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# Jacobi Stability Analysis of Lü System

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**Abstract:** In this paper, we analyse the non-linear dynamics of the Lü system from the view point of Kosambi-Cartan-Chern (KCC) theory. We reformulate the Lü system as a set of two second order non-linear differential equations and obtain five KCC-invariants which express the intrinsic geometric properties. The Jacobi stability of the Lü system at equilibrium points are investigated in terms of the eigenvalues of the deviation tensor. The equilibrium point $E_0$ is always Jacobi unstable, while the Jacobi stability of other equilibrium points $E_1$, $E_2$ depends on the parameters values.

**Keywords:** Finsler Space, geodesics, KCC-theory, KCC-invariants, Lü System.

**2010 AMS Classification Number:** 53B40, 53C22, 53C60.

## 1. Introduction

The Lyapunov stability is the mathematical concept for describing the stability of the solution of the dynamical system. The basic quantity of the theory is Lyapunov exponents, which measures exponential deviation from the given trajectories, is very difficult to determine analytically. An alternative approach which study the properties of dynamical system initiated by Kosambi[1], Cartan[2] and Chern[3] in 1940's is known as geometro-dynamical approach (KCC theory). The KCC theory is based on the basic idea that the second order dynamical system and geodesics equation in associated Finsler space are topologically equivalent. The KCC theory is a differential geometric theory for the deviations of the entire trajectories of the variational equations to nearby ones[4]. The KCC geometrical description of the dynamical systems is associated with non-linear connection and Berwald type connection to the differential system, and by using these connections the five geometrical invariants are obtained. The second KCC-invariant (deviation tensor) will be determine the Jacobi stability of the system. Nowadays, the Jacobi stability analysis has been used for the study of chaotic dynamical system. The KCC theory has applications to the geometric aspects of various systems, such as physical[5-11], biological[12-17] cosmology and gravitation[18-21] and general phenomena[22-27]. In the present paper, we discuss stability feature of the Lü system[28] by formulating the equations as a set of two second order differential equation and using KCC theory.

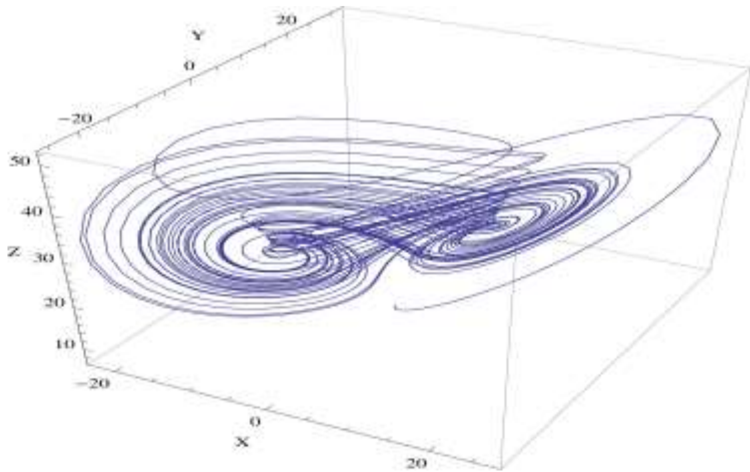



**Figure 1.** Chaotic attractor of system (1.1) for $a$ = 36, $b$ = 3, $c$ = 25

The Lü system[28] is given by

$$(1.1) \qquad \begin{cases} \dot{x} = a(y-x), \\ \dot{y} = -xz + cy, \\ \dot{z} = xy - bz, \end{cases}$$

where $a, b, c \in R^+$. This system yields a typical chaotic attractor for the parameters value $a = 36$, $b = 3$, $c = 25$ as shown in figure 1. This system is bridge between Lorenz and Chen systems[29]. The dynamical properties of the system (1.1), such as bifurcation, periodic windows and route of chaos have been studies in the literature[29]. The existense of homoclinic orbits of the equlibrium ponits is analyze in[30]. Adaptive synchronization with uncertain parameters and chaos synchronization of two identical Lü system were analysed theoretically and numerically by the auther's in[31-35]. In the literature

[36] shows that Lü system is a particular case of Lorenz system. In[37], a 4D hyperchaotic Lü system is constucted from 3D Lü system and they studied the no-hopf bifurcation, chaos and hyperchaotic behavior. The paper is oganised as follows. The first section is introductry. The section two presents the basics of KCC theory. In section three we obtain the nonlinear connections, Berwald connections, the KCC-invariants and conditions for the Jacobi stability at each equilibrium points. The fourth section discuss the behavior of deviation vector.

## 2. KCC-Theory and Jacobi Stability

Let us recall the basic concept and results of the KCC theory[4,7,8,38]. Consider a real smooth $n-$dimensional manifold $M$ and its tangents bundle $TM$. Let $\left(x^1, x^2,....x^n\right)=(x)$, $\left(\frac{dx^1}{dt}, \frac{dx^2}{dt},......\frac{dx^n}{dt}\right)=\left(\frac{dx}{dt}\right)=y$ and $t$ be $2n+1$ co-ordinates in an open connected subset $\Omega$ of the $(2n+1)-$dimensional euclidean space $R^n\times R^n\times R^1$. We regarded time $t$ as an absolute invariant. Let us consider the system of second order differential equation (SODE) of the form

$$\frac{d^2x^i}{dt^2}+2G^i(x, y, t)=0,\quad i=1,2,.....,n, \tag{2.1}$$

where $G^i$ is $C^\infty$ in a neighbourhood of initial conditions $\left((x)_0, (y)_0, t_0\right)\in\Omega$. The SODE (2.1) is equivalent to the equation of motion in Finsler space, given by the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)-\frac{\partial L}{\partial x^i}=F_i,\quad i=1,2,.....,n, \tag{2.2}$$

where $L$ and $F_i$ are the Lagrangian and external force respectively.

The intrinsic geometric properties of (2.1), under a non-singular coordinate transformations

$$\begin{cases}\bar{x}^i=f^i\left(x^1, x^2,......x^n\right), i=1,2,....,n\\ \bar{t}=t\end{cases} \tag{2.3}$$

are given by the five KCC invariants, named after Kosambi[1], Cartan[2] and Chern[3]. Under the above non-singular coordinate transformations (2.3), let us define KCC-covariant derivative of a contravariant vector field $\xi^i(x)$ on $\Omega$ by[4,39]

(2.4) $$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j,$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}$ are the coefficients of the non-linear connection. Throughout the paper the Einstein summation convention is used.
By putting $\xi^i = y^i$ and using equation (2.1), the covariant differential becomes

(2.5) $$\frac{Dy^i}{dt} = N^i_j y^j - 2G^i = -\varepsilon^i,$$

where the contravariant vector field $\varepsilon^i$ is called the first KCC- invariant, represents an external force.

Let us consider the variation of the trajectories $x^i(t)$ of system (2.1) according to

(2.6) $$\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta,$$

where $\eta$ denotes a parameter, with $|\eta|$ small and $\xi^i(t)$ are the components of contravariants vector defined along the curve $x^i = x^i(t)$. Substituting equation (2.6) into equation (2.1) and taking the limit as $\eta \to 0$, yields the variational equation, we get[4,14,22]

(2.7) $$\frac{d^2\xi^i}{dt^2} + 2N^i_j \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0.$$

Using the KCC-covariant differential (2.4), the above equation (2.7) becomes

(2.8) $$\frac{D^2\xi^i}{dt^2} = P^i_j \xi^j,$$

where

(2.9) $$P_j^i = -2\frac{\partial G^i}{\partial x^j} - 2G^l G_{jl}^i + y^l \frac{\partial N_j^i}{\partial x^l} + N_l^i N_j^l + \frac{\partial N_j^i}{\partial t}.$$

Here $G_{jl}^i = \frac{\partial N_j^i}{\partial y^l}$ is the Berwald connection[4,38]. The tensor $P_j^i$ is second KCC-invariant or deviation tensor of equation (2.1). The equation (2.8) is the Jacobi field equation when system (2.1) describes the geodesic equation in either Finsler or Riemannian geometry. The third, fourth and fifth invariants of the system (2.1) are defined as[4,38]

(2.10) $$P_{jk}^i = \frac{1}{3}\left(\frac{\partial P_j^i}{\partial y^k} - \frac{\partial P_k^i}{\partial y^j}\right), \quad P_{jkl}^i = \frac{\partial P_{jk}^i}{\partial y^l}, \quad D_{jkl}^i = \frac{\partial G_{jk}^i}{\partial y^l}.$$

The third, fourth and fifth invariant are called the torsion tensor, Riemann-curvature curvature tensor and Douglas curvature tensor respectively. Alternatively, we give another definition for the third and fourth invariants as[38]

(2.11) $$B_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j},$$

(2.12) $$B_{jkl}^i = \frac{\partial B_{kl}^i}{\partial y^i},$$

(2.13) $$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

**Jacobi stability of dynamical system:** Let us consider the trajectories $x^i = x^i(t)$ of (2.1) as curves in the Euclidean space $\left(R^n, \langle .,. \rangle\right)$, where $\langle .,. \rangle$ is the canonocal inner product of the $R^n$. We assume that the deviation vector $\xi$ satisfies the initial conditions

$$\xi(0) = 0, \ \dot{\xi}(0) = W \neq 0,$$

where $0 \in R^n$ is the null vector. Let us consider now an adapted inner product $\langle\langle .,. \rangle\rangle$ to the deviation tensor $\xi$ by

$$\langle\langle X, Y\rangle\rangle := \frac{1}{\langle W, W\rangle}.\langle X, Y\rangle,$$

for any vectors $X, Y \in R^n$. Obviously, $\|W\|^2 = \langle\langle W, W\rangle\rangle = 1$. Then, for $t \approx 0^+$, the trajectories of (2.1) are[14,22,40]

(i) bunching together if and only if the real part of the eigenvalues of $P^i_j(0)$ are strictly negative.

(ii) dispersing if and only if the real part of eigenvalues of $P^i_j(0)$ are strictly positive.

Now, we define the notion of Jacobi stability for SODE[14,22]. This kind of stability refers to the focusing tendency (in a small vicinity of $t_0$) of trajectories of (2.1) with respect to the variation (2.6) that satisfy the conditions

**Definition 2.1:** *The trajectory of (2.1) are called Jacobi stable at* $\left(x(t_0), \dot{x}(t_0)\right)$ *if and only if real parts of the eigenvalues of the deviation tensor* $P^i_j|_{t_0}$ *are strictly negative, and Jacobi unstable, otherwise.*

A basic result of the KCC theory is the following[41]:

*Two systems of the form (2.1) on* $\Omega$ *can be locally transfered, relative to equation (2.3), one into another, if and only if the five KCC-invariants* $\varepsilon^i$, $P^i_j$, $P^i_{jk}$, $P^i_{jkl}$, $D^i_{jkl}$ *are equivalent tensor. In particular, there exist coordinates* ($\bar{x}$) *for which the* $G^i(\bar{x}, \bar{y}, t)$ *vanish if and only if all KCC-invariants are zero.*

The matrix form of the deviation tensor in two dimensional space can be written as

(2.14) $$P^i_j = \begin{bmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{bmatrix},$$

with eigenvalues as

(2.15) $$\lambda_{\pm} = \frac{1}{2}\left[P^1_1 + P^2_2 \pm \sqrt{\left(P^1_1 - P^2_2\right)^2 + 4P^1_2 P^2_1}\right].$$

The eigenvalues are the solution of the quadratic equation

(2.16) $$\lambda^2 - \left(P_1^1 + P_2^2\right)\lambda + \left(P_1^1 P_2^2 - P_2^1 P_1^2\right) = 0.$$

Routh-Hurwitz criteria used to obtain the sign of eigenvalues of deviation tensor[7,42], according to which all the roots of polynomial $P(\lambda)$ are negative or have negative real parts if

(2.17) $$P_1^1 + P_2^2 < 0, \qquad P_1^1 P_2^2 - P_2^1 P_1^2 > 0.$$

## 3. Jacobi Stability of the Lü System

In this section, the Jacobi stability of the Lü system (1.1) is studied by using the KCC-theory. Firstly we transform the Lü system (1.1) into a system of second-order differential equation. From first equation of system (1.1), write $y$ as

$$y = \frac{\dot{x}}{a} + x.$$

By substituting $y$ into the second equation of the system (1.1), we get

(3.1) $$\ddot{x} - \left(\dot{x} + a(1-c)\right)x + axz = 0.$$

Differentiating third equation of the (1.1) with respect to $t$, we have

$$\ddot{z} = y\dot{x} + x\dot{y} - b\dot{z},$$

by substituting the value of $y$ from first equation of system (1.1), we get

$$\ddot{z} = \dot{x}\left(\frac{\dot{x}}{a} + x\right) + \left(\frac{\ddot{x}}{a} + x\right)x - b\dot{z}.$$

By substituting the value of $\ddot{x}$ in to above equation from (3.1), we get

(3.2) $$\ddot{z} + b\dot{z} - \frac{(\dot{x})^2}{a} - \left(\frac{c+a}{a}\right)x\dot{x} + x^2 z - cx^2 = 0.$$

Now, Let us change the notation as $x = x^1$, $\dot{x} = y^1$, $z = x^2$, $\dot{z} = y^2$, then the Lü system (1.1) is equivalent to the following system of second order differential equation

(3.3)
$$\frac{d^2x^1}{dt^2}-cy^1+a(1-c)x^1+ax^1x^2=0,$$
$$\frac{d^2x^2}{dt^2}+by^2-\frac{(y^1)^2}{a}+\left(\frac{c+a}{a}\right)x^1y^1+(x^1)^2x^2-c(x^1)^2=0.$$

3.1 **The non-linear connections, Berwald connections and the KCC invariants:** The system of second order differential equation formulation (3.3) can be written as

(3.4)
$$\frac{d^2x^i}{dt^2}+2G^i\left(x^j,y^j\right)=0,\quad i,\ j=1,2,$$

where

(3.5)
$$G^1=\frac{1}{2}\left[-cy^1+a(1-c)x^1+ax^1x^2\right],$$

and

(3.6)
$$G^2=\frac{1}{2}\left[by^2-\frac{(y^1)^2}{a}+\left(\frac{c+a}{a}\right)x^1y^1+(x^1)^2x^2-c(x^1)^2\right].$$

Therefore the components of non-linear connection, Berwald connectionare given as

$$N_1^1=-\frac{c}{2},\ N_2^1=0,\ N_1^2=-\frac{y^1}{a}+\left(\frac{c+a}{2a}\right)x^1,\ N_2^2=\frac{b}{2}.$$

$$G_{11}^1=G_{12}^1=G_{21}^1=G_{22}^1=G_{12}^2=G_{21}^2=G_{22}^2=0,\quad G_{11}^2=-\frac{1}{a}.$$

The components of the first KCC invariant are given as

$$\varepsilon^1=-\frac{c}{2}y^1+a(1-c)x^1+ax^1x^2,$$

$$\varepsilon^2=\frac{b}{2}y^2+\left(\frac{c+a}{2a}\right)x^1y^2+(x^1)^2x^2-c(x^1)^2.$$

The components of the deviation tensor of the Lü system are obtained by equation (2.9), as follows

$$
(3.7)\qquad \begin{cases}
P_1^1 = -a(1-c) - ax^2 + \dfrac{c^2}{4}, \\
P_2^1 = -ax^1, \\
P_1^2 = -x^1x^2 - \dfrac{c}{a}y^1 - \dfrac{b}{2a}x^1y^1 \\
\qquad + \left(\dfrac{bc}{4a} + \dfrac{b}{4} - \dfrac{3c}{4} - \dfrac{c^2}{4a} + 1\right), \\
P_2^2 = -\left(x^1\right)^2 + \dfrac{b^2}{4}.
\end{cases}
$$

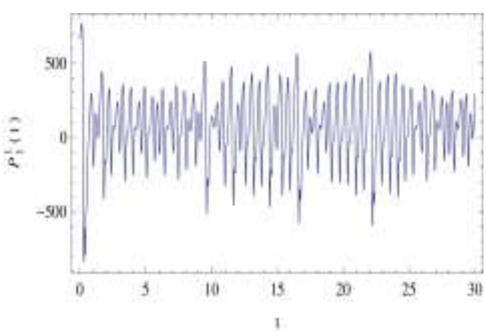
The trace and determinants of the deviation tensor, given by

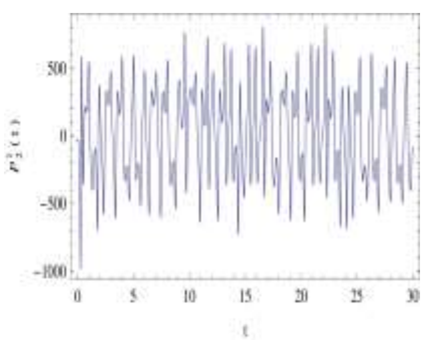
$$
trace(P) = -a(1-c) - ax^2 + \frac{c^2}{4} - \left(x^1\right)^2 + \frac{b^2}{4},
$$

and

$$
\det(P) = \left(\frac{c^2}{4} - a(1-c) - ax^2\right)\left(\frac{b^2}{4} - \left(x^1\right)^2\right) + ax^1\left(-x^1x^2 - \frac{c}{a}y^1 - \frac{b}{2a}x^1y^1 + \frac{bc}{4a} + \frac{b}{4} - \frac{3c}{4} - \frac{c^2}{4a} + 1\right)
$$

respectively.



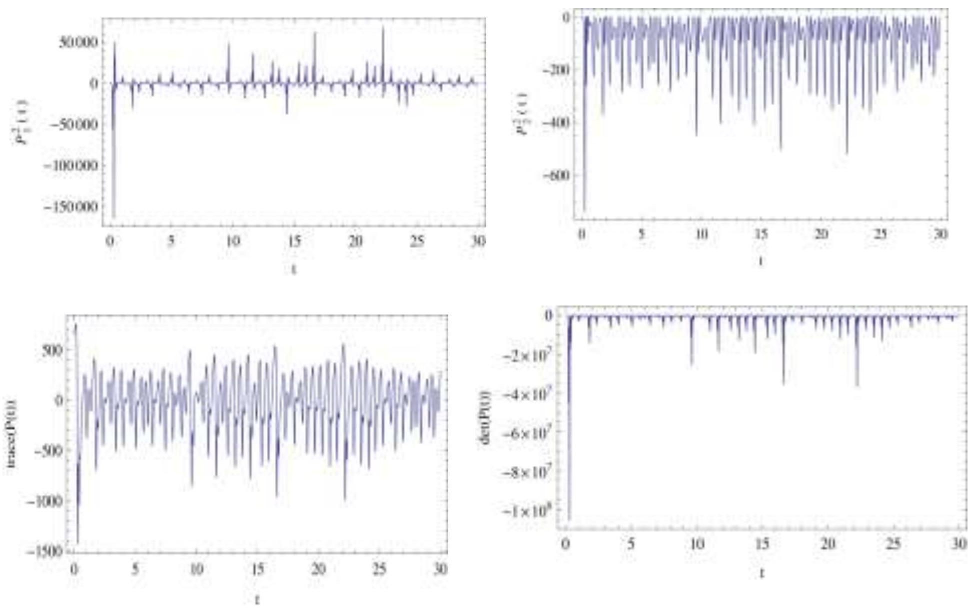


**Figure 2.** Time variation of deviation tensor component $P_1^1(t)$, $P_2^1(t)$, $P_1^2(t)$, $P_2^2(t)$, $trace(P(t))$ and $\det(P(t))$ for parameters value $a = 36$, $b = 3$, $c = 25$. The initial conditions for the numerical integration system are $x(0) = 1$, $y(0) = 0.5$, $z(0) = 10$

The time variation of the components of deviation tensor, trace and determinant of deviation tensor are represented in Figure 2.

The third invariant can be interpreted geometrically as a torsion tensor. The components of third invariant are obtained by equation (2.10) as $P_{jk}^i = 0$. The fourth invariant $P_{jkl}^i = \dfrac{\partial P_{jk}^i}{\partial y^l}$ and the fifth invariant $D_{jkl}^i = \dfrac{\partial G_{jk}^i}{\partial y^l}$ are identically zero.

**3.2 The Jacobi stability at the equilibrium point:** The system of equation (1.1) has equlibria

$$E_0(0, 0, 0),\ \ E_1\left(\sqrt{bc}, \sqrt{bc}, c\right),\ \ E_2\left(-\sqrt{bc}, -\sqrt{bc}, c\right).$$

In respect of system of equation (3.3) the equilibrium points are

$$E_0(0, 0),\ \ E_1\left(\sqrt{bc}, c\right)\ \text{ and }\ E_2\left(-\sqrt{bc}, c\right)$$

Now we obtain the KCC invariants at the equilibrium points. At the equilibrium points the first KCC invariant are given by

$$\varepsilon^1(E_0)=\varepsilon^2(E_0)=\varepsilon^2(E_1)=\varepsilon^2(E_2)=0,\ \varepsilon^1(E_1)=a,\ \ \varepsilon^1(E_2)=-a.$$

**Theorem 3.1:** *The equilibrium point* $E_0(0,0)$ *of system is Jacobi unstable.*

***Proof:*** The Jacobi matrix at the equilibrium point $E_0$ is

$$P^i_j=\begin{bmatrix} a(c-1)+\dfrac{c^2}{4} & 0 \\ 0 & \dfrac{b^2}{4}\end{bmatrix}, \tag{3.8}$$

It characteristics equation is

$$\lambda^2-\left(a(c-1)+\frac{c^2}{4}+\frac{b^2}{4}\right)\lambda+\left(a(c-1)+\frac{c^2}{4}\right)\frac{b^2}{4}=0.$$

This gives the eigenvalues as $\frac{b^2}{4}$ and $a(c-1)+\frac{c^2}{4}$. Since this gives one eigenvalue is nonnegative, thus the equilibrium point $E_0$ is Jacobi unstable.

The components of the second KCC invariant (deviation tensor) at the equilibrium points $E_1$ and $E_2$ are given as

$$P^1_1(E_1)=\frac{c^2-4a}{4},\qquad P^1_2(E_1)=-a\sqrt{bc},$$

$$P^2_1(E_1)=\frac{\sqrt{bc}}{4a}\{a(4+b-c)+(b-c)c\},\qquad P^2_2(E_1)=\frac{b(b-4c)}{4},$$

$$P^1_1(E_2)=\frac{c^2-4a}{4},\qquad P^1_2(E_2)=a\sqrt{bc},$$

$$P^2_1(E_2)=\frac{\sqrt{bc}}{4a}\{a(-4-b+c)+(-b+c)c\},\ P^2_2(E_2)=\frac{b(b-4c)}{4},$$

The eigenvalue of the deviation tensor at the equilibrium point are obtained by (2.15) and we get

$$\lambda_1(E_1)=\lambda_1(E_2)=\frac{1}{8}\left[-\sqrt{\begin{array}{l}16a^2+8ab^2+b^4-96abc-16ab^2-8b^3c-8ac^2\\+16abc^2-2b^2c^2+24bc^3-c^4-4a+b^2-4bc+c^2\end{array}}\right]$$

$$\lambda_2(E_2)=\lambda_2(E_1)=\frac{1}{8}\left[\sqrt{\begin{array}{l}16a^2+8ab^2+b^4-96abc-16ab^2-8b^3c-8ac^2\\+16abc^2-2b^2c^2+24bc^3-c^4-4a+b^2-4bc+c^2\end{array}}\right].$$

The eigenvalue of the second KCC invariant (deviation tensor) are the solution of the equation

$$\lambda^2-\left\{\frac{b^2+c^2-4a-4bc}{4}\right\}\lambda+\frac{(c^2-4a)}{16a}\left\{a(b^2-4bc)\right.$$
$$\left.-\sqrt{bc}\left(4a+ab-ac+bc-c^2\right)\right\}=0.$$

By using Routh-Hurwitz criteria we obtain the following result:

**Theorem 3.2:** *If the constant parameters* $a>0, b>0$, $c>0$ *and* $c^2-4a\neq 0$ *and satisfy simultaneously the constraints* $b^2+c^2<4(a+bc)$ *and* $a(b^2-4bc)>\sqrt{bc}\left(4a+ab-ac+bc-c^2\right)$ *respectively, then the equilibrium point* $E_1$ *and* $E_2$ *are Jacobi stable and Jacobi unstable, otherwise.*

## 4. Dynamics of Deviation Vector

The behaviour of the deviation vector $\xi^i$, $i=1,2$, giving the trajectories behaviour of the dynamical system near a fixed point is described by equation (2.7) are as

(4.1)
$$\frac{d^2\xi^1}{dt^2}-c\frac{d\xi^1}{dt}-\left(a(1-c)+ax^2\right)\xi^1+ax^1\xi^2=0,$$
$$\frac{d^2\xi^2}{dt^2}+b\frac{d\xi^2}{dt}+\left(\frac{(c+a)x^1-2y^1}{a}\right)\frac{d\xi^1}{dt}$$
$$\left(2x^1x^2-2cx^1+\left(1+\frac{c}{a}\right)\right)\xi^1+\left(x^1\right)^2\xi^2=0.$$

The deviation vector obtained from its components, is given as

$$\xi(t)=\sqrt{\left[\xi^1(t)\right]^2+\left[\xi^2(t)\right]^2}$$

The instability exponents analogy with the Lyapunov exponent is defined as[7]

$$\delta_i(E)=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\xi^i(t)}{\xi_{i0}},\ i=1,2,$$

and

$$\delta(E)=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\xi(t)}{\xi_{10}}.$$

Now, we investigate the behaviour of the system (4.1) near the equilibrium points of the Lü system.

**(A) Dynamics of the deviation vector near $E_0$:** The dynamics of the deviation vector near point $E_0$ describes by the differential equations

$$\frac{d^2\xi^1}{dt^2}-c\frac{d\xi^1}{dt}+a(1-c)\xi^1=0,$$

$$\frac{d^2\xi^2}{dt^2}+b\frac{d\xi^2}{dt}=0.$$

Solving above system of equation, we get

$$\xi^1(t)=\frac{1}{c^2-4a+4ac}\left(e^{\frac{1}{2}\left(c+\sqrt{c^2-4a+4ac}\right)t}-e^{\frac{1}{2}\left(c-\sqrt{c^2-4a+4ac}\right)t}\right)\xi_{10},$$

$$\xi^2(t)=\frac{\left(1-e^{-bt}\right)}{b}\xi_{20},$$

where the initial conditions are $\xi^1(0)=0$, $\dot{\xi}^1(0)=\xi_{10}$, $\xi^2(0)=0$, $\dot{\xi}^2(0)=\xi_{20}$. The time behavior of the component $\xi^2(t)$ of deviation vector is depend only on the coefficient $b$. The deviation vector will be

$$\xi(t)=\left[\frac{\left(e^{\frac{1}{2}\left(c+\sqrt{c^2-4a+4ac}\right)t}-e^{\frac{1}{2}\left(c-\sqrt{c^2-4a+4ac}\right)t}\right)^2}{\left(c^2-4a+4ac\right)^2}\xi_{10}^2+\frac{\left(1-e^{-bt}\right)^2}{b^2}\xi_{20}^2\right]^{\frac{1}{2}}.$$

The behavior of deviation vector and instability exponent over time variation are shown in the figure 3.

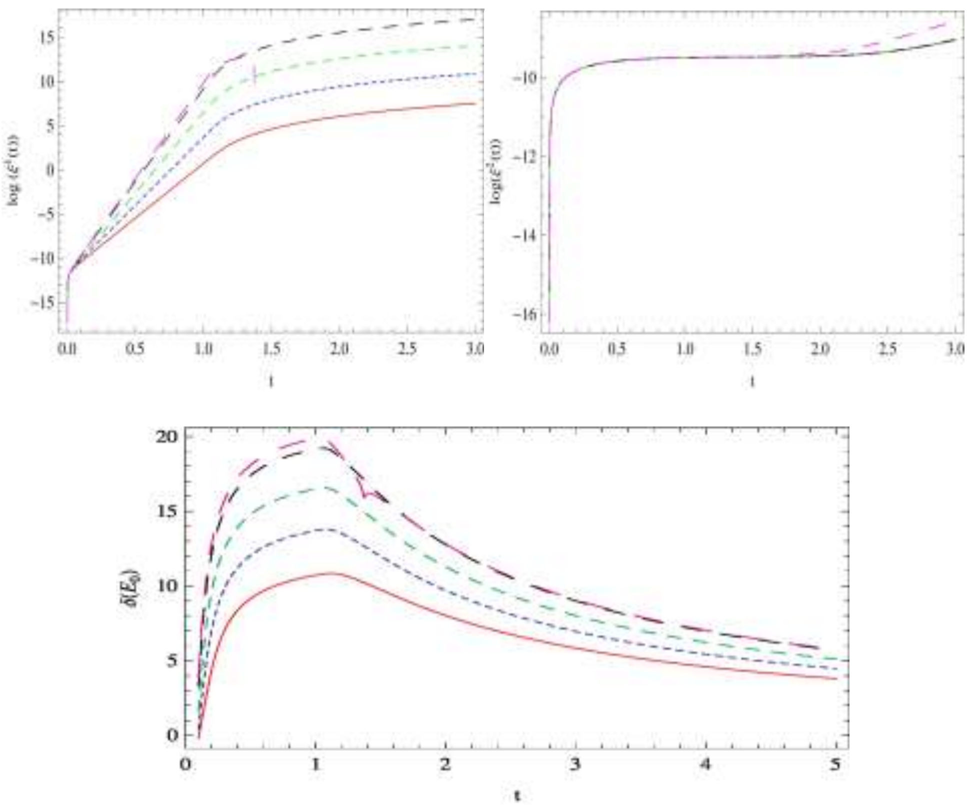



**Figure 3.** Time variation of $\xi^1(t)$, $\xi^2(t)$ and $\delta\left(E_0\right)$ respectively for parameters value $a=36$, $b=3$ and $c=13$ (solid, red), $c=18$ (dotted, Blue), $c=23$ ( dashed, Green), $c=28$ (long dashed, Black), $c=29.35$ (ultra long dashed, Magenta). The initial conditions for the numerical integration system are $\xi^1(0)=\xi^2(0)=0$, $\dot{\xi}^1(0)=10^{-10}$, $\dot{\xi}^2(0)=10^{-9}$

**(B) Dynamics of the deviation vector near $E_1$:** The dynamics of the deviation vector near point $E_1$ describes by the differential equations

$$\frac{d^2\xi^1}{dt^2} - c\frac{d\xi^1}{dt} + a\xi^1 + a\sqrt{bc}\,\xi^2 = 0,$$

$$\frac{d^2\xi^2}{dt^2} + b\frac{d\xi^2}{dt} + \frac{(a+c)\sqrt{bc}}{a}\frac{d\xi^1}{dt} + bc\xi^2 = 0.$$

The behaviour of the components of deviation vector and instability exponent over time variation are shown in the figure 4.

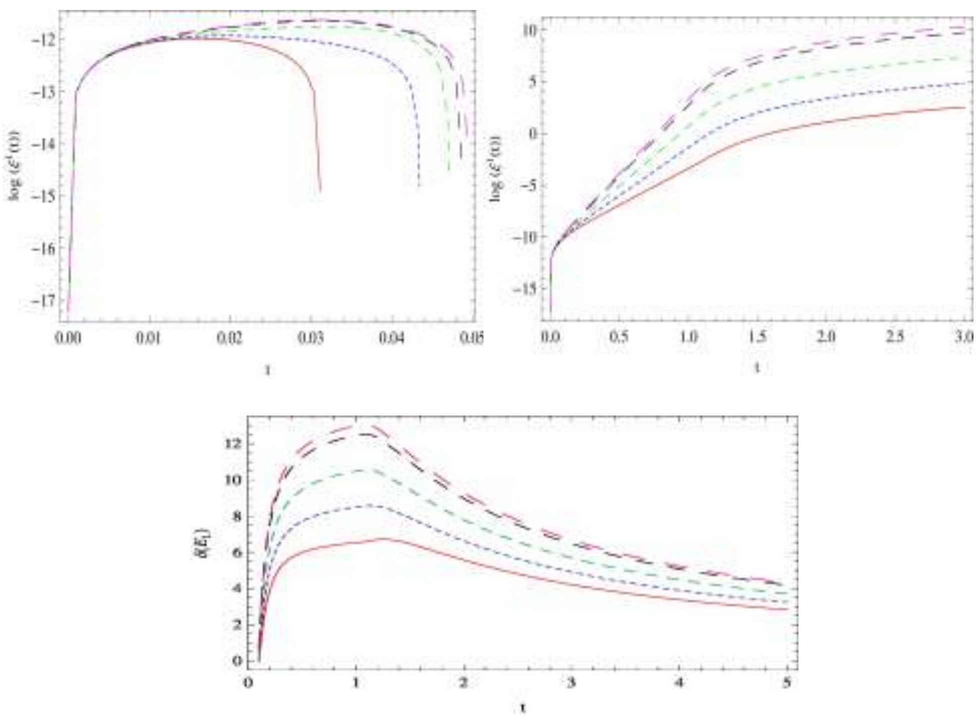


**Figure 4.** Time variation of $\xi^1(t)$, $\xi^2(t)$ and $\delta(E_1)$ respectively for parameters value $a = 36$, $b = 3$ and $c = 13$ (solid, red), $c = 18$ (dotted, Blue) $c = 23$ ( dashed, Green), $c = 28$ (long dashed, Black), $c = 29.35$ (ultra long dashed, Magenta). The initial conditions for the numerical integration system are $\xi^1(0) = \xi^2(0) = 0$, $\dot{\xi}^1(0) = 10^{-10}$, $\dot{\xi}^2(0) = 10^{-9}$

**(C) Dynamics of the deviation vector near $E_2$:** The dynamics of the deviation vector near point $E_2$ describes by the differential equations

$$\frac{d^2\xi^1}{dt^2} - c\frac{d\xi^1}{dt} + a\xi^1 - a\sqrt{bc}\,\xi^2 = 0,$$

$$\frac{d^2\xi^2}{dt^2} + b\frac{d\xi^2}{dt} - \frac{(a+c)\sqrt{bc}}{a}\frac{d\xi^1}{dt} + bc\xi^2 = 0.$$

The behaviour of the components of deviation vector and instability exponent over time variation are shown in the figure 5.

## 5. Conclusion

In this paper, some geometric properties of Lü system have been investigated by using KCC theory. First we have formulate the Lü system equivalent to a set of two second order nonlinear differential equation. We obtain the components the nonlinear connection, Berwald connection and five KCC invariant. The components of first KCC invariants vanish except $\varepsilon^1$ at the equilibrium points $E_1$ and $E_2$. The components of deviation tensor is obtained and the time variation is shown in figure 2. We have also shown the time variation of trace and determinant of deviation tensor. We find the Jacobi stability condition at the equilibrium points. The equilibrium point $E_0$ is Jacobi unstable. When the parameters $a > 0$, $b > 0$, $c > 0$ and $c^2 - 4a \neq 0$ and satify the contraints $b^2 + c^2 < 4(a+bc)$ and $a\left(b^2 - 4bc\right) >$ $\sqrt{bc}\left(4a + ab - ac + bc - c^2\right)$, the equilibrium point $E_1$, $E_2$ are Jacobi stable. At lastly we discuss the beahavior of deviation vector near the equilibrium points.

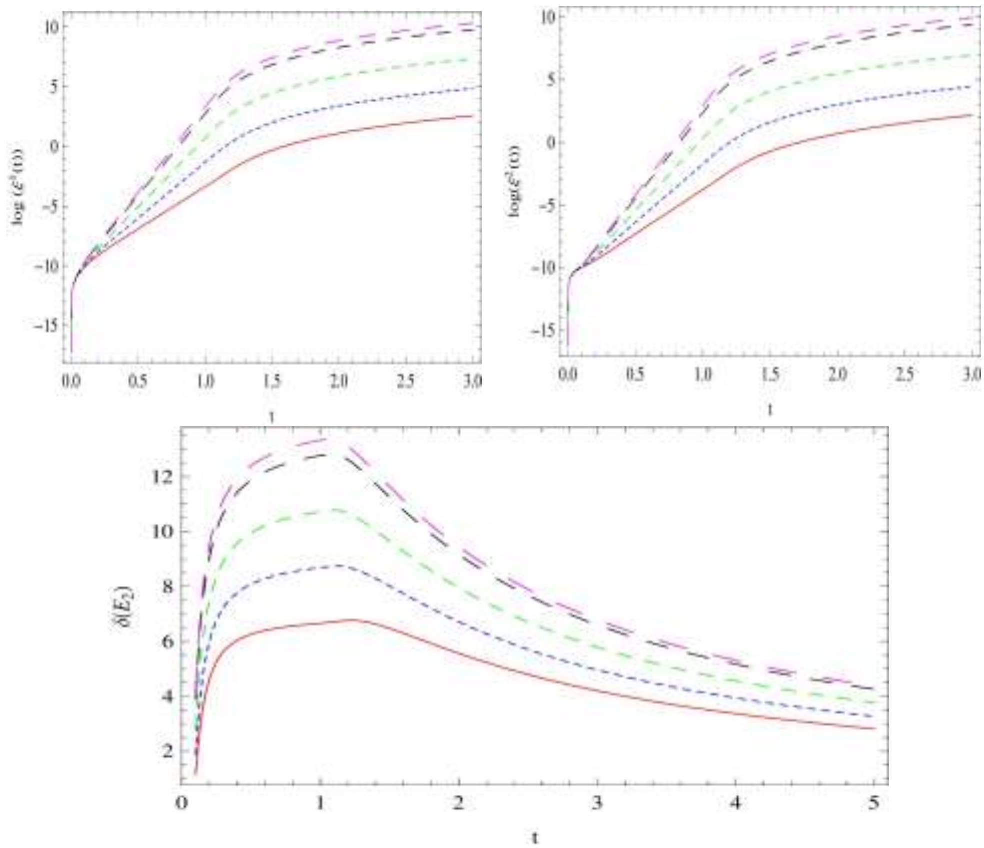



**Figure 5.** Time variation of $\xi^1(t)$, $\xi^2(t)$ and $\delta(E_2)$ respectively for parameters value $a=36$, $b=3$ and $c=13$ (solid, red), $c=18$ (dotted, Blue), $c=23$ ( dashed, Green), $c=28$ (long dashed, Black), $c=29.35$ (ultra long dashed, Magenta). The initial conditions for the numerical integration system $\xi^1(0)=\xi^2(0)=0$, $\dot{\xi}^1(0)=10^{-10}$, $\dot{\xi}^2(0)=10^{-9}$

The contraints $b^2+c^2<4(a+bc)$ and $a(b^2-4bc)>\sqrt{bc}\,(4a+ab-ac+bc$ $-c^2$, the equilibrium points $E_1$, $E_2$ are Jacobi stable. At lastly we discuss the beahavior of deviation vector near the equilibrium points.

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