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Cartesian Product of r-GF Structure Manifolds

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Abstract: Cartesian product of two manifolds has been defined and studied by Pandey[1]. In this paper we have taken Cartesian product of r-GF structure manifolds, where r is some finite integer, and studied some properties of curvature and Ricci tensor of such a product manifold.

Key words & Phases: r-GF Structure Manifolds, generalized almost contact structure, KH-structure.

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1. Introduction

Let $M_1, M_2, ..., M_r$ be r-GF structure manifolds each of class C^∞ and of dimension $n_1, n_2, ..., n_r$ respectively. Suppose $(M_1)m_1$, $(M_2)m_2, \ldots, (M_r)m_r$ be their tangent spaces at $m_1 \in M_1$ $, m_2 \in M_2, ..., m_r \in M_r$, then the product space $(M_1)m_1 \times (M_2)m_2 \times .... \times (M_r)m_r$ contains vector fields of the form $(X_1, X_2, ..., X_r)$, where $X_1 \in (M_1)m_1, X_2 \in (M_2)m_2, ..., X_r \in (M_r)m_r$. Vector addition and scalar multiplication on above product space are defined as follows:

(1.1) $$(X_1, X_2, ..., X_r) + (Y_1, Y_2, ..., Y_r) = (X_1 + Y_1, X_2 + Y_2, ..., X_r + Y_r),$$

(1.2) $$\lambda(X_1, X_2, ..., X_r) = (\lambda X_1, \lambda X_2, ..., \lambda X_r),$$

where $X_i, Y_i \in (M_i)m_i$, $i = 1, 2, ..., r$ and λ is a scalar.

Under these conditions the product space $(M_1)m_1 \times (M_2)m_2 \times ...... \times (M_r)m_r$ forms a vector space.

Define a linear transformation F on the product space

(1.3) $$F(X_1, X_2, ..., X_r) = (F_1X_1, F_2X_2, ..., F_rX_r),$$

where $F_1, F_2, ..., F_r$ are linear transformations on $(M_1)m_1, (M_2)m_2, ..., (M_r)m_r$ respectively.

If $f_1, f_2, ..., f_r$ be C^∞ functions over the spaces $(M_1)m_1, (M_2)m_2, ..., (M_r)m_r$ respectively, we define the C^∞ function $f_1, f_2, ..., f_r$ on the product space as

(1.4) $$(X_1, X_2, ..., X_r)(f_1, f_2, ..., f_r) = (X_1 f_1, X_2 f_2, ..., X_r f_r).$$

Let $D_1, D_2, ..., D_r$ be the connections on the manifolds $M_1, M_2, ..., M_r$ respectively. We define the operator D on the product space as

(1.5) $$D_{(X_1, X_2, ..., X_r)}(Y_1, Y_2, ..., Y_r) = (D_{1_{X_1}} Y_1, D_{2_{X_2}} Y_2, ..., D_{r_{X_r}} Y_r).$$

Then D satisfies all four properties of a connection and thus it is a connection on the product manifold.

2. Some Results

Theorem 2.1: *The product manifold* $M_1 \times M_2 \times .... \times M_r$ *admits a GF-structure if and only if the manifolds* $M_1, M_2, ..., M_r$ *are GF-structure manifolds.*

Proof: Suppose $M_1, M_2, ..., M_r$ are GF-structure manifolds. Thus there exist tensor fields $F_1, F_2, ..., F_r$ each of type (1,1) on $M_1, M_2 ...., M_r$ respectively satisfying

(2.1) $$F_i^2(X_i) = a^2 X_i, \qquad i = 1, 2, ..., r$$

where a is any complex number, not equal to zero.

In view of equation (1.3) it follows that there exists a linear transformation F on $M_1 \times M_2 \times .... \times M_r$ satisfying

(2.2) $$F^2(X_1, X_2, ..., X_r) = (F^2{}_1X_1, F_2{}^2X_2, ..., F^2{}_rX_r)$$

$$= a^2(X_1, X_2, ..., X_r).$$

Thus, the product manifold admits a GF-structure.

Let us define a Riemannian metric g on the product manifold $M_1 \times M_2 \times .... \times M_r$ as

$$a^2 g((X_1, X_2, ..., X_r), (Y_1, Y_2, ..., Y_r)) = a^2 g_1(X_1, Y_1) + a^2 g_2(X_2, Y_2) + ... + a^2 g_r(X_r, Y_r), \tag{2.3}$$

where $g_1, g_2, ..., g_r$ are the Riemannian metrics over the manifolds $M_1, M_2 ...., M_r$ respectively.

If $\xi_1, \xi_2, ..., \xi_r$ be vector fields and $\eta_1, \eta_2, ..., \eta_r$ be 1-forms on the GF-structure manifolds $M_1, M_2 ...., M_r$ respectively, then a vector field ξ and a 1-form η on the product manifold $M_1 \times M_2 \times .... \times M_r$ is defined.

We now prove the following results.

Theorem 2.2: *The product manifold* $M_1 \times M_2 \times .... \times M_r$ *admits a generalized almost contact structure if and only if the manifolds* $M_1, M_2 ...., M_r$ *possess the same structure.*

Proof: Let $M_1, M_2 ...., M_r$ are generalized almost contact manifolds. Thus there exists tensor fields F_i of type (1,1), vector fields ξ_i and 1-forms. η_i, $i = 1, 2, ..., r$ satisfying

$$F_i^2(X_i) = a^2 X_i + \eta_i(X_i)\xi_i, \tag{2.4}$$

for product manifold $M_1 \times M_2 \times .... \times M_r$.

$$F^2(X_1, X_2, ..., X_r) = (F^2{}_1 X_1, F_2{}^2 X_2, ..., F^2{}_r X_r),$$

by the help of equation (2.4), takes the form

$$F^2(X_1, X_2, ..., X_r) = a^2(X_1, X_2, ..., X_r) + (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, ..., \eta_r(X_r)\xi_r),$$

or

$$F^2(X) = a^2 X + \eta(X)\xi. \tag{2.5}$$

Hence the product manifold admits a generalized almost contact metric structure[2].

Theorem 2.3: *The product manifold* $M_1 \times M_2 \times .... \times M_r$ *admits a KH-structure if and only if the manifolds* $M_1, M_2 ...., M_r$ *are KH-structure manifolds.*

Proof: Suppose $M_1, M_2 ...., M_r$ are KH-structure manifolds. Thus

$$(2.6) \qquad \begin{aligned}(D_{1_{X_1}} F_1)(Y_1) &= (D_{2_{X_2}} F_2)(Y_2) \\ &= .... \\ &= (D_{r_{X_r}} F_r)(Y_r) \\ &= 0.\end{aligned}$$

As D is a connection on the product manifold, we have

$$(2.7) \qquad \begin{aligned}(D_{(X_1, X_2, ..., X_r)} F)(Y_1, Y_2, ..., Y_r) &= D_{(X_1, X_2, ..., X_r)} \{F(Y_1, Y_2, ..., Y_r) \\ &\quad - F\{D_{(X_1, X_2, ..., X_r)}(Y_1, Y_2, ..., Y_r)\}.\end{aligned}$$

In view of equation (1.3) and equation (1.5), this takes the form

$$\begin{aligned}(D_{(X_1, X_2, ..., X_r)} F)(Y_1, Y_2, ..., Y_r) &= D_{(X_1, X_2, ..., X_r)} (F_1 Y_1, F_2 Y_2, ..., F_r Y_r) \\ &\quad - F(D_{1_{X_1}} Y_1, D_{2_{X_2}} Y_2, ..., D_{r_{X_r}} Y_r) \\ &= -(D_{1_{X_1}} F_1 Y_1, D_{2_{X_2}} F_2 Y_2, ..., D_{r_{X_r}} F_r Y_r) \\ &\quad - (F_1 D_{1_{X_1}} Y_1, F_2 D_{2_{X_2}} Y_2, ..., F_r D_{r_{X_r}} Y_r) \\ &= ((D_{1_{X_1}} F_1)(Y_1), (D_{2_{X_2}} F_2)(Y_2), ...(D_{r_{X_r}} F_r)(Y_r) \\ &= 0.\end{aligned}$$

Thus, the product manifold is KH-structure manifold.

Theorem 2.4: *The product manifold* $M_1 \times M_2 \times .... \times M_r$ *of GF-structure manifolds* $M_1, M_2 ...., M_r$ *is almost Tachibana if and only if the manifolds* $M_1, M_2 ...., M_r$ *are separately Tachibana manifolds.*

Proof: Let a GF-structure manifolds $M_1, M_2, ..., M_r$ are almost Tachibana manifolds. Then

(2.8) $(D_{i_{X_i}} F_i)(Y_i) + (D_{i_{Y_i}} F_i)(Y_i) = 0, \quad i = 1, 2, ..., r$.

3. Curvature and Ricci Tensor

Let $X = (X_1, X_2, ..., X_r)$ and $Y = (Y_1, Y_2, ..., Y_r)$ be C^∞ vector fields on the product manifold $M_1 \times M_2 \times .... \times M_r$ and $F = (f_1, f_2, ..., f_r)$ be a C^∞ function. Then

(3.1) $$[(X_1, X_2, ..., X_r), (Y_1, Y_2, ..., Y_r)](f_1, f_2, ..., f_r)$$
$$= (X_1, X_2, ..., X_r)\{(Y_1, Y_2, ..., Y_r)(f_1, f_2, ..., f_r)\} - (Y_1, Y_2, ..., Y_r)$$
$$= ([X_1, Y_1] f_1, [X_2, Y_2] f_2, ..., [X_r, Y_r] f_r) .$$

Suppose $K_i(X_i, Y_i, Z_i)$, $i = 1, 2, ..., r$ be the curvature tensors of the GF-structure manifolds $M_1, M_2, ..., M_r$ respectively. If $K(X, Y, Z)$ be the curvature tensor of the product manifold $M_1 \times M_2 \times .... \times M_r$ .Then we have

(3.2) $$K(X, Y, Z) = [K_1(X_1, Y_1, Z_1), K_2(X_2, Y_2, Z_2), ..., K_r(X_r, Y_r, Z_r)].$$

If $W = (W_1, W_2, ..., W_r)$ be a vector field on the product manifold, then

(3.3) $$K'(X, Y, Z, W) = g(K(X, Y, Z, W)),$$

(3.4) $$K'(X, Y, Z, W) = K_1'(X_1, Y_1, Z_1, W_1) + K_2'(X_2, Y_2, Z_2, W_2) + ...$$
$$+ K_r'(X_r, Y_r, Z_r, W_r)$$

Thus, we have

Theorem 3.1: *The product manifold $M_1 \times M_2 \times .... \times M_r$ is of constant curvature if and only if GF-structure manifolds $M_1, M_2, ..., M_r$ are separately of constant curvature.*

Theorem 3.2: *The Ricci tensor of the product manifold $M_1 \times M_2 \times .... \times M_r$ is the sum of the Ricci tensor of the GF-structure manifolds $M_1, M_2, ..., M_r$.*

Theorem 3.3: *The product manifold* $M_1 \times M_2 \times .... \times M_r$ *is an Einstein space if and if only if the GF-structure manifolds* $M_1, M_2, ..., M_r$ *are separately Einstein spaces.*

Proof: Let the product manifold $M_1 \times M_2 \times .... \times M_r$ be an Einstein space. Thus

(3.5) $$Ric(X, Y) = Cg(X, Y),$$

where $C = \frac{K}{n}$, K being the scalar curvature and n being the dimension of the product manifold. Then

$$Ric(X_i, Y_i) = Cg_i(X_i, Y_i), \quad i = 1, 2, ..., r.$$

Therefore the manifolds $M_1, M_2, ..., M_r$ are also Einstein spaces.

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