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Riemannian Manifold with a Special Type of Semi-symmetric Non-metric Connection

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Abstract: In the present paper, we obtain certain results for a Riemannian manifold equipped with a special type of semi-symmetric non-metric connection. We also obtain the expressions for the curvature tensor, Ricci tensor and scalar curvature, and certain results related to them.

Keywords: Semi-symmetric non-metric connection, Riemannian manifold.

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1. Introduction

The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O. C. Andonie[1], M. C. Chaki and A. Konar[2], U. C. De[3] etc., while a special type of semi-symmetric metric connection on a weakly symmetric Riemannian manifold has been studied by U. C. De and Joydeep Sengupta[4]. P. N. Pandey and S. K. Dubey[5] discussed an almost Grayan manifold admitting a semi-symmetric metric connection while a Kähler manifold equipped with a semi-symmetric metric connection and a semi-symmetric non-metric connection, and an almost Hermitian manifold with a semi-symmetric recurrent connection have been studied by P. N. Pandey and B. B. Chaturvedi[6,7,8]. Nirmala S. Agashe and Mangala R. Chafle[9] have studied semi-symmetric non-metric connection on a Riemannian manifold in 1992.

Let (M^n, g) $(n>2)$ be an n- dimensional Riemannian manifold and D be the Riemannian connection. We define another linear connection ∇ for two arbitrary vector fields X and Y such that

(1.1) $$\nabla_X Y = D_X Y + a\omega(X)Y + b\omega(Y)X,$$

where ω is a 1- form associated with a vector field ρ by $\omega(X)=g(X,\rho)$, and a, b are non-zero real or complex numbers such that $a\neq b$.

Putting $g(Y,Z)$ in place of Y in (1.1), we have

$$(\nabla_X g)(Y,Z)=-a\omega(X)g(Y,Z)-b\omega(Y)g(X,Z)-b\omega(Z)g(X,Y), \tag{1.2}$$

which shows that the connection ∇ is non-metric.

The connection ∇ is said to be a special type of semi-symmetric non-metric connection if the torsion tensor T and the curvature tensor $\tilde{R}$ of the connection ∇ satisfy the following conditions:

$$(\nabla_X T)(Y,Z)=\omega(X)T(Y,Z), \tag{1.3}$$

and

$$\tilde{R}(X,Y)Z=0. \tag{1.4}$$

From (1.1), the torsion tensor T of the connection is given by

$$T(X,Y)=(a-b)[\omega(X)Y-\omega(Y)X]. \tag{1.5}$$

From (1.5), we have

$$(C_1^1 T)(Y)=-(a-b)(n-1)\omega(Y), \tag{1.6}$$

where C_1^1 denotes the operator of contraction.

Operating (1.6) by ∇_X, we get

$$(\nabla_X C_1^1 T)(Y)=-(a-b)(n-1)(\nabla_X\omega(Y)). \tag{1.7}$$

Contracting (1.3), we get

$$(\nabla_X C_1^1 T)(Y)=\omega(X)(C_1^1 T)(Y). \tag{1.8}$$

In view of (1.6), (1.8) becomes

$$(\nabla_X C_1^1 T)(Y)=-(a-b)(n-1)\omega(X)\omega(Y). \tag{1.9}$$

From (1.7) and (1.9), we have

(1.10) $$(\nabla_X \omega)(Y) = \omega(X)\omega(Y).$$

2. Certain Results

Putting $\omega(Y)$ in place of Y in (1.1) and using $\omega(X) = g(X, \rho)$, we get

(2.1) $$(\nabla_X \omega)(Y) = (D_X \omega)(Y) - b\omega(X)\omega(Y).$$

Using (1.10) in (2.1), we get

(2.2) $$(D_X \omega)(Y) = (b+1)\omega(X)\omega(Y).$$

Now, we propose:

Theorem 2.1. *In a Riemannian manifold equipped with the special type of semi-symmetric non-metric connection* ∇, *we have*

(i) $(D_X \omega)(Y) = 0$, *for all values of a if and only if* $b = -1$,

(ii) ω *is closed with respect to the special type of semi-symmetric non-metric connection* ∇, *i.e.*

$$\tilde{d}\omega(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) = 0,$$

(iii) ω *is also closed with respect to the Riemannian connection D, i.e.*

$$d\omega(X,Y) = (D_X \omega)(Y) - (D_Y \omega)(X) = 0.$$

Proof. In view of (2.2), we find that $(D_X \omega)(Y) = 0$ if and only if $b = -1$.

Now from (1.9), we have

(2.3) $$(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) = 0.$$

which implies

(2.4) $$\tilde{d}\omega(X,Y) = 0.$$

Hence ω is closed with respect to the special type of semi-symmetric non-metric connection.

Now interchanging *X* and *Y* in (2.1), we get

(2.5) $$(\nabla_Y \omega)(X) = (D_Y \omega)(X) - b\omega(Y)\omega(X).$$

Subtracting equation (2.5) from (2.1), we have

(2.6) $$\tilde{d}\omega(X,Y)=d\omega(X,Y).$$

Using (2.4) in (2.6), we have

(2.7) $$d\omega(X,Y)=0.$$

Hence ω is also closed with respect to Riemannian connection.

3. Curvature tensor

From (1.1), we have

(3.1) $$\nabla_Y Z = D_Y Z + a\omega(Y)Z + b\omega(Z)Y.$$

Replacing Y for $\nabla_Y Z$ in equation (1.1), we have

(3.2) $$\nabla_X \nabla_Y Z = D_X \nabla_Y Z + a\omega(X)\nabla_Y Z + b\omega(\nabla_Y Z)X.$$

Using (1.1) in (3.2), we have

(3.3) $$\begin{aligned}\nabla_X \nabla_Y Z = {} & D_X D_Y Z + a(D_X\omega)(Y)Z + a\,\omega(D_X Y)Z + a\,\omega(Y)D_X Z \\ & + b(D_X\omega)(Z)Y + b\omega(D_X Z)Y + b\,\omega(Z)D_X Y \\ & + a\omega(X)D_Y Z + a^2\omega(X)\omega(Y)Z + ab\omega(X)\omega(Z)Y \\ & + b\omega(D_Y Z)X + ab\omega(Y)\omega(Z)X + b^2\omega(Y)\omega(Z)X.\end{aligned}$$

Interchanging X and Y in the above equation, we get

(3.4) $$\begin{aligned}\nabla_Y \nabla_X Z = {} & D_Y D_X Z + a(D_Y\omega)(X)Z + a\,\omega(D_Y X)Z + a\,\omega(X)D_Y Z \\ & + b(D_Y\omega)(Z)X + b\omega(D_Y Z)X + b\,\omega(Z)D_Y X \\ & + a\omega(Y)D_X Z + a^2\omega(Y)\omega(X)Z + ab\omega(Y)\omega(Z)X \\ & + b\omega(D_X Z)Y + ab\omega(X)\omega(Z)Y + b^2\omega(X)\omega(Z)Y.\end{aligned}$$

From equation (1.1), we may write

(3.5) $$\nabla_{[X,Y]}Z = D_{[X,Y]}Z + a\,\omega([X,Y])Z + b\,\omega(Z)[X,Y].$$

Subtracting (3.4) and (3.5) from (3.3), we have

(3.6) $$\tilde{R}(X,Y)Z = R(X,Y)Z + b[\omega(X)Y - \omega(Y)X]\omega(Z).$$

Now, we propose

Theorem 3.1. *The Riemannian manifold equipped with the special type of semi-symmetric non-metric connection* ∇ *has the following relation between curvature tensor R and torsion tensor T*

$$R(X,Y)Z = -\frac{b}{(a-b)}T(X,Y)\omega(Z). \tag{3.7}$$

Proof. From (1.4) and (3.6), we have

$$R(X,Y)Z = b[\omega(Y)X - \omega(X)Y]\omega(Z). \tag{3.8}$$

Using (1.5) in (3.8), we have (3.7).

Now, we propose

Theorem 3.2. *In a Riemannian manifold equipped with the special type of semi-symmetric non-metric connection* ∇, *the following relations hold*

$$S(Y,Z) = b(n-1)(\nabla_Y \omega)(Z), \tag{3.9}$$

$$r = b(n-1)\,\omega(\rho), \tag{3.10}$$

$$S(Y,\rho) = r\,\omega(Y), \tag{3.11}$$

where S and r are Ricci and scalar curvature respectively.

Proof. Contracting equation (3.8), we get

$$S(Y,Z) = b(n-1)\omega(Y)\omega(Z). \tag{3.12}$$

Using (1.10) in (3.12), we have (3.9). Contracting (3.12), we get (3.10). Now putting $Z=\rho$ in (3.12), we have

$$S(Y,\rho) = b(n-1)\omega(Y)\omega(\rho). \tag{3.13}$$

Using (3.10) in (3.13), we have (3.11). Now from (3.12), we can write

$$\begin{aligned}(\nabla_X S)(Y,Z) + S(\nabla_X Y,Z) + S(Y,\nabla_X Z) = b(n-1)[(\nabla_X \omega)(Y)\omega(Z) \\ + \omega(\nabla_X Y)\omega(Z) + (\nabla_X \omega)(Z)\omega(Y) + \omega(\nabla_X Z)\omega(Y)].\end{aligned} \tag{3.14}$$

Using (1.10) and (3.12) in (3.14), we have

$$(\nabla_X S)(Y,Z) = 2b(n-1)\omega(X)\omega(Y)\omega(Z). \tag{3.15}$$

Again from (3.12), we may write

$$(3.16)\quad (D_X S)(Y,Z)+S(D_X Y,Z)+S(Y,D_X Z)=b(n-1)[(D_X\omega)(Y)\,\omega(Z) + \omega(D_X Y)\,\omega(Z)+(D_X\omega)(Z)\,\omega(Y)+\omega(D_X Z)\,\omega(Y)].$$

In view of (2.2) and (3.12), (3.16) becomes

$$(3.17)\qquad (D_X S)(Y,Z) = 2b(b+1)(n-1)\omega(X)\omega(Y)\omega(Z).$$

Now, we propose:

Theorem 3.3. *In a Riemannian manifold equipped with the special type of semi-symmetric non-metric connection* ∇, *the Ricci tensor S satisfies the following*

$$(3.18)\qquad (\nabla_X S)(Y,Z)=(\nabla_Y S)(Z,X)=(\nabla_Z S)(X,Y),$$

$$(3.19)\qquad (D_X S)(Y,Z)=(D_Y S)(Z,X)=(D_Z S)(X,Y),$$

$$(3.20)\qquad (D_X S)(Y,Z)=0 \text{ if and only if } b=-1 \text{ or } (\nabla_X\omega)(Y)=0,$$

$$(3.21)\qquad (div W)(X,Y,Z)=0,$$

where W denotes the projective curvature tensor defined by

$$(3.22)\qquad W(X,Y,Z)=R(X,Y,Z)-\frac{1}{(n-1)}[S(Y,Z)X-S(X,Z)Y]$$

and '*div*' *denotes the divergence.*

Proof. Interchanging *X, Y, Z,* in cyclic order in (3.15), we have (3.18). Interchanging *X, Y, Z,* in cyclic order in (3.17), we get (3.19). In view of (1.10), (3.17) becomes

$$(3.23)\qquad (D_X S)(Y,Z)=2b(n-1)(b+1)(\nabla_X\omega)(Y).$$

From (3.23), we can easily get (3.20).
It is known that in an n-dimensional Riemannian manifold (n>2)

$$(3.24)\qquad (div W)(X,Y,Z)=\left(\frac{n-2}{n-1}\right)[(D_X S)(Y,Z)-(D_Y S)(X,Z)].$$

Using (2.2) in (3.24), we have (3.21).

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