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Special Types of Projective Motions in a Projective Finsler Space

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Abstract: The studies of a Projective Finsler space have been carried out by Misra and Meher[1]. Special types of projective motions have been introduced using a contra, a concurrent, a special concircular and torse-forming transformations in a recurrent Finsler space and Finsler space in detail by Misra[2-4], Misra and Meher[5], Pandey[6-9], Misra and Mishra[10]. Takano[11] studied the existence of affine motion in a non-Riemannian K* space and obtained several significant results. He also studied the existence of projective motion in a Riemannian space of birecurrent curvature. The purpose of the present paper is to study special types of projective motions in a projective Finsler space.

Key words: Projective Finsler space, Projective motion, Special transformations.

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1. Introduction

Let F_n be an n-dimensional Finsler space equipped with Berwald's connection parameters $G^i_{jk}(x,\dot{x})$. These parameters are homogeneous functions of degree zero in their directional arguments. Hence, from the Euler's theorem, following relations hold

(1.1) $$G^i_{jkh}\dot{x}^j = 0,$$

where $G^i_{jkh} = \dot{\partial}_j G^i_{kh}$.

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The covariant derivative of a vector field $X^i(x,\dot{x})$ is defined by

(1.2) $$\mathcal{B}_k X^i = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk},$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$ and $\dot{\partial}_k \equiv \frac{\partial}{\partial \dot{x}^k}$.

The projective connection parameters are defined by Misra and Meher[1] as

(1.3) $$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1}(\delta^i_j G^r_{kr} + \delta^i_k G^r_{jr} + \dot{x}^i G^r_{jkr}),$$

where Π^i_{jk} is symmetric in its lower indices and is homogeneous of degree zero in its directional arguments .
The connection parameters Π^i_{jk} satisfy the following contractions:

(1.4) $$\begin{cases} (a) \quad \Pi^i_{ij} = \Pi^i_{ji} = 0, \\ (b) \quad \Pi^i_i = 0, \\ (c) \quad \Pi^i_{ikh} = \Pi^i_{khi} = \Pi^i_{kih} = 0, \end{cases}$$

where

$$\Pi^i_{jkh} = \dot{\partial}_j \Pi^i_{kh}$$
$$= G^i_{jkh} - \frac{1}{n+1}(\delta^i_k G^r_{jhr} + \delta^i_h G^r_{jkr} + \delta^i_j G^r_{khr} + \dot{x}^i G^r_{jkhr}).$$

Furthermore, the projective connection parameters also satisfy

(1.6) $$\Pi^i_{jkh}\dot{x}^j = \Pi^i_{jkh}\dot{x}^k = \Pi^i_{jkh}\dot{x}^h = 0,$$

The entities Π^i_{jkh} are positively homogeneous of degree -1 in their directional arguments. The entities Π^i_{jkh} constitute a tensor and are symmetric in all of its lower indices.
Transvecting (1.3) by $\dot{x}^k$, we get

(1.7) $$\begin{cases} (a) \quad \Pi^i_{jk}\dot{x}^k = \Pi^i_j = G^i_j - \frac{1}{n+1}(\delta^i_j G^r_r + \dot{x}^i G^r_{rj}), \\ (b) \quad \Pi^i_j \dot{x}^j = \Pi^i = 2G^i - \frac{2}{n+1}\dot{x}^i G^r_r, \end{cases}$$

The projective covariant derivative of a vector field $X^i(x,\dot{x})$ is defined by Misra[3]

(1.8) $$\mathcal{P}_j X^i = \partial_j X^i - (\dot{\partial}_m X^i)\Pi^m_{pj}\dot{x}^p + X^m \Pi^i_{mj}.$$

The projective covariant derivative gives rise to the commutation formula

(1.9) $$\left(\mathcal{P}_j\mathcal{P}_k - \mathcal{P}_k\mathcal{P}_j\right)X^i = Q^i_{mjk}X^m - (\dot{\partial}_m X^i)Q^m_{pjk}\dot{x}^p,$$

where

(1.10) $$Q^i_{jkh}(x,\dot{x}) = 2(\partial_{[j}\Pi^i_{k]h} + \Pi^s_{r[k}\Pi^i_{j]hs}\dot{x}^r + \Pi^i_{r[j}\Pi^r_{k]h}),$$

which are analogous to a curvature tensor. The entities Q^i_{jkh} are homogeneous functions of degree zero in their directional arguments. It may be noted that Q^i_{jkh} are equal to Q^i_{hjk} this will mean that Q^i_{jkh} is symmetric in their lower indices j and h, and Q^i_{jkh} is skew-symmetric in its lower indices j and k

(1.11) $$Q^i_{jkh}(x,\dot{x}) = -Q^i_{kjh}(x,\dot{x}).$$

The contraction of indices in Q^i_{jkh} yields

(1.12) $$\begin{cases} (a) \quad Q^i_{jki} = 0, \\ (b) \quad Q^i_{ikh} = Q_{kh}, \\ (c) \quad Q^i_{jih} = -Q^i_{ijh} = -Q_{jh}, \\ (d) \quad Q^i_{ik} = Q_{k}. \end{cases}$$

On the other hand the contraction with respect to i, j yields the quantities

(1.13) $$Q^i_{ikh} = Q_{kh} = \dot{\partial}_i \Pi^i_{kh} - \dot{x}^r \Pi^s_{ri}\Pi^i_{khs} - \Pi^i_{rk}\Pi^r_{ih},$$

which are analogous to Ricci tensor of Riemannian geometry. The contracted quantities Q_{kh} are symmetric in their indices. Applying the definition of projective covariant derivative, Q_{kh} may also be written as

(1.14) $$Q_{kh} = \mathcal{P}_i \Pi^i_{kh} + \Pi^i_{rk}\Pi^r_{ih}.$$

The following homogeneous properties of entities Q^i_{jkh} are given by

(1.15) $$\begin{cases}(a) \quad Q^i_{jkh}\dot{x}^h = Q^i_{jk} = 2(\partial_{[j}\Pi^i_{k]} + \Pi^i_{r[j}\Pi^i_{k]}),\\ (b) \quad Q_{kh}\dot{x}^h = Q_k.\end{cases}$$

Definition 1.1: A Finsler space equipped with the entities Π^i_{jk}, Q^i_{jkh} etc. is called a projective Finsler space and is denoted by P-F_n .

Let us consider an infinitesimal point transformation

(1.16) $$\bar{x}^i = x^i + \varepsilon v^i,$$

where v^i denotes the components of a contravariant vector field, independent of the directional arguments and ε is an infinitesimal constant. The Lie-derivative of a vector $X^i(x,\dot{x})$ with respect to (1.16) is expressed by

(1.17) $$\pounds X^i = v^s \mathcal{P}_s X^i - X^s \mathcal{P}_s v^i - (\dot{\partial}_s X^i)(\mathcal{P}_m v^s)\dot{x}^m,$$

where £ stands for the Lie-derivative. The Lie-derivative of the entities Q^i_{jkh} and the connection parameters Π^i_{jk} are given by

(1.18) $$\begin{aligned}\pounds Q^i_{jkh} = v^s \mathcal{P}_j Q^i_{skh} - Q^s_{jkh}\mathcal{P}_s v^i + Q^i_{skh}\mathcal{P}_j v^s + Q^i_{jsh}\mathcal{P}_k v^s\\ + Q^i_{jks}\mathcal{P}_h v^s - (\dot{\partial}_s Q^i_{jkh})(\mathcal{P}_m v^s)\dot{x}^m,\end{aligned}$$

and

(1.19) $$\pounds \Pi^i_{jk} = \mathcal{P}_j \mathcal{P}_k v^i + v^h Q^i_{hjk} + \Pi^i_{jkh}(\mathcal{P}_r v^h)\dot{x}^r + \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r),$$

respectively. The processes of Lie-derivative and projective covariant derivative are connected by

(1.20) $$\mathcal{P}_j \pounds \Pi^i_{kh} - \mathcal{P}_k \pounds \Pi^i_{jh} = \pounds Q^i_{jkh} - \Pi^i_{hsj}(\pounds \Pi^s_{kr})\dot{x}^r + \Pi^i_{hsk}(\pounds \Pi^s_{jr})\dot{x}^r.$$

If a projective Finsler space admits a projective motion then the Lie-derivatives of the projective connection Π^i_{jk} and curvature tensor type entites Q^i_{jkh} satisfy

(1.21) $$\pounds \Pi^i_{jk} = 0,$$

and

(1.22) $$\pounds Q^i_{jkh} = 0.$$

Let us now consider an infinitesimal transformation of the type (1.16). As indicated by Takano[11] the projective covariant derivative of the vector v^i appearing in (1.16) may assume a number of values which are being respectively given as

$$(1.23)\qquad \begin{cases} (a) & \mathcal{P}_j v^i = 0, \\ (b) & \mathcal{P}_j v^i = c\delta^i_j, \qquad \text{c being a non-zero constant} \\ (c) & \mathcal{P}_j v^i = \rho(x,\dot{x})\delta^i_j, \qquad \rho \neq \text{constant} \\ (d) & \mathcal{P}_j v^i = \phi_j(x,\dot{x})v^i, \qquad \phi \neq 0 \\ (e) & \mathcal{P}_j v^i = \rho(x,\dot{x})\delta^i_j + \phi_j(x,\dot{x})v^i, \quad \mathcal{P}_j\phi_k - \mathcal{P}_k\phi_j = 0 \\ (f) & \mathcal{P}_j v^i = \rho(x,\dot{x})\delta^i_j + \phi_j(x,\dot{x})v^i, \end{cases}$$

The vector v^i appearing in (1.23) assumes different names like contra vector field / concurrent vector field / special concircular vector field / recurrent vector field / concircular vector field and torse-forming vector field respectively. The transformations with such different v^i are respectively named as contra / concurrent / special concircular / recurrent / concircular and torse-forming transformations. The functions ϕ_j are positively homogeneous of degree zero in directional arguments and satisfy[10]

$$(1.24)\qquad \dot{x}^k\phi_k = \dot{x}^k\dot{\partial}_k\phi = \phi.$$

2. Contra Transformation Defining Projective Motion

The Lie-derivative of the connection coefficient with respect to an infinitesimal transformation (1.16) is given by (1.19). Thus, if contra transformation defines a projective motion, we have from (1.19), (1.21) and (1.23a) the following after a little simplification

$$(2.1)\qquad v^h Q^i_{hjk} = \frac{1}{n+1}(\delta^i_j\partial_k\partial_r v^r + \delta^i_k\partial_j\partial_r v^r).$$

The contraction of (2.1) with respect to the indices i and k gives

$$(2.2)\qquad \partial_j\partial_r v^r = 0.$$

where we have taken (1.12a) into account. Using the equation (2.2) in (2.1), we get

(2.3) $$v^h Q^i_{hjk} = 0.$$

We, therefore, have

Theorem 2.1: *A P-F_n admitting a projective motion of the type (1.23a) satisfies (2.3).*

3. Concurrent Transformation Defining Projective Motion

In this section, we consider the case when a concurrent transformation defines a projective motion. With the help of the equations (1.19), (1.21) and (1.23b), we get

(3.1) $$v^h Q^i_{hjk} + c\Pi^i_{jkh}\dot{x}^h - \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r) = 0.$$

Using the equation (1.5) in (3.1), we get

(3.2) $$v^h Q^i_{hjk} = \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r).$$

The contraction of (3.2) with respect to the indices i, k and using equation (1.12a) thereafter, we obtain

(3.3) $$\partial_j \partial_r v^r = 0.$$

Using equation (3.3) in (3.2), we get

(3.4) $$v^h Q^i_{hjk} = 0.$$

Therefore, we have

Theorem 3.1: *A P-F_n admitting a projective motion of the type (1.23b) satisfies (3.4).*

Using the equations (1.22), and (1.23b) in (1.18), we get

(3.5) $$v^s \mathcal{P}_j Q^i_{skh} + 2cQ^i_{jkh} = 0.$$

Thus, we have

Theorem 3.2: *If the concurrent transformation (1.16) defines a projective motion in P-F_n, then the relation (3.5) holds.*

4. Special Concircular Transformation Defining Projective Motion

In this section, we consider the case when a special concircular transformation defines a projective motion. With the help of the equations (1.19), (1.21) and (1.23c), we get

$$\rho_j \delta^i_k + v^h Q^i_{hjk} + \rho \Pi^i_{jkh} \dot{x}^h = \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r), \tag{4.1}$$

where $\mathcal{P}_j \rho = \rho_j$.

Using equation (1.6) in (4.1), we get

$$\rho_j \delta^i_k + v^h Q^i_{hjk} = \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r). \tag{4.2}$$

Contracting (4.2) with respect to the indices i and k and using (1.12a) thereafter, we obtain

$$\partial_j \partial_r v^r = n\rho_j. \tag{4.3}$$

Similarly, if equation (4.2) is contracted with respect to the indices i and j then (4.1) in view of the equation (1.12c), yields

$$\rho_k - v^h Q_{hk} = \partial_k \partial_r v^r. \tag{4.4}$$

Thus eliminating $\partial_j \partial_r v^r$ from (4.3) and (4.4), we get

$$v^h Q_{hj} + (n-1)\rho_j = 0. \tag{4.5}$$

We, therefore have

Theorem 4.1: *If the special concircular transformation (1.16) defines a projective motion in projective Finsler space then (4.5) necessarily holds.*

Using the equations (1.15c), (1.22) and (1.23c) in (1.18), we get

$$v^s \mathcal{P}_j Q^i_{skh} + 2\rho Q^i_{jkh} = 0. \tag{4.6}$$

Thus, we have

Theorem 4.2: *If the special concircular transformation (1.16) defines a projective motion in projective Finsler space then the relation (4.7) necessarily holds.*

5. Recurrent Transformation Defining Projective Motion

In this section, we consider the case when a recurrent transformation defines a projective motion. With the help of the equations (1.19), (1.21), (1.23d) and (1.24) we get

$$(5.1)\qquad (\mathcal{P}_j\phi_k+\phi_j\phi_k)v^i+v^hQ^i_{hjk}+\Pi^i_{jkh}v^h\phi=\frac{1}{n+1}(\delta^i_j\partial_k\partial_rv^r+\delta^i_k\partial_j\partial_rv^r).$$

Contracting (5.1) with respect to the indices $i,\ j$ and using (1.12c), (1.4c) together with the skew-symmetric property of Q^i_{jkh}, we deduce

$$(5.2)\qquad (\mathcal{P}_i\phi_k+\phi_i\phi_k)v^i-v^hQ_{hk}=\partial_k\partial_rv^r.$$

Similarly, if (5.1) is contracted with respect to the indices i and k then in view of the equation (1.12a) and (1.4c), we obtain

$$(5.3)\qquad (\mathcal{P}_j\phi_i+\phi_j\phi_i)v^i=\partial_j\partial_rv^r.$$

Eliminating $\partial_j\partial_rv^r$ from (5.2) and (5.3), we have

$$(5.4)\qquad (Q_{hj}+\mathcal{P}_j\phi_h-\mathcal{P}_h\phi_j)v^h=0.$$

Thus, we have

Theorem 5.1: *If the recurrent transformation (1.16) defines a projective motion in* $P\text{-}F_n,$ *then the relation (5.4) holds.*

Theorem 5.2: *The invariance property of* ϕ_j *under the projective covariant operator* $\mathcal{P}_i$ *implies that the projective motion in projective Finsler space satisfies the relations*

$$(5.5)\qquad \begin{cases}(a)\quad v^hQ_{hj}=0,\\ (b)\quad \partial_k\partial_rv^r-\phi_i\phi_kv^i=0.\end{cases}$$

Proof : As ϕ_1 is a projective covariant constant (5.5a) can be easily found out from (5.4). In view of (5.5a) and (5.2), we have (5.5b).

Using the equations (1.15c), (1.22), and (1.23d) in (1.18), we get

(5.6) $$\left[\rho_j Q^i_{skh} + Q^i_{skh}\phi_j + Q^i_{jsh}\phi_k + Q^i_{jks}\phi_h + (\dot\partial_s Q^i_{jkh})\phi_m \dot x^m\right] v^s = Q^s_{jkh}\phi_s v^i.$$

Contracting (5.6) with respect to the indices i and j and using the equations (1.12b) and (1.24), we get

(5.7) $$v^s \mathcal{P}_i Q^i_{skh} + Q_{sh}\phi_k v^s + Q_{ks}\phi_h v^s = -(\dot\partial_s Q_{kh})\phi v^s.$$

Similarly, if contracting (5.6) with respect to the indices i, k and using (1.12c), we get

(5.8) $$v^s \mathcal{P}_j Q_{sh} + Q_{sh}\phi_j v^s + Q_{js}\phi_h v^s = -(\dot\partial_s Q_{jh})\phi v^s.$$

Thus eliminating $(\dot\partial_s Q_{jh})\phi v^s$ from (5.7) and (5.8), we get

(5.9) $$v^s(\mathcal{P}_i Q^i_{sjh} - \mathcal{P}_j Q_{sh}) = 0.$$

Thus, we have

Theorem 5.3: *If the recurrent transformation (1.16) defines a projective motion in P-F_n, then the relation (5.9) holds.*

6. Concircular Transformation Defining Projective Motion

In this section, we consider the case when a concircular transformation defines a projective motion. With the help of the equations (1.19), (1.21), and (1.23e), we get

(6.1) $$\begin{aligned} &\delta^i_k\rho_j + \mathcal{P}_j(\phi_k v^i) + v^h Q^i_{hjk} + \rho\Pi^i_{jkh}\dot x^h + \Pi^i_{jkh} v^h \phi_r \dot x^r \\ &= \frac{1}{n+1}(\delta^i_j \partial_k\partial_r v^r + \delta^i_k \partial_j \partial_r v^r), \end{aligned}$$

where $\mathcal{P}_j\rho = \rho_j$.

Using the equations (1.5), (1.23e) and (1.24) in (6.1), we obtain

(6.2) $$\begin{aligned} &\delta^i_k\rho_j + (\mathcal{P}_j\phi_k)v^i + \delta^i_j\rho\phi_k + \phi_j\phi_k v^i + v^h Q^i_{hjk} + v^h \Pi^i_{jkh}\phi \\ &= \frac{1}{n+1}(\delta^i_j \partial_k\partial_r v^r + \delta^i_k \partial_j \partial_r v^r). \end{aligned}$$

Contracting (6.2) with respect to the indices i, j and using (1.4c) and (1.12c) together with the skew-symmetric property of Q^i_{jkh}, we deduce

(6.3) $$\rho_k + (\mathcal{P}_i\phi_k + \phi_i\phi_k)v^i + n\rho\phi_k - v^h Q_{hk} = \partial_k\partial_r v^r.$$

Similarly contracting (6.2) with respect to the indices i and k, in view of the equations (1.4c) and (1.12a), we obtain from (6.2)

(6.4) $$n\rho_j + (\mathcal{P}_j\phi_i + \phi_i\phi_j)v^i + \phi_j\rho = \partial_j\partial_r v^r.$$

Thus eliminating $(\partial_j\partial_r v^r)$ from (6.3) and (6.4) and using (1.23), we have

(6.5) $$v^h Q_{hj} + (n-1)(\rho_j - \rho\phi_j) = 0.$$

We, therefore have

Theorem 6.1: *If the concircular transformation (1.16) defines a projective motion in P-F$_n$ then the relation (6.5) holds.*

Imposing further conditions on the functions ρ and ϕ_1, we have

Theorem 6.2: *The invariance property of ρ and ϕ_i, under the projective covariant operator $\mathcal{P}_m$ implies that the projective motion in projective Finsler space satisfies the relations*

(6.6) $$\begin{cases}(a) & v^h Q_{hj} = (n-1)\rho\phi_j, \\ (b) & v^h Q_{hk} - n\rho\phi_k - \phi_i\phi_k v^i + \partial_k\partial_r v^r = 0.\end{cases}$$

Proof: As ϕ_1 and ρ are projective covariant constants, (6.6a) can be easily found out from (6.5) . Using (6.6) in (6.3), we can see the truth of the theorem.

Using the equations (1.15c), (1.22) and (1.23e) in (1.18), we obtain

(6.7) $$v^s\mathcal{P}_j Q^i_{skh} + 2\rho Q^i_{jkh} - Q^s_{jkh}\phi_s v^i + Q^i_{skh}\phi_j v^s + Q^i_{jsh}\phi_k v^s + Q^i_{jks}\phi_h v^s = -(\dot\partial_s Q^i_{jkh})\phi v^s.$$

Contracting (6.7) with respect to the indices i and j and using (1.12b), we get

(6.8) $$v^s\mathcal{P}_i Q^i_{skh} + 2\rho Q_{kh} + Q_{sh}\phi_k v^s + Q_{ks}\phi_h v^s = -(\dot\partial_s Q_{kh})\phi v^s.$$

Similarly, contracting (6.7) with respect to the indices i, k and using the equation (1.12c), we obtain

(6.9) $$v^s\mathcal{P}_j Q_{sh} + 2\rho Q_{jh} + Q_{sh}\phi_j v^s + Q_{js}\phi_h v^s = -(\dot\partial_s Q_{jh})\phi v^s.$$

Thus eliminating $(\dot{\partial}_s Q_{jh})\phi v^s$ in (6.8) and (6.9), we get

$$v^s(\mathcal{P}_i Q^i_{sjh} - \mathcal{P}_j Q_{sh}) = 0. \tag{6.10}$$

Thus, we have

Theorem 6.3: *If the concircular transformation (1.16) defines a projective motion in P-F_n, then the relation (6.10) holds.*

7. Torse-forming Transformation Defining Projective Motion

In this section, we consider the case when a torse-forming transformation defines a projective motion. With the help of the equations (1.5), (1.16), (1.19), (1.21) and (1.23f), we get

$$\delta^i_k \rho_j + (\mathcal{P}_j \phi_k) v^i + \delta^i_j \rho \phi_k + \phi_j \phi_k v^i + v^h Q^i_{hjk} + v^h \Pi^i_{jkh} \phi = \frac{1}{n+1}(\delta^i_j \partial_k \partial_r v^r + \delta^i_k \partial_j \partial_r v^r). \tag{7.1}$$

where $\mathcal{P}_j \rho = \rho_j$.

Contracting (7.1) with respect to the indices i and j and using (1.4c), (1.12c), together with the skew-symmetric property of Q^i_{jkh}, we deduce

$$\rho_k + (\mathcal{P}_i \phi_k + \phi_i \phi_k) v^i + n\rho\phi_k - v^h Q_{hk} = \partial_k \partial_r v^r. \tag{7.2}$$

Similarly, if contracting (7.1) with respect to the indices i and k, then in view of the equations (1.4c) and (1.12a), we obtain from (7.1)

$$n\rho_j + (\mathcal{P}_j \phi_i + \phi_i \phi_j) v^i + \phi_j \rho = \partial_j \partial_r v^r. \tag{7.3}$$

Thus eliminating $(\partial_j \partial_r v^r)$ from (7.2) and (7.3), we have

$$v^r Q_{hj} + (n-1)(\rho_j - \rho\phi_j) + (\mathcal{P}_j \phi_i - \mathcal{P}_i \phi_j) v^i = 0. \tag{7.4}$$

We, therefore have

Theorem 7.1: *If the torse-forming transformation (1.16) defines a projective motion in projective Finsler space then the relation (7.4) necessarily holds .*

Imposing further conditions on the functions ρ and ϕ_1, we have

Theorem 7.2: *The invariance property of ρ and ϕ_l under the projective covariant operator P_m implies that the projective motion in projective Finsler space satisfies the relations*

(7.5)
$$\begin{cases}(a) \quad v^h Q_{hj} = (n-1)\rho\phi_j, \\ (b) \quad v^h Q_{hk} - n\rho\phi_k - \phi_i\phi_k v^i + \partial_k\partial_r v^r = 0.\end{cases}$$

Proof : As ϕ_l and ρ are projective covariant constants, (7.5a) can be easily found out from (7.4). Using (7.5a) in (7.2), we can see the truth of the theorem.

Using equations (1.15c), (1.16), (1.22), (1.23f) and (1.24) in (1.18), we get

(7.6)
$$\begin{aligned} &v^s \mathcal{P}_j Q^i_{skh} + 2\rho Q^i_{jkh} - Q^s_{jkh}\phi_s v^i + Q^i_{skh}\phi_j v^s + Q^i_{jsh}\phi_k v^s \\ &+ Q^i_{jks}\phi_h v^s = -(\dot{\partial}_s Q^i_{jkh})\phi v^s. \end{aligned}$$

Contracting (7.6) with respect to the indices i and j and using (1.12b), we get

(7.7)
$$v^s \mathcal{P}_i Q^i_{skh} + 2\rho Q_{kh} + Q_{sh}\phi_k v^s + Q_{ks}\phi_h v^s = -(\dot{\partial}_s Q_{kh})\phi v^s.$$

Similarly, contracting (7.6) with respect to the indices i and k and using the equation (1.12c), we get

(7.8)
$$v^s \mathcal{P}_j Q_{sh} + 2\rho Q_{jh} + Q_{sh}\phi_j v^s + Q_{js}\phi_h v^s = -(\dot{\partial}_s Q_{jh})\phi v^s.$$

Thus eliminating $(\dot{\partial}_s Q_{jh})\phi v^s$ from (7.7) and (7.8), we get

(7.9)
$$v^s(\mathcal{P}_i Q^i_{sjh} - \mathcal{P}_j Q_{sh}) = 0.$$

Thus, we have

Theorem 7.3: *If the torse-forming transformation (1.16) defines a projective motion in P-F_n, then the relation (7.9) necessarily holds .*

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