i m_j + n_i n_j) = -(\dot{\partial}_j \sigma_1) l_i \cdot$$

Since L. H. S. of equation (3.6) is symmetric in i and j therefore

$$(\dot{\partial}_i \sigma_1) l_j = (\dot{\partial}_j \sigma_1) l_i \cdot$$

Contracting this equation by l^j, we get $(\dot{\partial}_i \sigma_1) = (\dot{\partial}_j \sigma_1) l^j l_i$. Since σ_1 is positively homogeneous of degree zero in y^i, therefore $(\dot{\partial}_j \sigma_1) l^j = 0$, which implies $\dot{\partial}_i \sigma_1 = 0 \cdot$

Thus equation (3.6) shows that σ_1 = 0. Hence σ_i = 0 which shows that σ is constant, i.e. the transformation is homothetic.

Hence we conclude that, for non homothetic transformation σ_2 and σ_3 do not vanish simultaneously. So we consider here three cases of non homothetic transformation.

Case (i): Let $\sigma_2 \neq 0$ and $\sigma_3 \neq 0$. In this case, from (3.5), we have

$$\text{(3.7)} \qquad \frac{H_{;2}}{H_{;3}} = \frac{I_{;2}}{I_{;3}} = \frac{J_{;2}}{J_{;3}} = -\frac{\sigma_3}{\sigma_2} \quad \text{and} \quad h_1 = -\sigma_2 v_2 - \sigma_3 v_3 .$$

Conversely if (3.7) holds then from (2.13), (3.2), (3.3) and (3.4), we get (3.1). So (M, $\bar{L}$) is a Landsberg space. Hence we have the following:

Theorem (3.3): *A three dimensional Landsberg space is σ-conformally flat if and only if (3.7) holds.*

Case (ii): Let $\sigma_2 = 0$ and $\sigma_3 \neq 0$. In this case, from (3.5), we have

(3.8) $$H;_3 = 0,\ I;_3 = 0,\ J;_3 = 0 \text{ and } h_1 + \sigma_3 v_3 = 0.$$

Conversely if (3.8) holds for $\sigma_2 = 0$ and $\sigma_3 \neq 0$, then from (2.13), (3.2), (3.3) and (3.4) we get (3.1). So (M, $\bar{L}$) is a Landsberg space. Therefore it follows that

Theorem (3.4): *If σ_i is orthogonal to m^i, then a three dimensional Landsberg space is σ-conformally flat if and only if $H;_3 = 0$, $I;_3 = 0$, $J;_3 = 0$ and* $h_1 = -\sigma_3 v_3$.

Case (iii): Let $\sigma_2 \neq 0$ and $\sigma_3 = 0$. In this case, from (3.5), we have

(3.9) $$H;_2 = 0,\ I;_2 = 0,\ J;_2 = 0 \text{ and } h_1 + \sigma_2 v_2 = 0.$$

Conversely if (3.9) holds for $\sigma_2 \neq 0$ and $\sigma_3 = 0$, then from (2.13), (3.2), (3.3) and (3.4) we get (3.1). So (M, $\bar{L}$) is a Landsberg space. Therefore it follows that

Theorem (3.5): *If σ_i is orthogonal to n^i, then a three dimensional Landsberg space is σ-conformally flat if and only if* $H;_2 = 0$, $I;_2 = 0$, $J;_2 = 0$ and $h_1 = -\sigma_2 v_2$.

4. Conformally flat Berwald space

We consider the case when the Finsler space (M, $\bar{L}$) be a Berwald space. We shall rewrite $\bar{H}_{\perp k} = \left(\dfrac{\partial \bar{H}}{\partial x^k} - \bar{G}^r_k \dfrac{\partial \bar{H}}{\partial y^r}\right)$. Since H, I, J and connection vector h_i are only functions of the variable (y^i), this equation is equivalent to

$$\bar{H}_{\perp k} = -\bar{G}^r_k \frac{\partial H}{\partial y^r}.$$

Therefore, from (2.8), we get

(4.1) $$\begin{aligned}\bar{H}_{\perp k} = -L\{l^r(\sigma_1 l_k + \sigma_2 m_k + \sigma_3 n_k) - m^r(\sigma_2 l_k - \sigma_4 m_k + \sigma_5 n_k)\\ - n^r(\sigma_3 l_k + \sigma_5 m_k - \sigma_6 n_k)\}H|_r.\end{aligned}$$

Since $H|_r = L^{-1}\left(H;_1 l_r + H;_2 m_r + H;_3 n_r\right)$ and $H;_1 = 0$, we have

(4.2) $$\bar{H}_{\perp k} = H;_2(\sigma_2 l_k - \sigma_4 m_k + \sigma_5 n_k) + H;_3(\sigma_3 l_k + \sigma_5 m_k - \sigma_6 n_k).$$

Similarly, we get

(4.3) $$\bar{I}_{\perp k} = I;_2(\sigma_2 l_k - \sigma_4 m_k + \sigma_5 n_k) + I;_3(\sigma_3 l_k + \sigma_5 m_k - \sigma_6 n_k),$$

(4.4) $$\bar{J}_{\perp k} = \mathrm{J};_2(\sigma_2 l_k - \sigma_4 m_k + \sigma_5 n_k) + J;_3(\sigma_3 l_k + \sigma_5 m_k - \sigma_6 n_k).$$

Now, we discuss all the three cases which are discussed in previous section.

Case (i). Let $\sigma_2 \neq 0$ and $\sigma_3 \neq 0$.

If the three dimensional Landsberg space (M, $\bar{\mathrm{L}}$) is conformally flat, then from equations (4.2), (4.3), (4.4), (2.12) and (3.7), we get

(4.5) $$\bar{H}_{\perp k} = (\sigma_5 H;_3 - \sigma_4 H;_2)\, m_k + (\sigma_5 H;_2 - \sigma_6 H;_3)\, n_k,$$

$$\bar{I}_{\perp k} = (\sigma_5 I;_3 - \sigma_4 I;_2)\, m_k + (\sigma_5 I;_2 - \sigma_6 I;_3)\, n_k,$$

$$\bar{J}_{\perp k} = (\sigma_5 J;_3 - \sigma_4 J;_2)\, m_k + (\sigma_5 J;_2 - \sigma_6 J;_3)\, n_k,$$

and $$h_j = \{h_2 - (\sigma_4 v_2 - \sigma_5 v_3 + \sigma_3 + H\sigma_5 + J\sigma_6 - J\sigma_4 + I\sigma_5)\}m_j + \{h_3 + (\sigma_5 v_2 + \sigma_6 v_3 + 2\sigma_5 J + \sigma_6 I - \sigma_4 I + \sigma_2)\}n_j.$$

From Theorem (3.1) it follows that the space (M, $\bar{\mathrm{L}}$) is a Berwald space if $\bar{H}_{\perp k} = \bar{I}_{\perp k} = \bar{J}_{\perp k} = 0$ and $\bar{h}_i = 0$.

Therefore from (4.5) it follows that (M, $\bar{\mathrm{L}}$) is a Berwald space, if

(4.6) $$\sigma_5 H;_3 - \sigma_4 H;_2 = 0,\quad \sigma_5 H;_2 - \sigma_6 H;_3 = 0,\quad \sigma_5 I;_3 - \sigma_4 I;_2 = 0,$$

$$\sigma_5 I;_2 - \sigma_6 I;_3 = 0,\quad \sigma_5 J;_3 - \sigma_4 J;_2 = 0,\quad \sigma_5 J;_2 - \sigma_6 J;_3 = 0.$$

$$h_2 - (\sigma_4 v_2 - \sigma_5 v_3 + \sigma_3 + H\sigma_5 + J\sigma_6 - J\sigma_4 + I\sigma_5) = 0$$

and $$h_3 + (\sigma_5 v_2 + \sigma_6 v_3 + 2\sigma_5 J + \sigma_6 I - \sigma_4 I + \sigma_2) = 0.$$

Conversely if (4.6) holds, then from (4.5) we get $\bar{H}_{\perp k} = \bar{I}_{\perp k} = \bar{J}_{\perp k} = 0$ and $\bar{h}_i = 0$. Hence (M, $\bar{\mathrm{L}}$) is a Berwald space.

Theorem (4.1): *A three dimensional conformally flat Landsberg space is a Berwald space if and only if the equations (4.6) are satisfied.*

Case (ii): Let $\sigma_2 = 0$ and $\sigma_3 \neq 0$. In this case if a three dimensional Lansberg space is σ- conformally flat then from Theorem (3.4) we get $H;_3 = 0$, $I;_3 = 0$, $J;_3 = 0$ and $h_1 = -\sigma_3 v_3$. Therefore from (4.6) and (2.8)′, a three dimensional σ- conformally flat Landsberg space is a Berwald space if

$$(4.7)(a) \qquad H;_2 = 0,\ I;_2 = 0,\ J;_2 = 0,$$

$$(4.7)(b) \qquad h_2 = \sigma_1 v_2 + \sigma_3(1 - Jv_2 + v_3 I - HI + 2J^2 + IJ),$$

$$(4.7)(c) \qquad h_3 = \sigma_1 v_3 + \sigma_3\left(Iv_2 + Jv_3\right).$$

Since H, I, J are positively homogeneous of degree zero in y^i, therefore $H;_1 = 0$, $I;_1 = 0$, $J;_1 = 0$. Hence main scalars H, I, J are functions of position only.

Conversely, if H, I, J are functions of position only and (4.7)(b) and (4.7)(c) hold for $\sigma_2 = 0$ and $\sigma_3 \neq 0$, then from (4.5) we get $\overline{H}_{\perp k} = \overline{I}_{\perp k} = \overline{J}_{\perp k} = 0$ and $\overline{h}_i = 0$. Hence (M, $\overline{L}$) is a Berwald space. Therefore we get

Theorem (4.2): *If σ_i is orthogonal to m^i then a three dimensional σ-conformally flat Landsberg space is a Berwald space if and only main scalars are functions of position only and (4.7)(b) and (4.7)(c) are satisfied.*

Case (iii): Let $\sigma_2 \neq 0$ and $\sigma_3 = 0$. In this case if a three dimensional Lansberg space is σ- conformally flat then from Theorem (3.5) we get $H;_2 = 0$, $I;_2 = 0$, $J;_2 = 0$ and $h_1 = -\sigma_2 v_2$. Therefore from (4.6) and (2.8)′, a three dimensional σ- conformally flat Landsberg space is a Berwald space if

$$(4.8)(a) \qquad H;_3 = 0,\ I;_3 = 0,\ J;_3 = 0,$$

$$(4.8)(b) \qquad h_2 = \sigma_1 v_2 + \sigma_2(Hv_2 - Jv_3),$$

$$(4.8)(c) \qquad h_3 = \sigma_1 v_3 - \sigma_2(v_2 - Iv_3 + 2J^2 + I^2 - HI + 1)$$

Hence H, I, J are functions of positions only along with (4.8)(a) and (4.8)(b).

Conversely, if H, I, J are functions of position only and (4.8)(b) and (4.8)(c) hold for $\sigma_2 = 0$ and $\sigma_3 = 0$, then from (4.5) we get $\overline{H}_{\perp k} = \overline{I}_{\perp k} = \overline{J}_{\perp k} = 0$ and $\overline{h}_i = 0$. Hence (M, $\overline{\text{L}}$) is a Berwald space. Therefore we get

Theorem (4.3): *If σ_i is orthogonal to n^i then a three dimensional σ-conformally flat Landsberg space is a Berwald space if and only main scalars are functions of position only and (4.8)(b) and (4.8)(c) are satisfied.*

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