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The SIS Model for Japanese Encephalitis

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Abstract: Various effects of disease JE (Japanese Encephalitis) causing death on the host population are studied in an endemic model of SIS type. The basic problem discussed in this paper is to describe the spread of an infection within a population. Thresholds are identified which determine when the population survives and when disease remains endemic. It is further assumed that there is no substantial development of immunity and that removed infectious are in effect cured of disease. The stability criteria for trivial and non-trivial equilibrium are studied. The analysis of Singh[1] and Baily[2] can be obtained by particularizing the generalized parameter.

Key Words: Japanese Encephalitis, epidemiological model, SIS Model.

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1. Introduction

There are many infectious diseases in which infection transmission is caused by direct contact of susceptibles and infectious, while there are some diseases which are also transmitted indirectly by flow of bacteria from infectious into the environment. JE is one of them which transmitted to man by the bite of infected mosquitoes.

Among the animal hosts pigs are major vertibrate hosts for JE viruses. Cuisine mosquitoes notable C. trilaeniorhynchus, C. vishnui gelidus with some anophelines have been incriminated as the vectors of JE.

In simplest epidemiological models often it is assumed that the total population size is constant. Some epidemiological models with varying population size assume constant immigration and emigration proportional to the size of the population so that population approaches an equilibrium size.

When we want to study any given disease in greater depth more realistic models are needed. It may be observed that the treatment of two populations interacting in host - vector manner i.e. with cross infection between groups but non-within groups can be regarded as a special case.

We shall therefore adopt an intermediate position of confining attention in present paper to two populations which are more conveniently thought of as representing the human host and mosquito vector.

2. Formation of Mathematical model

Let us consider an SIS model in which there are two interacting populations, the first representing the human hosts, consists at time t of x susceptibles, y infectious and z recovered or immunized, we shall assume total population size to be $x + y + z = n$ and second population representing the intermediate host or vector are x', y' and z' with $x' + y' + z' = n'$.

In mosquito population removals involve death only, with no isolation or recovery. So the class of removals z' can be ignored and we can put $x' + y' = n'$, where n' is constant over the time interval considered. Since the female mosquitoes appear unaffected by their parasite load. We assume that the death rate operates equally on suscepetibles and infectious. In order to maintain constant mosquito population, we must also have a birth rate γ' to balance the death rate.

Let us consider for human population:

β = infection rate.

n = total population size.

x = total no. of susceptibles.

y = total no. of infected individual.

z = total no. of recovered individual.

λ = birth rate.

μ = rate of infection transferred back to susceptible.

v = rate of removals transferred back to susceptible.

Let us consider all three human groups viz. susceptibles, infectious and removals having same death rate λ, balanced by birth rate λ which produces only susceptibles. It follows that basic differential equation describing the rate of change of human population is

$$\frac{dx}{dt} = -\beta x y' + (\lambda+\mu) y + (\lambda+\nu) z,$$

(2.1) $$\frac{dy}{dt} = \beta x y' - (\gamma+\lambda+\mu) y,$$

$$\frac{dz}{dt} = \gamma y - (\lambda+\nu) z.$$

A similar argument for the mosquito population is

$$\frac{dx'}{dt} = -\beta' x' y + \nu' y',$$

(2.2) $$\frac{dy'}{dt} = \beta' x' y - \gamma' y'.$$

The linearly independent equations in (2.1) and (2.2) are

(2.3) $$\frac{dx}{dt} = -\beta x y' + (\lambda+\mu) y + (\lambda+\nu) z,$$

(2.4) $$\frac{dy}{dt} = \beta x y' - (\gamma+\lambda+\mu) y,$$

(2.5) $$\frac{dy'}{dt} = \beta' x' y - \gamma' y'.$$

Let us assume that μ is very small. Then we have

(2.6) $$\frac{dx}{dt} = -\beta x y' + \lambda y + (\lambda+\nu) z,$$

(2.7) $$\frac{dy}{dt} = \beta x y' - (\gamma+\lambda) y,$$

(2.8) $$\frac{dy'}{dt} = \beta' x' y - \gamma' y'.$$

3. Stability Analysis

We will employ stability analysis to locate the possible equilibrium point and to decide which equilibrium point stable.

4. Equilibrium States

The equilibrium points occur where

$$\frac{dx}{dt}=0, \qquad \frac{dy}{dt}=0, \qquad \frac{dy'}{dt}=0.$$

From (2.6), we have

$$-\beta xy'+\lambda y+(\lambda+\nu)z=0,$$

i.e.,

(4.1) $$\beta xy'=\lambda y+(\lambda+\nu)z\ .$$

From (2.7),

$$\beta xy'-(\gamma+\lambda)y=0\ ,$$

i.e.,

(4.2) $$\beta xy'=(\gamma+\lambda)y.$$

From (2.8), we have

$$\beta' x' y-\gamma' y'=0,$$

i.e.,

(4.3) $$\beta' x' y=\gamma' y'\ .$$

Substituting $z=n-x-y$ in (4.1) and $x'=n'-y'$ in (4.3), we obtain

$$\beta xy'=\lambda\, y+\left(\lambda+\nu\right)(n-x-y),$$

or $$\beta xy'=\lambda\, y+\left(\lambda+\nu\right)(n-x)-(\lambda+\nu)\, y\ ,$$

(4.4) $$\beta xy'=(\lambda+\nu)(n-x)-\nu y.$$

and

$$\beta'(n'-y')y=\gamma' y',$$

$$\beta' n' y-\beta' y' y=\gamma' y',$$

(4.5) $$\beta' n' y=y'(\beta' y+\gamma')\ .$$

Eliminating $\beta xy'$ between (4.2) and (4.4), we get

$$(\gamma+\lambda)\, y=(\lambda+\nu)(n-x)-\nu y,$$

(4.6) $$(\lambda+\nu)x+(\gamma+\lambda+\nu)y=(\lambda+\nu)n.$$

Multiplying corresponding sides of equation (4.2) and (4.5), we find

$$\beta\beta' x n' y y' = (\gamma+\lambda)(\beta' y+\gamma') y y',$$

$$\beta\beta' x n' = (\gamma+\lambda)(\beta' y+\gamma'),$$

or $$\beta\beta' n' x = (\gamma+\lambda)\beta' y+(\gamma+\lambda)\gamma',$$

(4.7) $$\beta\beta' n' x-(\gamma+\lambda)\beta' y = (\gamma+\lambda)\gamma'.$$

Solving (4.6) and (4.7), for x and y by Cramer's rule, we obtain

$$x=\frac{\begin{vmatrix}(\lambda+\nu)n & (\gamma+\lambda+\nu)\\ (\gamma+\lambda)\gamma' & -(\gamma+\lambda)\beta'\end{vmatrix}}{\begin{vmatrix}(\lambda+\nu) & (\gamma+\lambda+\nu)\\ \beta\beta' n' & -(\gamma+\lambda)\beta'\end{vmatrix}}$$

and $$y=\frac{\begin{vmatrix}(\lambda+\nu) & (\lambda+\nu)n\\ \beta\beta' n' & (\gamma+\lambda)\gamma'\end{vmatrix}}{\begin{vmatrix}(\lambda+\nu) & (\gamma+\lambda+\nu)\\ \beta\beta' n' & -(\gamma+\lambda)\beta'\end{vmatrix}}$$

i.e. $$x=\frac{-(\lambda+\nu)(\gamma+\lambda)n\beta'-\gamma'(\gamma+\lambda)(\gamma+\lambda+\nu)}{-(\lambda+\nu)(\gamma+\lambda)\beta'-\beta\beta' n'(\gamma+\lambda+\nu)}$$

and $$y=\frac{(\lambda+\nu)(\gamma+\lambda)\gamma'-\beta\beta' n n'(\lambda+\nu)}{-(\lambda+\nu)(\gamma+\lambda)\beta'-\beta\beta' n'(\gamma+\lambda+\nu)},$$

(4.8) $$x=\frac{(\gamma+\lambda)\{n\beta'(\lambda+\nu)+\gamma'(\gamma+\lambda+\nu)\}}{\beta'\{n'\beta(\gamma+\lambda+\nu)+(\lambda+\nu)(\gamma+\lambda)\}},$$

(4.9) $$y=\frac{(\lambda+\nu)\{nn'\beta\beta'-\gamma'(\gamma+\lambda)\}}{\beta'\{n'\beta(\gamma+\lambda+\nu)+(\lambda+\nu)(\gamma+\lambda)\}}.$$

Putting the value of x and y in equation (4.2), we have

$$y'=\frac{(\gamma+\lambda)y}{\beta x},$$

$$y' = \frac{(\lambda+\nu)\{nn'\beta\beta' - \gamma'(\gamma+\lambda)\}}{\beta\{n\beta'(\gamma+\nu)+\gamma'(\gamma+\lambda+\nu)\}} ,$$

(4.10)
$$y' = \frac{(\lambda+\nu)\{nn'\beta\beta' - \gamma'(\gamma+\lambda)\}}{\beta\{n\beta'(\gamma+\nu)+\gamma'(\gamma+\lambda+\nu)\}} .$$

Thus we see that there are three possible equilibrium points

(4.11)
$$\begin{bmatrix} x = \bar{x} = 0 \\ y = \bar{y} = 0 \\ y' = \bar{y}' = 0 \end{bmatrix}$$

and

$$x = \bar{\bar{x}} = \frac{(\gamma+\lambda)\{n\beta'(\lambda+\nu)+\gamma'(\gamma+\lambda+\nu)\}}{\beta'\{n'\beta(\gamma+\lambda+\nu)+(\lambda+\nu)(\gamma+\lambda)\}},$$

$$y = \bar{\bar{y}} = \frac{(\lambda+\nu)\{nn'\beta\beta' - \gamma'(\gamma+\lambda)\}}{\beta'\{n'\beta(\gamma+\lambda+\nu)+(\lambda+\nu)(\gamma+\lambda)\}},$$

$$y' = \bar{\bar{y}}' = \frac{(\lambda+\nu)\{nn'\beta\beta' - \gamma'(\gamma+\lambda)\}}{\beta\{n\beta'(\lambda+\nu)+\gamma'(\gamma+\lambda+\nu)\}}.$$

Since all the parameters in the expression for x, y and y' are positive, it follows that

(1) If $nn'\beta\beta' \leq \gamma'(\gamma+\lambda)$ then equilibrium point is at $y = 0$ and $y' = 0$.

(2) If $nn'\beta\beta' > \gamma'(\gamma+\lambda)$ then both equilibrium points.

5. Stability

Now, we decide whether the solution will move to (y, y') or to $(\bar{y}, \bar{y}')$ when both are possible. For this, we will linearize the generating equation (2.7), (2.8) around the equilibrium at $(0,0)$

$$\frac{dy}{dt} = -(\gamma+\lambda)\, y ,$$

$$\frac{dy'}{dt} = -\gamma' y'.$$

Define the matrix quantities

$$x = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad M = \begin{bmatrix} -(\gamma+\lambda) & 0 \\ 0 & \gamma' \end{bmatrix}$$

Rewriting the linearized equation in compact form, we have

$$\frac{dy}{dx} = My.$$

An ordinary differential equation with constant coefficients always has exponential solution.

Substituting $y = \begin{Bmatrix} B \\ A \end{Bmatrix} e^{\lambda t} = Ke^{\lambda t}$ in the matrix equation, we have $\left[M - \lambda I\right] K e^{\lambda t} = 0$.

Since $e^{\lambda t} \neq 0$, in general, the determinant of coefficient matrix must be zero.

$$\left[M - \lambda I\right] = \begin{bmatrix} -(\gamma+\lambda) & 0 \\ 0 & -\gamma' \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

$$\left[M - \lambda I\right] = \begin{bmatrix} -(\gamma+2\lambda) & 0 \\ 0 & -(\gamma'+\lambda) \end{bmatrix}.$$

Let $\qquad \left|M - \lambda I\right| = 0$

$$\Rightarrow \qquad \begin{vmatrix} -(\gamma+2\lambda) & 0 \\ 0 & -(\gamma'+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \qquad (\gamma+2\lambda)(\gamma'+\lambda) = 0$$

$$\Rightarrow \qquad 2\lambda^2 + (\gamma+2\gamma')\lambda + \gamma\gamma' = 0$$

$$\Rightarrow \qquad \lambda = \frac{-(\gamma+2\gamma')\pm\sqrt{(\gamma+2\gamma')^2-8\gamma\gamma'}}{4}$$

$$\Rightarrow \qquad \lambda = \frac{-(\gamma+2\gamma')\pm\sqrt{(\gamma-2\gamma')^2}}{4}$$

$$\Rightarrow \qquad \lambda = \frac{-(\gamma+2\gamma')\pm(\gamma-2\gamma')}{4}$$

$$\Rightarrow \qquad \lambda = -\frac{\gamma}{2}, -\gamma' .$$

It is apparent that both roots are always real and

$$i)\ \lambda_1<0,\ \lambda_2<0 \quad if \quad nn'\beta\beta' \le \gamma'(\gamma+\lambda),$$

$$ii)\ \lambda_1>0,\ \lambda_2>0 \quad if \quad nn'\beta\beta' > \gamma'(\gamma+\lambda),$$

where the A's and B's parameterize the initial distribution from (0,0). It is clear from the nature of roots λ_1 and λ_2 that (0,0) is a stable equilibrium if $nn'\beta\beta' \le \gamma'(\gamma+\lambda)$,

or,
$$\frac{(nn'\beta\beta')}{\gamma'(\gamma+\lambda)} \le 1$$

or,
$$R = \frac{(nn'\beta\beta')}{\gamma'(\gamma+\lambda)} \le 1$$

6. Result

If $R \le 1$ the origin is asymptotically stable and is in fact the only equilibrium points if $nn'\beta\beta' < \gamma'(\gamma+\lambda)$, While if $R > 1$ the origin is unstable, but the equilibrium point given by (4.11) is asymptotically stable. It can also be shown that there are no periodic solutions contained entirely relevant region given by $0 \le y \le n$, $0 \le y' \le n'$.

The control of disease depends upon reducing the basic reproduction rate $R = \dfrac{(nn'\beta\beta')}{\gamma'(\gamma+\lambda)}$ to below unity.

Putting $z = 0$ and $\gamma = 0 = \nu$, we get the result obtained by Singh et.al.[1].

References

1. L. S. Singh and S. K. Singh, Simple Epidemic model for encephalitis, *V. J. M. S.*, **25**(2005) 9-16.
2. N. T. J. Baily, *The mathematical theory of infection disease and its applications*, 1975.
3. M. Brown, Differential equations and their applications, *Applied Mathematical Sciences* (15), Springer-Verlag, New York, 1975.
4. L. Eldestin-Keshet, *Mathematical models in biology*, Rando House, New York, 1981.
5. J. N. Kapur, *Mathematical modeling in biology and Medicine*, Affiliated press, 1991.
6. R. M. May and R. M. Anderson, Population biology of infectious disease II, *Nature*, **280** (1979) 455-461.
7. D. Mollisson, Simplifying simple epidemic model, *Nature*, **310** (1984) 224-225.
8. National Institute of Virology, *Pune, Japanese Encephalitis in India ICMR New Delhi*, 1980.
9. WHO, Japanese Encephalitis, Technical introduction and Guidelines for treatment SEA/CD/79, WHO, New Delhi, 1979.