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On Para-Sasakian Manifolds

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Abstract: In this paper, we have studied the properties of Curvature tensor in a para-sasakian manifolds and the condition that if p-sasakian manifold M is concircularly flat, then $V(\xi, X)S = 0$. It has also been obtained that the p-sasakian manifolds is of constant curvature if $R(\xi, X)R = 0$, where V, R and S are the concircular, Riemannian and Ricci curvature tensors respectively, ξ is a characteristic vector field and $X \in TM$.

Key words: p-sasakian manifold, curvature tensors**.**

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1. Introduction

Let M^n be an n-dimensional C^∞ manifolds. If there exists a tensor field F of type (1, 1), a vector field ξ and a 1-form η in M^n satisfying

(1.1) $$\bar{\bar{X}} = X - \eta(X)\xi, \qquad \bar{X} = F(X), \qquad \eta(\xi) = 1.$$

Then M^n is called an almost para-contact manifold. Let g be the Riemannian metric satisfying

(1.2) $$g(X, \xi) = \eta(X),$$

(1.3) $$\eta(FX) = 0, \qquad F\xi = 0, \qquad rank\, F = (n-1),$$

(1.4) $$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y).$$

Then the set (F, ξ, η, g) satisfying (1.1), (1.2), (1.3) and (1.4) is called an almost para-contact Riemannian structure. The manifold with such structure is called an almost para-contact Riemannian manifold[1].

If we define $`F(X, Y) = g(\bar{X}, Y)$, then in addition to the above relations the following are satisfied

(1.5) $$`F(X, Y) = `F(Y, X),$$

(1.6) $$`F(\bar{X}, \bar{Y}) = `F(X, Y).$$

Let us consider an n-dimensional differentiable manifold M with a positive definite metric g which admits 1-forms η satisfying

(1.7) $$(\nabla_{X\eta})(Y) - (\nabla_{Y\eta})(X) = 0,$$

and

(1.8) $$(\nabla_X \nabla_{Y\eta})(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z),$$

where ∇ denote the covariant differentiation with respect to g.

Moreover, if we put

(1.9) $$\eta(X) = g(X, \xi) \qquad \text{and} \qquad \nabla_{X\xi} = \bar{X}.$$

Then it can be easily verified that the manifold in consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called a p-sasakian manifold[2]. For a p-sasakian manifold the following relations hold

(1.10) $$R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

(1.11) $$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

(1.12) $$R(\xi, X)\xi = X - \eta(X)\xi,$$

(1.13) $$S(\xi, X) = -(n-1)\eta(X),$$

(1.14) $$Q\xi = -(n-1)\xi,$$

(1.15) $$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

(1.16) $$\eta(R(X, Y)\xi) = 0,$$

(1.17) $$\eta(R(\xi,X)Y) = \eta(X)\eta(Y) - g(X,Y),$$

where R and S are the curvature tensors and Ricci tensors and Q is the Ricci operator.

An almost para-contact Riemannian manifold M is said to be η-Einstein[3] if the Ricci tensor S is of the form $S = ag + b\eta\otimes\xi,$ where a and b are smooth functions on M. In this case, we have

(1.18) $$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y).$$

In particular, if b=0, then M is an Einstein manifold. For any vector field X, Y, Z if S is the Ricci curvature and Q is the Ricci operator, then

$$S(X,Y) = g(QX,Y).$$

2. Curvature Tensors

By definition the concircular curvature tensor V and the projective curvature tensor P are given by

(2.1) $$V(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

and

(2.2) $$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y],$$

where R and r are Riemannian curvature tensor and scalar curvature of M^n respectively.

The endomorphisms $X\wedge Y$, $X\wedge_S Y$ and the homeomorphism $R(X,\xi)R$ are defined by

(2.3) $$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$

(2.4) $$(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y,$$

(2.5) $$(R(X,\xi)R)(U,Z)W = R(X,\xi)R(U,Z)W - R(R(X,\xi,U)Z)W$$
$$-R(U,R(X,\xi,Z)W) - R(U,Z)R(X,\xi,W).$$

If we take $(R(X,\xi)R)(U,Z)W = 0$, then from (2.5), we have

(2.6) $R(X,\xi)R(U,Z)W - R\big(R(X,\xi,U)Z\big)W - R\big(U,R(X,\xi,Z)W\big)$
$$- R(U,Z)R(X,\xi,W) = 0.$$

Putting $U=\xi$ in (2.6) and using (1.11) and (1.12), we have

$$R(X,Z,W) = g(X,W)Z - g(Z,W)X.$$

By suitable contraction, we have

$$r = -n(n-1).$$

Thus we can state the following theorem

Theorem (2.1): *If a p-Sasakian manifold satisfies the conditions*

$$(R(X,\xi)R)(U,Z)W = 0,$$

then the manifold is of constant curvature.

Theorem (2.2): *If a p-Sasakian manifold M^n is concircularly flat, then the scalar curvature $r=-n(n-1)$.*

Proof: As M^n is concircularly flat, V(X, Y, Z) vanishes identically and from (2.1), we have

$$R(X,Y,Z) = \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

$$R(X,Y,Z,U) = \frac{r}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Putting X = U = ξ and using (1.11) and (1.4), we get

$$\left[1 + \frac{r}{n(n-1)}\right]g(FY,FZ) = 0.$$

Now, $g(FY,FZ)=0$ is not possible in general since g is non-singular and this proves the theorem.

We now state the following results as a corollary of Theorem (2.2)

Corollary (2.1): *A concircularly flat p- Sasakian manifold is necessarily a manifold of constant curvature-1.*

Theorem (2.3): *An n-dimensional p- Sasakian manifold M satisfies*

$V(\xi, X).S = 0$ *if and only if either M has a scalar curvature* $r = -n(n-1)$ *or M is an Einstein manifold with the scalar curvature* $r = -n(n-1)$.

Proof: The condition $V(\xi, X).S = 0$ implies that

(2.7) $$S(V(\xi, X)\,Y, \xi) + S(Y, V(\xi, X)\,\xi) = 0.$$

Now from (2.1) and (1.8), we have

$$\left[1 + \frac{r}{n(n-1)}\right](-g(X,Y)S(\xi,\xi) + \eta(Y)S(X,\xi) - \eta(X)S(Y,\xi) + S(X,Y) = 0.$$

Using (2.13), we get

$$\left[1 + \frac{r}{n(n-1)}\right](S - (1-n)g) = 0.$$

Therefore either the scalar curvature r of M is $r = -n(n-1)$, which is of constant curvature or $S = (1-n)g$ which implies that M is an Einstein manifold with the scalar curvature $r = -n(n-1)$. The converse statement of the theorem is trivial.

Using the fact of theorem (2.2) that a p-Sasakian manifold is concircularly flat then the scalar curvature $r = -n(n-1)$. We have following corollary of Theorem (2.3).

Corollary (2.2): *If a p- Sasakian manifold* M^n *is concircularly flat then* $V(\xi, X).S = 0$.

3. p- Sasakian Manifold satisfying $R(\xi, X)\,S = 0$

Let $(R(X,\xi)S)(Y,Z) = 0$, implies that

$$S(R(X,\xi)Y, Z) + S(Y, R(X,\xi)Z) = 0.$$

Putting $Z = \xi$ and using (1.11), (1.12) and (1.13), we get

(3.1) $$S(X,Y) = -(n-1)\,g(X,Y).$$

Hence the scalar curvature $r = -n(n-1)$, which gives

$$S(X,Y) = \frac{r}{n}g(X,Y).$$

i.e. M is an Einstein Manifold.

Thus we can state the following theorem:

Theorem (3.1): *If a p- Sasakian manifold satisfies the condition*

$$(R(X,\xi)S)(Y,Z)=0,$$

Then the manifold is an Einstein manifold.

Corollary (3.1): *If a p- Sasakian manifold is an Einstein manifold the scalar curvature r has a constant value equal to* $-n(n-1)$.

Definition (3.1): A para- Sasakian manifold M satisfying

$$S(X,Y)=b\,g(X,Y)-c\,g(FX,Y), \tag{3.2}$$

where b is a constant and c is a function on M, is called a p-Einstein manifold.

Theorem (3.2): *A p- Sasakian manifold M is a para- Einstein manifold if and only if*

$$(R(X,\xi)S)(Y,Z)=-c[\eta(Z)\,g(FY,X)+\eta(Y)g(FX,Z)].$$

Proof: we have,

$$(R(X,\xi)S)(Y,Z)=-S(R(X,\xi)Y,Z)-S(Y,R(X,\xi)Z). \tag{3.3}$$

Let M be a para- Einstein. Then using (3.2), we get

$$\begin{aligned} S(R(X,\xi)Y,Z) &= b\eta(Z)\,g(X,Y)+c\eta(Y)g(FX,Z) \\ &\quad -b\eta(Y)\,g(X,Z) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} S(Y,R(X,\xi)Z) &= b\eta(Y)\,g(X,Z)+c\eta(Z)g(FY,X) \\ &\quad -b\eta(Z)\,g(Y,X). \end{aligned} \tag{3.5}$$

Using (3.4), (3.5) in (3.3), we get

$$(R(X,\xi)S)(Y,Z)=-c[\eta(Z)\,g(FY,X)+\eta(Y)g(FX,Z)]. \tag{3.6}$$

Conversely, let (3.6) hold, then putting Z = ξ in (3.6), we get

$$-S(R(X,\xi)Y,\xi)-S(Y,R(X,\xi)\xi)=-cg(FY,X),$$

$$(n-1)g(X,Y)+S(Y,X)=-cg(FY,X),$$

$$S(Y,X)=-(n-1)g(X,Y)-c'F(Y,X),$$

which shows that M is a para- Einstein manifold.

From Theorem (3.1) a p-Sasakian manifold satisfying the conditions $R(X,\xi)S=0,$ is an Einstein manifold. This condition and Theorem (3.2) give c = 0, and thus a para- Einstein manifold reduces to Einstein manifold.

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